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A SURVEY OF ANALYSIS OF SHELLS BY THE DISPLACEMENT METHOD

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This paper presents a study of the current state of the art in the analysis of elastic shells by the direct stiffness method. The considerations deal chiefly with linear, static problems. The capabilities and weaknesses of the conical shell element and a newly developed curved element are studied in depth. Criteria are given which predict the necessary smallness of elements required to obtain shell stress analyses with acceptable levels of accuracy. The applicability of the current analysis methods to vibrations and to nonlinear problems is discussed. The development of a polygonal curved shell element, which is the major gap in the current set of analysis tools, is discussed briefly.

INTRODUCTION

This paper is concerned with the analysis of shells and shell structures by the displacement, or direct stiffness, method. The discussions are limited to elastic shells and deal chiefly with linear, static problems. Shell vibrations and geometrically nonlinear behavior are mentioned briefly. The paper is in part a survey of existing analysis methods, and in part a study in depth of two of these methods, the latter for the purpose of illustrating some of the important features of finite element shell analysis. The weaknesses of current methods and the gaps in the state of the art are discussed.

There are currently three major approaches to shell analysis by the displacement method. In the order of their introduction, they make use of (a) triangular or quadrilateral flat plate elements, (b) the conical shell element, and (c) the axisymmetric solid element. The first approach was given by Greene, Strome, and Weikel, (Reference 1). With it, the shell is replaced by an assemblage of flat plates which are individually of triangular or quadrilateral shape. Figure 1 shows such an idealization of a shell. Each plate element is connected to those surrounding it and is permitted to deform both in bending and in a plane stress state. Continuity of displacement and slope between neighboring elements is maintained along their common boundaries. The element used by these authors is in reality a sandwich plate element whose cover sheets are membranes and whose core resist only transverse shear.

The second approach is restricted to axisymmetric shells, and utilizes an element which is a frustum of a cone. The shell is represented by a stacked assemblage of these conical elements such as that shown in Figure 2. The conical element is permitted to deform in the membrane and bending states, and continuity of displacement and slope is enforced on the nodal circles along which neighboring elements are connected. The first use of the conical element was by Meyer and Harmon (Reference 2). These authors used for the deformation state of the shell element the analytical solution of edge loaded conical shells, and they restricted their considerations to edge loading problems. Their treatment made use of the force rather than the displacement method of finite element analysis. Shortly thereafter, Grafton and Strome, (Reference 3), presented the conical element within the framework of the direct stiffness method. They used for the displacement state of the shell element simple

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polynomial forms, which, as will be discussed later, is in agreement with the basic philosophy of the direct stiffness method. This formulation has found much use in practical shell analysis problems. In their derivation, Grafton and Strome approximated the integral of the strain energy of the shell, and subsequent work has shown this approximation to reduce the accuracy of the method. The next use of the conical element was by Popov, Penzien, and Lu, (Reference 4). These authors made use of the analytical solution of edge loaded conical shells, as did Meyer and Harmon, but presented a displacement rather than a force method formulation. They applied the method to problems other than those of edge loaded shells. In these problems, the use of the analytical solutions for the displacement state departs from the basic philosophy of the direct stiffness method, and inferior results may be expected. All of the above mentioned authors considered the case of axisymmetrical deformations. The extension to unsymmetrical deformations, by means of Fourier expansion, was given by Percy, Fian, Klein, and Navaratna, (Reference 5). These authors used the polynomial displacement forms of Grafton and Strome and removed the strain energy integral approximation of the latter, demonstrating improved accuracy as a result. They also studied the effect of including higher order polynomials in the displacement forms. The result of this is to cause a likeness between the Grafton and Strome type of conical element formulation and that of Harmon and Meyers, and Popov, et al. For edge loading problems, Percy, et al, demonstrated an increased accuracy for the case of higher order polynomials. For other types of problems, however, as mentioned above, deterioration of the accuracy may be expected.

The third major method of shell analysis applies to axisymmetric shells and is of use principally for thick shells or in regions of thin shells where junctures between adjoining shell structures require a considerable thickening of the shell wall. The method utilizes a ring element of triangular cross section which herein will be called the ring-wedge element. Figure 3 shows the element form and the representation of an axisymmetric solid, here a shell juncture region, by an assembalge of elements. The sectioned view is customarily used for this purpose. The method was first published by Clough and Rashio, (Reference 6), for axisymmetrical deformations, and was extended to the unsymmetrical case by Wilson, (Reference 7). The ring-wedge element has found considerable use in practical applications.

A fourth method has been developed, (Reference 8), in order to remedy difficulties encountered with the conical element. This method utilizes a double curved element in place of the singly curved conical element. With it, a shell is represented by an assemblage of curved elements in which the slope as well as the coordinates of the actual shell surface are matched by those of the assemblage at the nodal circles. The representation of an axisymmetrical shell by an assemblage of such elements is as shown in (Figure 2) except that the shell elements are curved in the meridional direction. The results of calculations made with the new element will be discussed in this paper.

In order to set the stage for later discussions, it is necessary at this time to consider some conceptual points relative to the analysis of shells by the direct stiffness method. The basis of this discussion arises in part from the fact that the direct stiffness method is closely related to, and in most cases is, the method of stationary potential energy, i.e., the Ritz method. To illustrate this point, consider the plane stress problem. Customarily, the plane structure, or plate, is represented by an assemblage of triangular elements. Each element is permitted to deform such that each of the three strains in its plane is constant over the element. As a result of this, its sides remain straight lines, and continuity of deflection is exactly preserved. Consequently, the total deformation of the assemblage is the sum of a set of continuous localized deformation states, each of which is, within the triangular regions, described by constant strains. This situation is completely within the scheme of the Ritz method. The important conclusion which may be drawn from this is that there is actually no approximation of the structure itself. Instead there is only approximation of its displacements. In subsequent discussions herein we will distinguish between approximation of the structure and of its displacement state. In the case of shells this has important consequences.

A second conceptual point of the direct stiffness method concerns the displacement approximation itself. The basic idea of the method is that through the use of many small elements it will be possible to represent complicated total structure deformations with very simple element deformations, e.g., the constant strains used in the plane stress problem. Experience has upheld this viewpoint, and, in fact, in some instances where complicated element displacements have been used, numerical results of low accuracy have been obtained.

This condition has always occurred when the structure itself, in addition to its deformation state, has been approximated. The approximation of the structure has the effect of disproportionately emphasizing the participation of the complicated deformation forms in the determination of problem solutions. In the direct stiffness method, approximation of the structure should be avoided if at all possible. If this is not possible, care should be taken to choose displacement states of the simplest obtainable forms.

Since the direct stiffness method is in reality an energy method, it is necessary to evaluate the loadings to be used in calculations by means of the energy principle. That is, generalized loads based on the approximate deformation shapes must be determined. The often used loadings based on the simple replacement of a distributed load over a small region by an "equivalent" concentrated force at the center of the region are not satisfactory in many problems. Reference 9 discusses the determination and use of generalized loads in the direct stiffness method.

THE CONICAL ELEMENT

The discussion of the capabilities of the various shell elements begins most conveniently with the conical elements. A consideration in some depth of this element sheds light on the whole subject of stiffness method shell analysis. There are many numerical results available from which to draw conclusions, and the element has been given much theoretical consideration, References 2 through 5. Moreover, the simplicity of application of the Grafton and Strome, and Percy, et al, type of conical element makes it important to deduce under what circumstances it can be used with satisfactory results.

The calculation of a doubly curved shell, such as a spherical or elliptical shell, by means of the representation by an assemblage of conical elements involves in general both of the approximations discussed in the previous section. This is a clear case in which the structure itself is approximated. In the conical element of References 3 and 5 the displacement state is also approximated. In the element of References 2 and 4 the displacements are exact for the case of edge loaded conical elements but approximate for other cases. Therefore, in the application of the conical shell element to a general class of problems, what is actually being computed is an approximate solution to the problem of an assemblage of conical frustums. The ability of this solution to yield an estimate of the stresses and deformations of the doubly curved shell then depends on two factors: (1) the similarity of behavior of the conical assemblage to that of the doubly curved shell, and (2) the accuracy lost in the calculation of the conical assemblage due to the approximation of its displacement functions. These two items prompt the question of under what conditions the approximate solution to the conical assemblage problem, using the polynomial displacement forms of References 3 and 5, might be a better estimate of the behavior of a doubly curved shell than would be a solution of the conical assemblage using the exact displacement functions of References 2 and 4. The total answer to this question can only be obtained through the numerical solution of many problems for which exact solutions are available. Some of these results will be given herein. Before proceeding to these results, however, it is first useful to examine briefly and in a qualitative way the comparative behavior to be expected from a conical assemblage and a doubly curved shell.

The similarity of behavior of a conical assemblage and a doubly curved shell is considered in this paper for three types of problems: (1) the edge bending or influence coefficient problem,

(2) the predominately membrane problem, and (3) problems in which the membrane and bending behaviors are inter-related parts of a total solution. The following discussions will consider the problems in this order. It is recalled that the local bending behavior of a doubly curved shell of revolution is, in analytical solutions, often approximated by that of the tangent conical shell. This result follows from the customary procedure of solving edge bending problems by retaining only the highest derivatives in the pertinent differential equations. For nonshallow shells and shells with large radius to thickness ratios, results obtained in this way have been shown to be accurate, (Reference 10). For shallow shells and shells with small radius to thickness ratios, it appears that this would again be true. However, to the writers' knowledge, this has not been demonstrated.* Based on these facts concerning analytical solutions, it appears that for thin, nonshallow shells, a conical assemblage will behave like the corresponding doubly curved shell, and the conical element idealization should yield excellent numerical results. For shallow shells, or shells with small radius to thickness ratios the accuracy of the conical element remains to be established. In both cases, the necessary smallness of conical elements required to achieve a stated accuracy requirement needs to be determined. For loadings of the type which produce large membrane stresses, in particular the meridional stress, we can reason that the behavior of a conical assemblage departs significantly from that of a doubly curved shell. The prominence of the meridional radius of curvature in the analytical membrane solution of such problems motivates this view.

To illustrate the behavior in question, consider the membrane solution of a pressure loaded conical shell. Both the normal and the meridional displacements are found to depend on the included angle of the cone and to vary as the square of the meridional length measured from the apex. Consider a two element conical assemblage such as that shown in Figure 4. The elements have different distances from their respective apexes and different cone angles, and hence in a membrane state they have different displacements and slopes at the common nodal circle. The deflections are shown schematically by the solid curves in the figure. The dominating influence in this problem is due to the nearly membrane forces exerted by each element on the other. These are indicated by the arrows shown superimposed on the elements in the figure. It is seen that each force is directed more inward than an actual membrane force on the element would be. Consequently, each element deflects inward relative to the membrane displacement state near the nodal circle. This is indicated by the dotted curves in the figure. As a result, there is a large incompatibility of slope between the elements, resulting in bending moments as shown in the figure. We conclude that for pressure loaded doubly curved shells of revolution, solutions obtained by means of the conical element will yield erroneous meridional moments. It is not expected that the deflections predicted by the conical element will be as seriously in error as the moments, but it remains to be established whether their accuracy will in fact be satisfactory. Numerical results given later will show numerical values of these quantities and indicate the range of validity of solutions obtained by the conical element idealization. We proceed now to detailed discussions of the two conical element formulations, those of (1) Grafton and Strome, and Percy, et al, and (2) Meyer and Harmon, and Popov, et al, beginning with the latter.

1. The Meyer and Harmon, and Popov et al, Conical Element.

In the formulations of these authors the conical shell element is permitted to deform according to the mathematical functions which are the analytical solutions of the differential equations of conical shells. The particular solutions used were derived for the conical frustum subjected to loads and moments applied to its edges. Consequently the stiffness method solution based on these shape functions is exactly that of a conical assemblage loaded

*It has been shown in Reference 10, that the approximate edge bending solution for conical shells, the Geckeler solution (Reference 11), is not a good approximation for shallow double curved shells.

at the nodal circles. This solution differs from that of a conical assemblage loaded by distributed loads such as pressurization, since the authors did not make use of the correct generalized loads corresponding to their displacement functions. Therefore their solutions yield a double approximation, in which the first is the approximation of a conical assemblage to a doubly curved shell, and the second is the approximation that all loads are represented by concentrated nodal circle loads.

For the edge bending problem, in which the loading in question is actually applied on a nodal circle, the second approximation becomes exact, and we have only to consider the approximation of the shell itself. In the nonshallow region and when the radius to thickness ratio is large, earlier discussions suggest that this conical element should yield excellent results provided only that small enough elements are used that the elements are close to being tangent to the shell surface. Reference 2 presents numerical results in which relatively large conical elements were used and excellent influence coefficient data obtained. In the shallow region and for small radius to thickness ratios the suitability of a conical assemblage as well as the necessary smallness of the elements is in question. The data shown in Reference 2 do not adequately cover this matter since, following Reference 10, the examples given are not in the pertinent range of geometries.

We next consider the essentially membrane solution to a pressure load problem. In order to discuss this problem we present results which are nearly equivalent to those which would result from the conical element formulation of Popov, et al, but have been computed by a different approach. What has been done here is illustrated in Figure 5. A doubly curved shell is approximated by a coarse conical element representation, in which each large element is further subdivided into six smaller conical elements. Each conical element is treated by the formulation of References 3 and 5*. The freedom of deformation afforded the large conical elements due to their further subdivision causes the solution obtained here to be a close approximation the the correct solution of a conical assemblage problem, provided the correct generalized loads are used. We present for comparison also the solutions resulting when the loads are applied only at the nodal circles joining the large conical pieces. This solution should be a close approximation to one resulting from the formulation of Popov, et al.

The comparison of these results is shown in Figures 6 and 7. The problem in question is a hemisphere loaded by internal pressure. The boundary conditions at the base are as shown in the diagram; note that the rotation is constrained to vanish. The dashed curve in Figure 6 shows the exact deflection of the spherical shell which corresponds to this conical element assemblage. It is seen that the two deflection solutions oscillate about the spherical shell solution and tend to be mirror images of each other. In the solution corresponding to static equivalent loads an accumulating error in the meridional displacement distorts the mirror image tendency and necessitates an early termination of the deflection plot. The four conical elements in this example are so large that accurate results cannot be expected, even for the deflections. The example was chosen to illustrate the nature of the solution rather than the level of accuracy inherent in the deflection predictions of the Popov, et al, conical element. With smaller elements, for these boundary conditions, the deflection solution should become quite accurate. The more critical item in this example is illustrated in Figure 7. Here the meridional moments are shown, and the edge bending behavior induced by the junctures between the large conical elements is clearly seen. The mirror image tendency is pronounced in this case, illustrating the differences in solutions to be expected from the use of generalized and static equivalent loadings. It will be seen subsequently that these moments are unacceptably large and do not reduce to acceptable values even for small element sizes. In this example the bending fiber stress is about 21,000 psi, compared to a membrane stress of

*The corrected strain energy integration is used here and in all other applications of this element in this paper.

5000 psi. Comparisons of these results with those given in Reference 4 shows that here the meridional moments are somewhat smaller than those which would be predicted by the method of Reference 4. This is due to the representation here of the large conical elements by only six subelements. In Reference 4, on the other hand, the behavior of the large conical elements would be represented exactly.

For the case in which the slope is unconstrained at the base of the shell there occur two changes in the results discussed above. First, the moment at the base of the shell necessarily vanishes. In the interior of the shell, however, the moments are unchanged from the values shown in Figure 7. Second, the displacement at the base of the shell increases by more than an order of magnitude.

It is found that only for very fine idealizations does the deflection error at the base of such a shell, where the rotation is unconstrained, reduce to a tolerable level. This problem is representative of the important practical one in which an engineering design has been performed to obtain as nearly as possible a membrane state of stress, and the stress analysis is being conducted to determine the fiber stresses and deflections of the structure. The failure of this conical element to handle this problem adequately is a serious flaw in the method. It will be seen that this failure occurs also in the conical element of References 3 and 5.

For problems in which the boundary constraints of a pressurized shell are such that significant edge bending occurs, the conical element of these authors (and also that of References 3 and 5) is again found deficient. Even for fine idealizations, the error in the meridional moment due to the edge bending induced by the element junctures causes appreciable error in the predicted support moments. In cases in which the support bending stresses are very large, the errors due to the idealization are not relatively so important, and the conical element can be used, provided a very fine idealization is employed.

The ideal application of this element is for influence coefficient calculations in cases where analytical solutions are cumbersome or unavailable. This is the application suggested by Meyer and Harmon, Reference 2.

2. The Conical Element of Grafton, Strome, and Percy, et al.

In this case again we have an approximation of the shell by a conical assemblage. We have here additionally an approximation of the displacements of the elements in which very simple functions are used. All of these authors represent the meridional displacement by a linear function and the normal displacement by a cubic function of meridional length. Hence here again we have solutions in which a double approximation is involved. Whereas in the conical element of the previous section we had solutions of a conical assemblage loaded approximately but permitted to deform exactly corresponding to those loads, here we have solutions of a conical assemblage permitted to deform only approximately but loaded in a manner consistent with those deformations.

The philosophy involved in the use of this element is that many small elements, each deforming in a simple way, should provide a good approximation to the actual structure. In any region of the structure in which the character of the deformation varies rapidly it is necessary to use a fairly large number of such elements. Clearly this is the case in edge bending calculations.

Results presented by Grafton and Strome and later modified by Percy, et al, demonstrate the capability of this conical element formulation to give accurate edge effect calculations. Using the corrected strain energy integration of the later work, it was found that excellent influence coefficient results were obtained with elements whose meridional length was equal to twenty times the thickness of the shell wall. Appreciable inaccuracy was obtained with elements as large as fifty times the thickness. The approximate strain energy integration of Grafton and Strome yields results of less accuracy, and, based on the comparative data given in Reference 5, the corrected integration appears to be necessary. These results were obtained for a cylindrical shell, but will be of fairly general applicability for influence coefficients in all nonshallow shells. For shallow shells and shells with small radius to thickness ratios, the accuracy of the conical element in influence coefficient calculations will be established in this paper. In order to do this we will determine the smallness of the conical elements required in order to predict influence coefficients to a prescribed level of accuracy. The analytical edge bending solution for axisymmetric deformations of shells of revolution, as given by Novozhilov, (Reference 11), shows that the rapidity of variation along the meridian of edge bending behavior depends on the thickness and on the radius to thickness ratio of the shell. More specifically, if we define a normalized meridional length variable to be the true meridional length divided by the thickness, the rapidity of variation of the edge bending behavior with respect to this normalized length depends on the square root of the ratio of the hoop principle radius of curvature of the shell to the thickness. This suggests that the necessary conical element size for influence coefficient calculations takes the form

$$\frac{\Delta S}{t} = K \sqrt{\frac{R_2}{t}} \tag{1}$$

where ΔS is the required meridional length of an element, K is a constant, t is the thickness of the shell, and R_2 is the principle radius of curvature of the element in the hoop direction. To define the shallow and nonshallow regions we introduce a quantity X_0 which is similar to one used in Reference 10,

$$X_0 = \sqrt[4]{12(1-\nu^2)} \sqrt{\frac{R_2}{t}} \sin\phi_0 \tag{2}$$

in which ν is Poisson's ratio and ϕ_0 is the angle between the axis of symmetry and the normal to the shell surface at the boundary. We find that for $X_0 > 5$, which is the case of nonshallow shells, the influence coefficients are predicted within 2 percent of the exact* values with K = 1. For $X_0 \lessgtr 5$, the case of shallow shells, the same accuracy is obtained with K = 1/2. These results have been determined from the solutions of a range of example problems for spherical shells, of which several are given in Tables 1 and 2. The criterion of Equation 1 was found reliable for all cases and is expected to have general applicability for influence coefficient problems. The criterion refers to a group of, say, four or more equally small elements near the boundary. The accuracy of the calculations is significantly improved by the subdivision of the single element adjacent to the boundary into two smaller elements. This effect is clearly shown by the tabulated results, in which case a is derived from case b in this manner.

It is noted in passing that the conical element of References 2 and 4 will provide better influence coefficient accuracy with fewer elements than that of References 3 and 5. The reason for this, of course, is the use in this element of the exact conical shell edge bending solution in the displacement functions of the element.

To sum up the situation with regard to the use of the conical elements for influence coefficient problems, we can state that they provide excellent accuracy with relatively few elements. There appears to be no reason to use the formulations of Meyer and Harmon, and Popov, et al, since these are very complicated to program for digital computation and excellent accuracy is practically obtainable with the formulations of References 3 and 5. The corrected strain energy integration should be used. As a side benefit of the use of the conical element for these problems it is noted that the distribution of stresses away from the loaded edge of the shell is obtained along with the influence coefficients themselves.

*The exact values are taken as those predicted by Gellatly, Reference 12.

We proceed to discuss the solutions of predominantly membrane problems, taking as examples the pressurization of a spherical shell. These problems have been seen to be the weakness of the conical element, and we expect errors in the subject element similar to those occurring in the element of References 2 and 4. Figure 8 shows results computed for the same spherical shell as was discussed in the previous section. The rotation at the base of the shell is constrained to vanish. The comparison between this Figure and Figures 6 and 7 illustrates the effect on the solution of the use of the simple polynomial shape functions of the present formulation. It is seen that the deflections have improved markedly over the results of the previous section. The meridional moment remains unacceptably in error, however, and, in fact, is found to have nearly the same numerical value (but opposite sign) as that predicted by the method of Reference 4. With the addition of only a few more elements, the deflection results, for this boundary condition, converge rapidly to highly accurate values. This is characteristic of the conical element of References 3 and 5. The moments also converge with the addition of more elements, as shown in Figure 9, but in this case the covergence is unsatisfactory. For example, taking the finest elementization indicated by the figure, the erroneous bending stress is still nearly half as large as the membrane stress itself.

A more important practical problem is the one in which the rotation at the boundary is unconstrained. This is the problem of an essentially membrane shell supported by membrane boundary conditions. The results for this case differ from those of Figures 8 and 9 in two ways. First, the moment necessarily vanishes at the boundary, and second, the displacement near the boundary becomes unacceptably in error. In the interior of the shell, both the displacement and the moment remain as shown in Figures 8 and 9. In Figure 10 the displacements near the boundary are shown. It is seen that the removal of the erroneous moment at the boundary gives rise to a sort of "tucking" behavior on the part of the deflections. This recalls the qualitative discussion relative to Figure 4. Even for a reasonable fine idealization, the deflection error at the boundary is seen to remain quite large.

The behavior of the deflection near the boundary indicates that its accuracy in the case of vanishing boundary rotation, Figure 8, is actually fortuitous. What actually happens is that if either the rotation or the normal deflection at the boundary is constrained to take its correct value, the other also takes its correct value. Clearly, in the majority of practical problems, such special boundary constraints would not be encountered. Hence, it should be expected in general that the conical element will give inaccurate displacements near boundaries unless very fine idealizations are used. It has been found that large displacement errors can be corrected only by carrying out the fine idealization a considerable distance from the boundary.

The erroneous meridional moments predicted by the conical element can be expressed in terms of the geometry of the shell and the meridional membrane stress. It is found that

$$\left[M_\phi \right]_{error} \approx 0.09\,(\Delta\phi)^2\, N_\phi\, R_1 \tag{3}$$

where $\Delta\phi$ is the change in slope angle along the meridian between elements, measured in radians, R_1 is the meridional radius of curvature in inches, and N_ϕ is the meridional membrane stress, pounds per inch. $\Delta\phi$ is equal to half the sum of the subtended angles of the elements adjacent to the nodal circle in question. The sign of the moment is as shown in Figure 4.

The stress intensities due to this moment and due to the membrane stress have the ratio, based on Equation 3

$$\frac{\left[\sigma_\phi\right]_{error}}{\left[\sigma_\phi\right]_{membrane}} \approx \frac{1}{2}\left(\frac{R_1}{t}\right)(\Delta\phi)^2 \tag{4}$$

where t is the thickness of the shell. It is seen that the stress error ratio, Equation 4, is proportional to the radius to thickness ratio of the shell as well as to the $(\Delta\phi)^2$ corresponding to the fineness of the idealization. If we arbitrarily take the permissible stress error to be 5 percent, the required fineness of the idealization is

$$\Delta\phi \leq \sqrt{\frac{t}{10 R_1}} \approx \frac{1}{3}\sqrt{\frac{t}{R_1}} \quad \text{or} \quad \frac{\Delta S}{t} \approx \frac{1}{3}\sqrt{\frac{R_1}{t}} \tag{5}$$

This is a surprising result, since it is seen from Equation 1 that the required fineness of idealization for edge bending calculations is a less severe requirement. The results given by Equations 3, 4, and 5 are of general applicability for problems in which N_ϕ and R_1 do not vary extremely rapidly compared to the size of the conical elements used in an analysis. They will be seen to apply to other problem types in the discussions which follow.

The errors predicted by the formulae will carry into the edge zone in problems where the boundary constraints induce severe edge bending behavior. Results illustrating this are shown in Figure 11. The figure illustrates the case of a pressure loaded spherical shell with its boundary fixed against displacement and rotation. The correct edge moment for this problem is 600 inch-pounds per inch. The conical element is seen to predict the basic behavior of the shell, which is a combination of severe bending near the support and membrane behavior in the interior, only for very small elements. The erroneous moments in the interior of the shell are seen to carry into the edge zone and affect the value of the support moment. The moment errors are consistent with Equation 3. The calculated numerical values of the support moment exceed the correct value by almost exactly the plotted moment which occurs in the interior of the shell.

In this type of structure, if the design is based entirely on the stresses occurring at the boundary, then the conical element can be utilized in the analysis provided very small elements are used. On the other hand, if the design attempts to achieve a constant margin of safety in the stresses, for example by careful variation of the thickness or the geometry of the shell, then the conical element will probably be incapable of satisfactory analysis with feasible element sizes.

THE DOUBLY CURVED SHELL ELEMENT

Recently there has been developed by the writers a shell element in which the meridian is curved, (Reference 8). The purpose in deriving this element was to gain an improved capability, compared to that of the conical element, to solve stress analysis problems in which the shell loading is distributed, such as due to pressurization. Results of computations indicate that this goal has been achieved.

The curved element is constructed so that, in an assemblage of elements, the tangent to the meridian curve is continuous. The result of such a construction is an element assemblage whose coordinates, slope, and hoop principal radius of curvature are everywhere continuous functions and are identical to those of the actual shell at the nodal circles. The meridional radius of curvature is a discontinuous function whose jumps occur at the nodal circles, but which, even for coarse elementizations, is a close approximation to the meridional radius of curvature of the shell itself.

If the curved element is applied to the analysis of the pressurized hemispherical shell, it is found that the moments and deflections are predicted to a high order of accuracy. The errors which occur decrease to negligible size for idealizations in which as few as four elements are used for the hemisphere. Moreover, the results are found to be unaffected by whether the slope at the support is unconstrained or is constrained to vanish. Hence the curved element correctly recognizes the membrane nature of this problem. For the pressurized spherical shell with fixed lower boundary, discussed in the previous section and shown in Figure 11, calculations were carried out with the same element sizes as were used for the conical element. The results are shown in Figure 12. The moment correctly decreases to negligible values in the interior of the shell. The support moment is predicted within 1 percent accuracy by the two finer idealizations. It is noted that with the curved element it is not necessary to use the fine idealization except in the near vicinity of the boundary.

In order to demonstrate the applicability of the curved element for nonspherical shells, we consider a 2 : 1 ellipsoidal shell. Figure 13 shows comparative results computed with the curved element and the conical element for this shell, loaded by pressure, and constrained to have vanishing rotation at the supports. In this problem the meridional moment should not vanish in the interior of the shell, since the shell is not in a completely membrane state. The curved element results converge rapidly to the moments shown in the figure for the 25 element assemblage, yielding the correct moment* at the support. The exact solution for this problem in the interior of the shell is not known to the writers, but the rapid convergence of the curved element results suggest that it is given by the lowest curve in the figure. The moment errors shown for the conical element are consistent with Equation 3. If the values of $\Delta\phi$, R_1, and N_ϕ corresponding to the nodal circles are inserted in Equation 3 and the resulting moment predictions are measured upward from the lowest curve in the figures, the conical element results are closely duplicated. The rise in the conical element moment at the right side of the figure, for the 25-element assemblage, corresponds to the change there to a coarser idealization. This, also, is predicted well by Equation 3.

The calculation of influence coefficients with the curved element shows comparable accuracy as with the conical element, since here the accuracy depends primarily on the ability of the element deformation shapes to represent the rapidly varying edge bending behavior, rather than on the closeness with which the element assemblage represents the shape of the actual shell. It is found that the element size criterion, Equation 1, is valid for the curved element, with the influence coefficient predictions less accurate by about 1/2 percent than for the conical element.

It has not been possible to determine an element size criterion for the curved element analogous to Equations 3, 4, and 5. The errors in such solutions with the curved element are governed not by the adequacy of the structural representation, as was the case for the conical element, but by the adequacy of the displacement shapes used in the derivations of the curved element.

THE FLAT PLATE ELEMENT

In the use of an assemblage of quadrilateral or triangular flat elements to represent a shell, we have a method which is not restricted to shells of revolution. Such a method was given by Greene, Strome, and Weikel, (Reference 1, Figure 1), which is taken from Reference 1, illustrates the flat element idealization for a shell of revolution which was studied by these authors. It is noted that a fine idealization by means of the flat elements implies the use of many elements in two directions in the shell surface rather than only meridionally as in the case of the conical element. Consequently a very large number of elements may be required for such analyses.

A shell of arbitrary shape can be readily formed by an assemblage of triangular elements, while the quadrilateral elements are restricted to cases in which there can be conveniently found groups of four points (nodes) which lie on a plane. Such a circumstance is only possible when the nodes of a quadrilateral element are located on two lines of curvature in the shell surface. This limits the use of flat quadrilateral elements to problems in which it is possible

*The correct support moment is computed from the results of Gellatly, (Reference 13).

to determine a coordinate system on the shell surface which consists of the network of lines of curvature. Shells of revolution are obvious examples of such cases.

In the case of flat plate elements we have a situation similar to that encountered in the conical element, in that both the shell structure and the deformation are approximated. The conclusions reached in our earlier discussions of the conical element concerning the qualitative effects of these types of approximations would appear to apply here also. It is expected that with fine idealizations the flat elements should have the ability to yield accurate influence coefficient data. It is expected that for problems where distributed loads induce large membrane stresses, there will result erroneous moments and, in the case of membrane support conditions, erroneous deflections near the boundary. In addition, since here we have sharply changing slope in the hoop as well as in the meridional direction, we expect erroneous hoop moments to occur when the hoop membrane stress is large. These suspicions cannot be examined with the data given in Reference 1, for reasons discussed below. However, the authors do show the expected deflection error near the boundary where membrane boundary conditions are prescribed. This is shown in Figure 14, which has been taken from Reference 1. The dotted curves have been added by the writers in order to show a similarity between these results and those of Figure 10.

The analogy between the conical elements discussed herein and the flat element of Reference 1 is incomplete, since in Reference 1 the authors made use of restricted types of displacement forms which differ markedly from those of the conical element. The flat elements used were restricted to remain flat in the deformed state, with all deflection of the shell occurring as a result of slope changes taking place at the interelement boundaries. These slope changes were permitted by the transverse shear flexibility of the elements. This type of displacement function departs sufficiently from actual shell bending behavior that a discussion in depth of this flat element is not possible. It is clear that the full potential, as well as a clear understanding of the weaknesses, of the flat element approach has not been reached due to a lack of continued study of this type of element.

The real value of the flat plate element idealization is its applicability to cases other than shells of revolution. For example, it could be used to compute stresses near nozzle cutouts in rocket motor case heads or to provide a complete shell analysis of the complicated motor cases of clustered boosters. Unfortunately, these problems generally involve large membrane stresses, so that erroneous moments would be expected in solutions obtained by assemblages of flat elements. The further development of this approach would be an instructive, and possibly fruitful, subject for research.

THE RING-WEDGE ELEMENT

This element, shown in Figure 3, has been used little in shell problems. It was developed for the stress analysis of solid propellant grains, primarily, and has been used for these and other problems involving massive axisymmetrical bodies. It should be most useful in the stress analysis of regions of shell structures where the approximations of thin-shell theory, particularly the Kirchhoff hypothesis, are unsatisfactory. The figure shows such an example, a thickened juncture between portions of a rocket motor case. The stress concentration resulting from the juncture configuration is of concern in such designs. In fiberglass motor cases, the interstage shell is often simply bonded to the pressurized portion of the case, and failure of the bond is of concern in the design. For these and other such problems the ring-wedge element is ideally suited.

The writers have used the ring-wedge for a large number of stress analyses of solid propellant grains. The results in the prediction of stress concentrations in critical regions of the solid grain have been excellent. Accuracies have been checked by computing solutions with varying fineness of the element mesh and the results have been found to converge rapidly with decreasing element size. The nature of these problems is sufficiently similar to those of shell stress concentrations and behavior of thickened junctures that the success of the ring-wedge element in these cases can be forecast confidently.

The deformations to which the ring-wedge is subjected in its derivation involve strains in the axial-radial plane which are constant over the element. Since the behavior of shells, in the problems in question, involve stresses which vary roughly linearly over the wall thickness, it is clear that a number of elements should be provided through the wall. Figure 3 illustrates this point. Away from the region of concern the shell will behave in a manner properly covered by the methods discussed in earlier sections, and it will not be necessary to carry the ring-wedge idealization into these regions. At the location of transfer from the ring-wedge to the usual shell idealization, shown in the figure, it is necessary to constrain the ring-wedge elements to the conditions that the normals to the shell mid-surface remain straight and unextended, i.e., the hypotheses of thin-shell behavior. The handling of such constraints within the direct stiffness method is described in Reference 14.

It has been found that the accuracy of stress prediction with the ring-wedge element is enhanced considerably by the proper arrangement of the idealization. The preferred arrangement begins with the construction of an orthogonal network of curves which follow approximately the trajectories of the principal stresses of the problem. On completion of the resulting network of quadrilateral elements, diagonal lines are inserted to arrive at the desired array of elements, such as shown in Figure 3. The diagonals are seen to form a diamond pattern, so that the network of diagonals also yields an array of quadrilaterals, though these are not generally formed by smooth curves and are not orthogonal. In the figure, the stress trajectory gridwork is constructed on the basis of a problem in which the motor case is in tension due to pressurization while the interstage shell carries a compressive load. It is seen that this gridwork provides an orderly array of small elements at the fillet where the stress concentration is expected. By virtue of the stress trajectory pattern, it also accommodates a kind of smooth flow of stress into and through the concentration region. These two features appear to provide the advantages of this type of idealization. Due to the bunching of the mesh in some regions and the widening of it elsewhere, it is necessary to add or remove mesh lines in places. It has been found that the accuracy of the stress prediction is enhanced if these lines are added or removed in pairs, one on either side of a mesh line which has been established for some distance. This is illustrated in the figure.

ADDITIONAL TOPICS

The foregoing discussions have dealt with a limited view of the stress and deformation analysis of shell structures. The problems and methods considered included only the linear, static behavior of elastic shells, and the major emphasis was on the symmetrical deformations of shells of revolution. The principal aim has been to demonstrate some of the basic characteristics of shell analysis by the direct stiffness method, using for illustration several of the existing analysis methods. It is not possible in this section to extend the scope of the work significantly. However, a few brief comments are in order on the problems of vibrations of shells and nonlinear deformations of shells. This section will conclude with a brief discussion of future developments needed.

Dynamic Response

All of the methods of shell analysis discussed previously can be implemented in differential equation integration programs to solve transient dynamic response problems, or in eigenvalue-eigenvector programs to determine natural modes and frequencies of shell structures. It is the applicability of the conical element for this purpose which is of concern here. The use of this element for problems of a membrane type, i.e., those with a smooth loading function, has been discussed in some detail. It has been seen that the combination of a polygonal meridian and a large meridional membrane stress produces a serious error in the meridional bending moment in the shell. In the simple problems discussed, the errors due to the approximation of the structure have tended to produce canceling effects in the determination of deflections. Hence, in most cases, the deflection calculations have been quite accurate. However, if the loading conditions are such that the meridional membrane stress varies fairly rapidly along the meridian, it appears that the cancellation effect will not be reliable, and erroneous deflections may be expected. The inertia forces of the shell vibration problem constitute a loading of this type, particularly for the higher modes of shell structures. Hence the ability of the conical element to predict vibration modes and frequencies of curved shells of revolution is questionable and remains to be established. For cylindrical and conical shells, on the other hand, the conical element should provide adequate solutions. The same conclusion appears to apply to the flat element idealizations. In any investigations of this matter, care should be taken to use "consistent" masses at the nodes or nodal circles. The derivation of consistent masses is discussed in Reference 15.

Nonlinear Problems

The extension of the direct stiffness method to geometrically nonlinear problems has been discussed in Reference 16 and extended in Reference 17, for beam and plate elements. In these cases the nonlinear behavior is associated with stresses of the membrane type, as opposed to bending type. The extension to the nonlinear case can be made in both the conical and doubly curved shell elements with relative ease. This appears to be a fairly important application for very thin shells, and for shells of special shape, such as the toroid, since in these cases the nonlinear effects of the membrane stress state have a significant or even dominant effect on the edge bending boundary conditions, (Reference 18).

Future Work

Out of the many developments which could be made in digital computer shell analyses by the direct stiffness method, the writers wish to mention only one. This is the development of a doubly curved shell element of triangular or quadrilateral planform. The class of problems solvable by this element includes all shells with cutouts, sections of shells, and complete shells which are not surfaces of revolution. In addition, this appears to be the most satisfactory approach to the solution of problems of stringer stiffened shells. The lack of such an element is the one really large gap in the current set of analysis tools. The development of the element will be a major accomplishment, however, and a satisfactory formulation and solution may not be available for some time. This development will probably have to wait for the implementation of the concepts of the generalized direct stiffness method, (Reference 19). Some progress is being made in this direction, (References 20 and 21).

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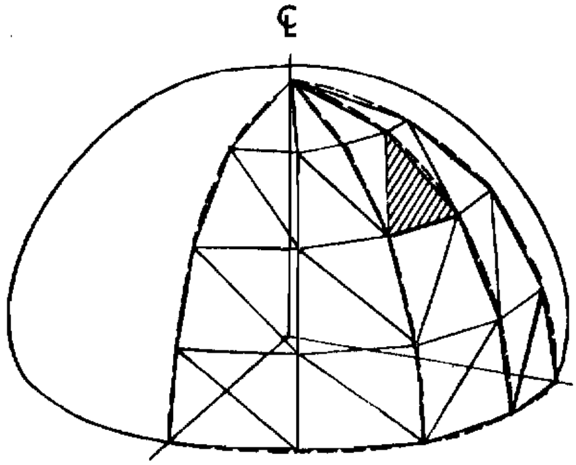


Figure 1. Flat Element Shell Idealization

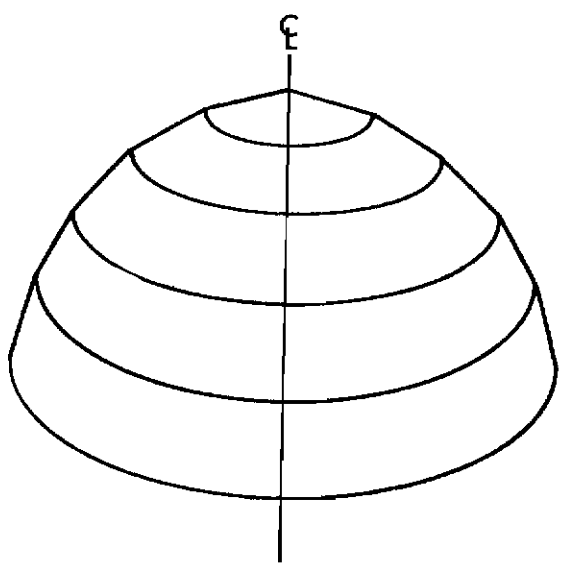


Figure 2. Conical Element Shell Idealization

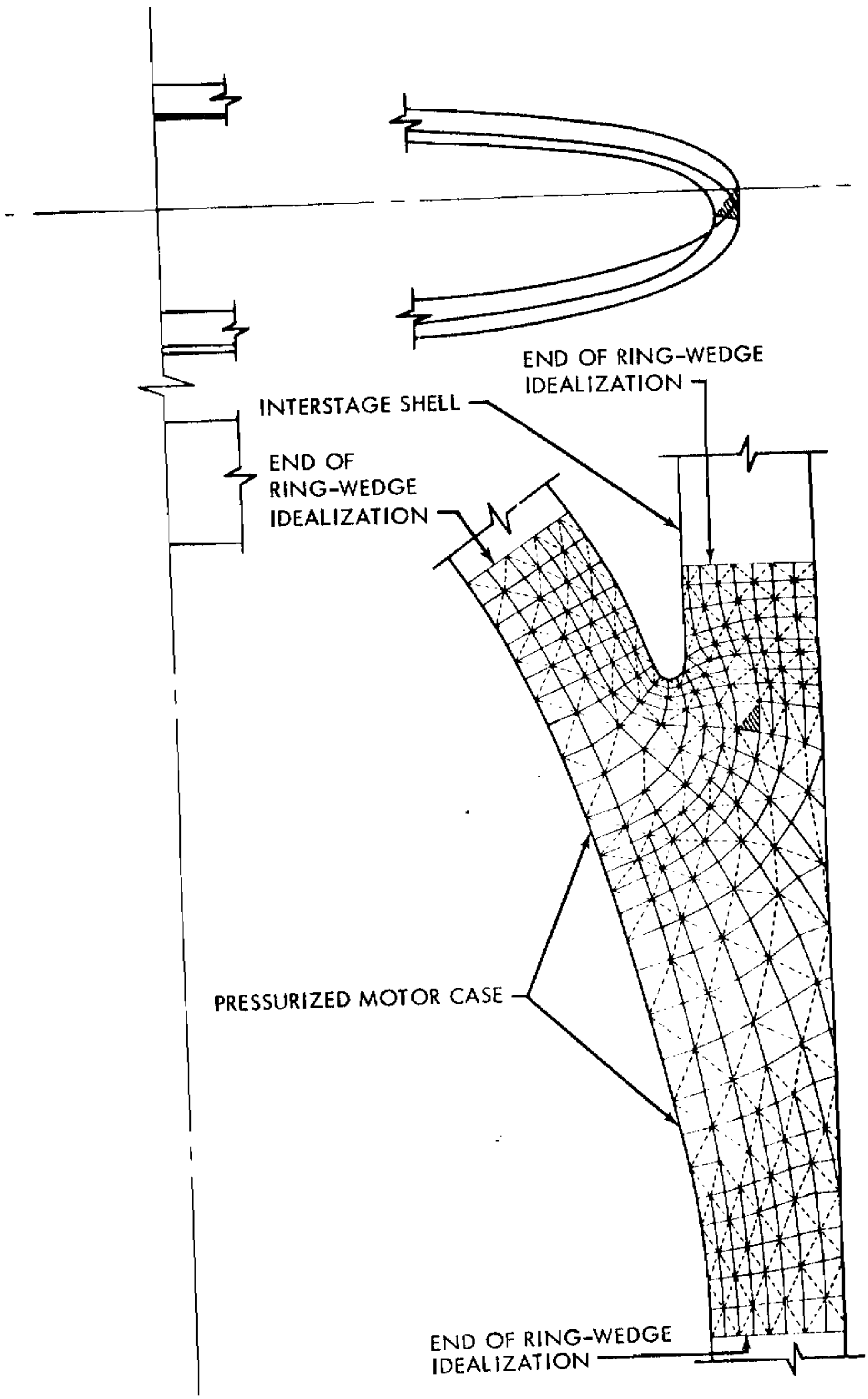


Figure 3. Ring-Wedge Element Idealization of Thickened Shell

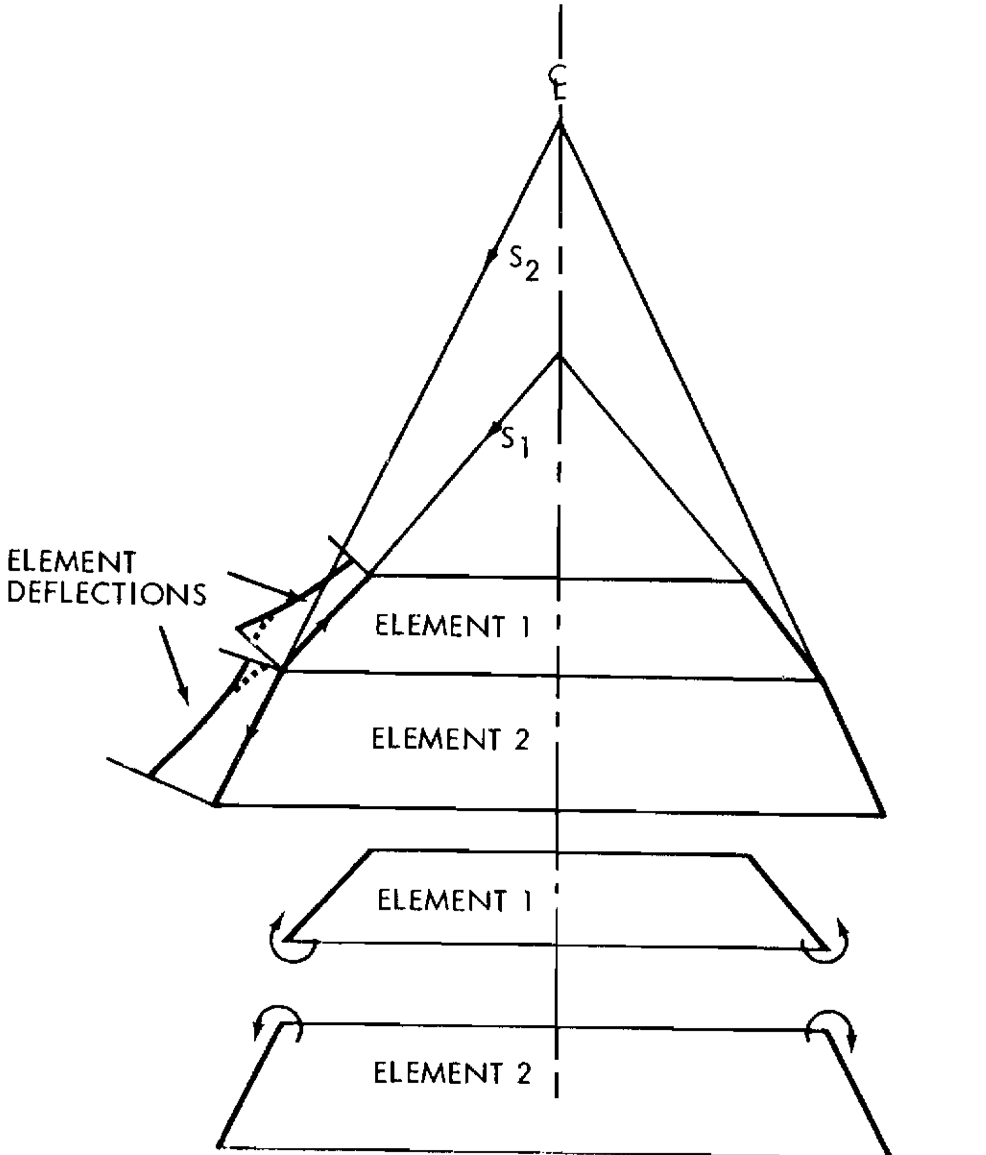


Figure 4. Meridional Moments in Shell With Polygonal Meridian

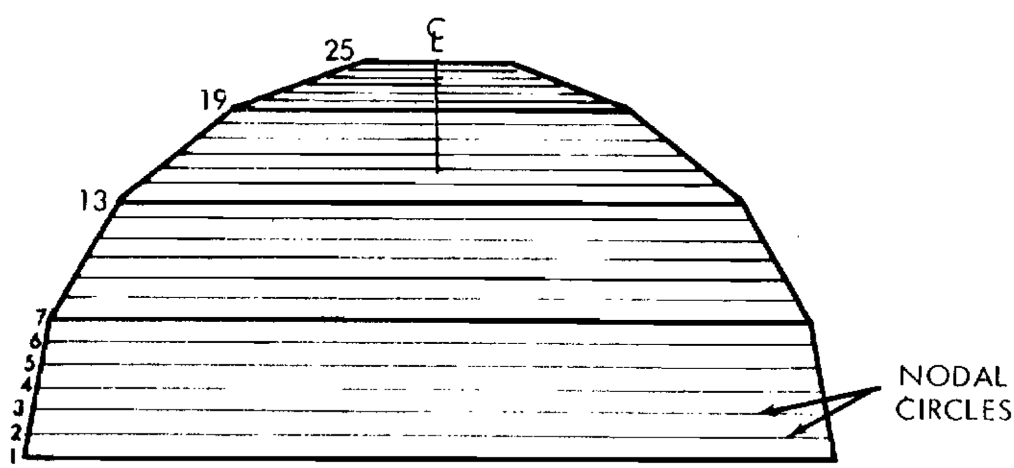


Figure 5. Conical Element Idealization of Shell With Polygonal Meridian

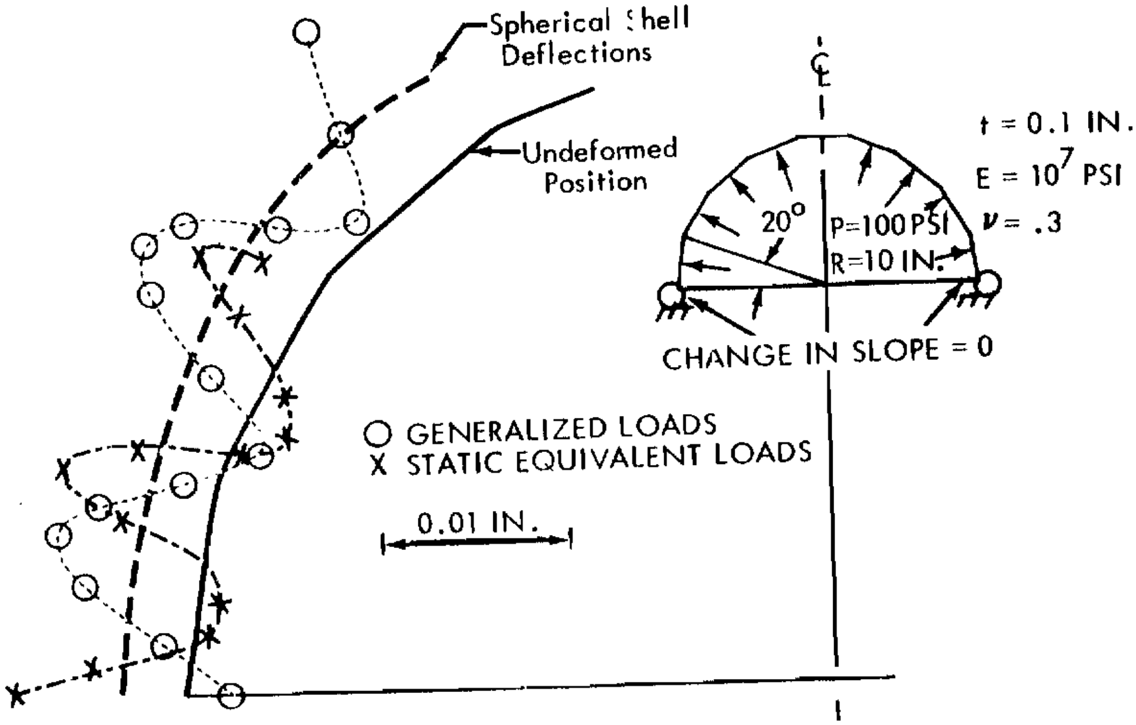


Figure 6. Deflections of Shell With Polygonal Meridian

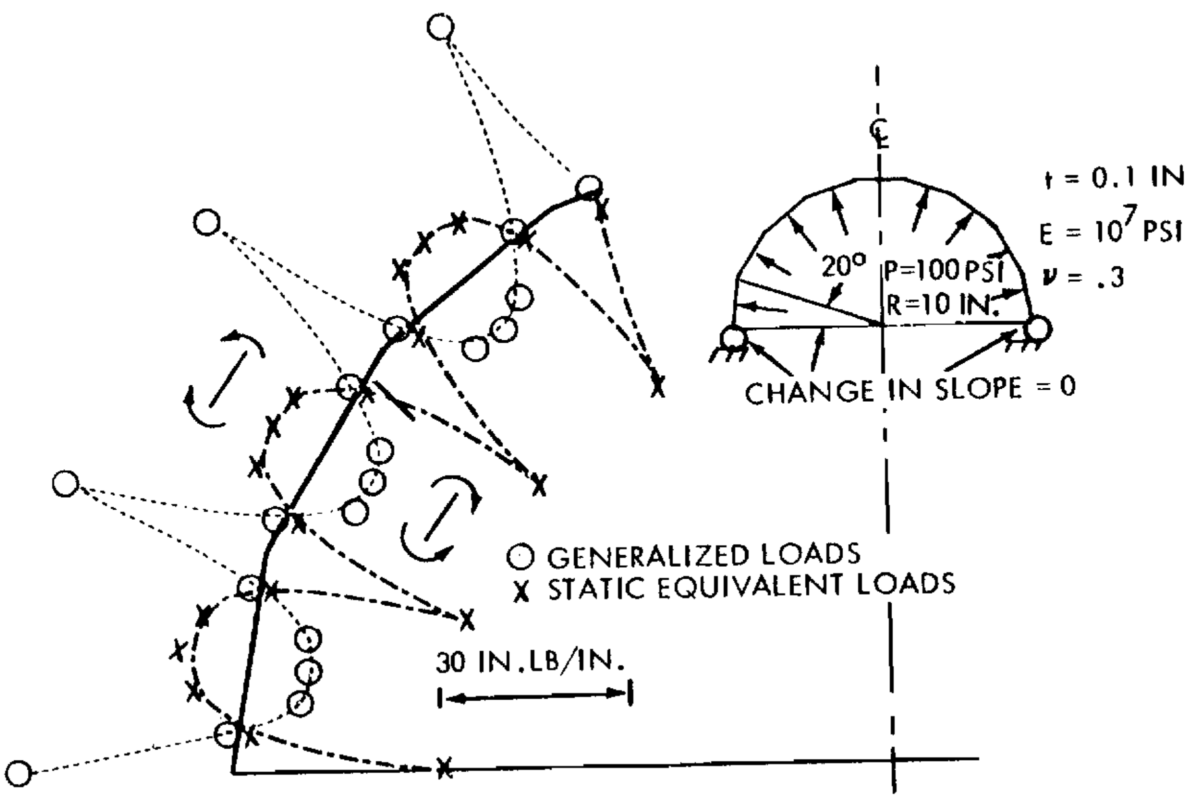


Figure 7. Meridional Moments in Shell With Polygonal Meridian

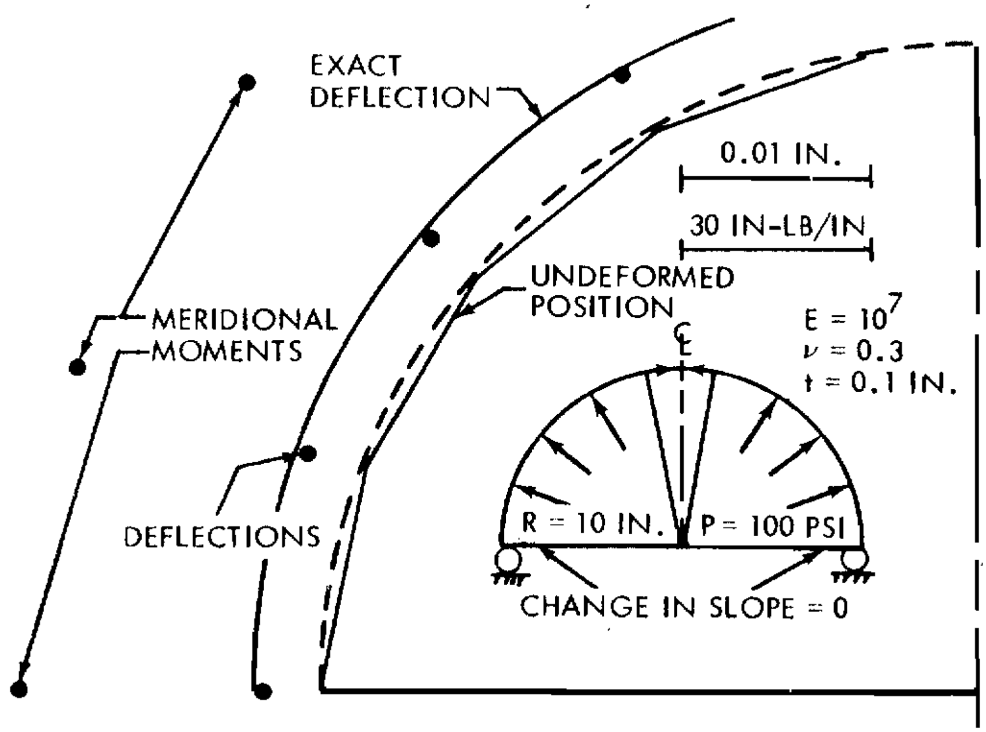


Figure 8. Deflections and Meridional Moments in Pressurized Spherical Shell Predicted by Four-Element Conical Idealization

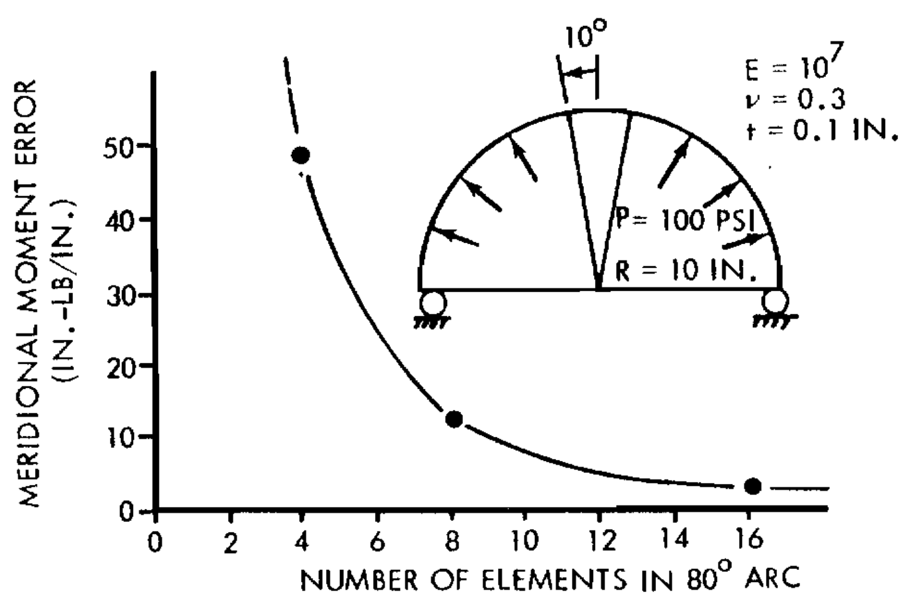


Figure 9. Meridional Moment Error Versus Number of Conical Elements for Pressurized Spherical Shell

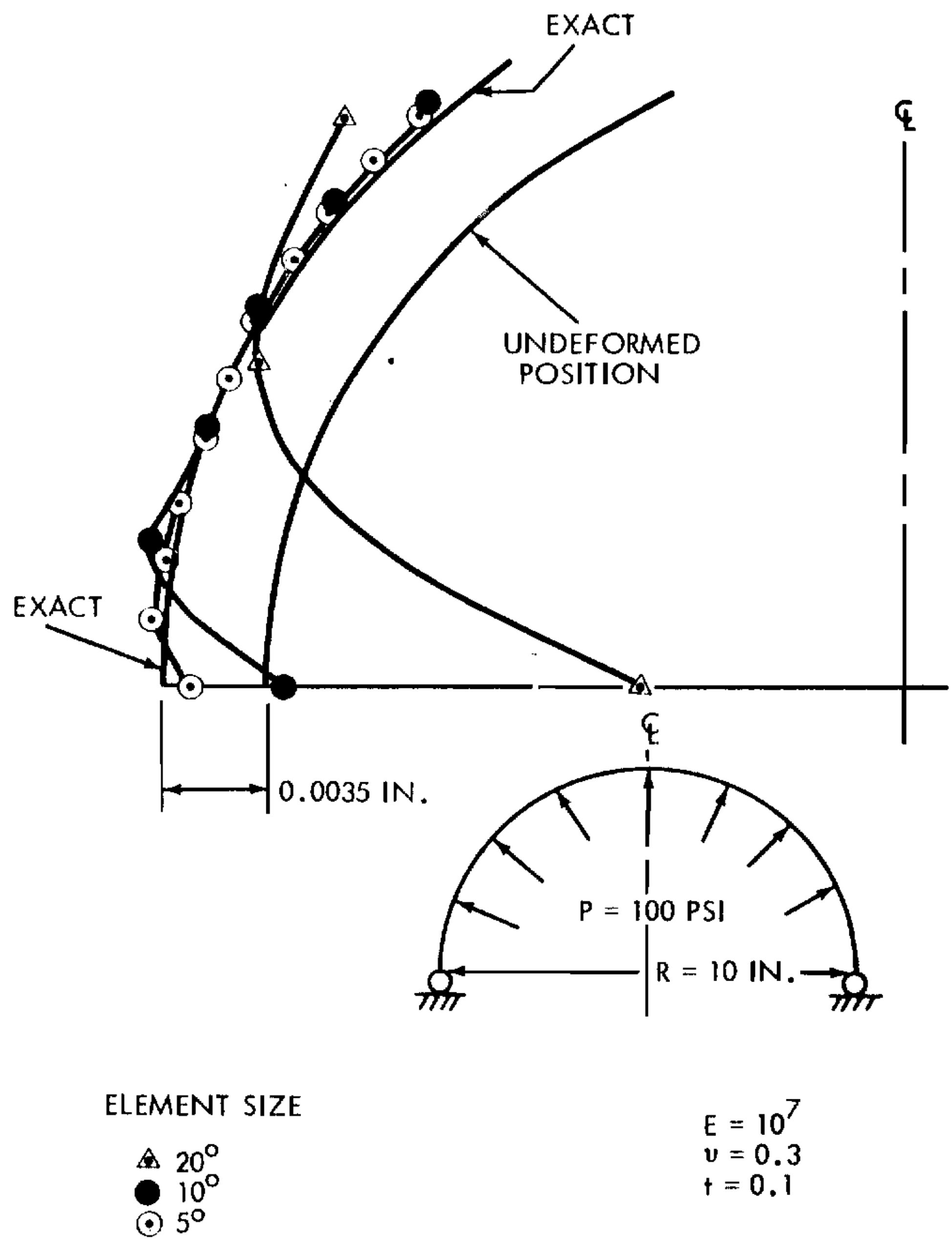


Figure 10. Deflections at Unconstrained Boundary of Pressurized Spherical Shell – Conical Element Idealization

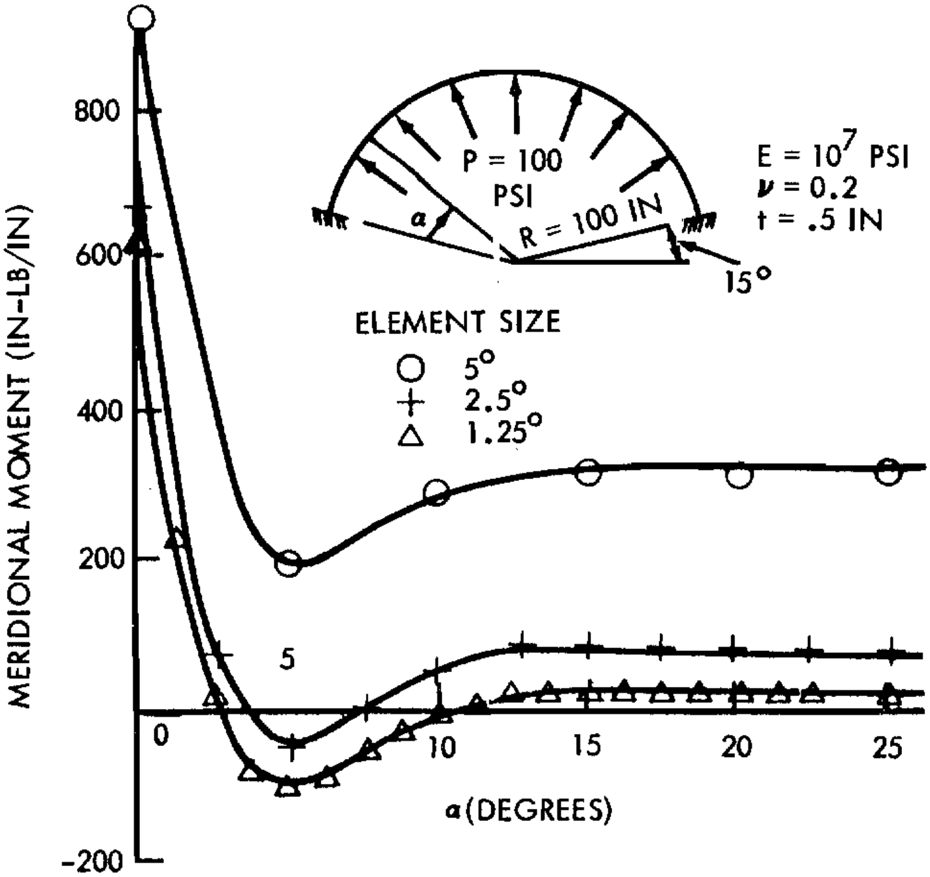


Figure 11. Prediction of Meridional Moments in Pressurized Spherical Cap — Conical Element

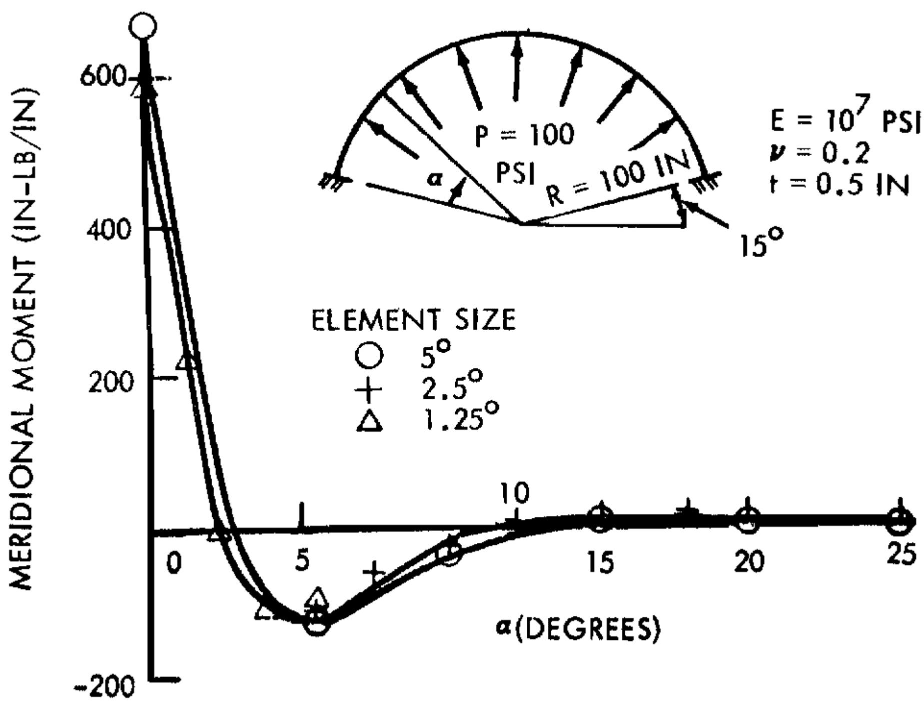


Figure 12. Prediction of Meridional Moments in Pressurized Spherical Cap — Curved Element

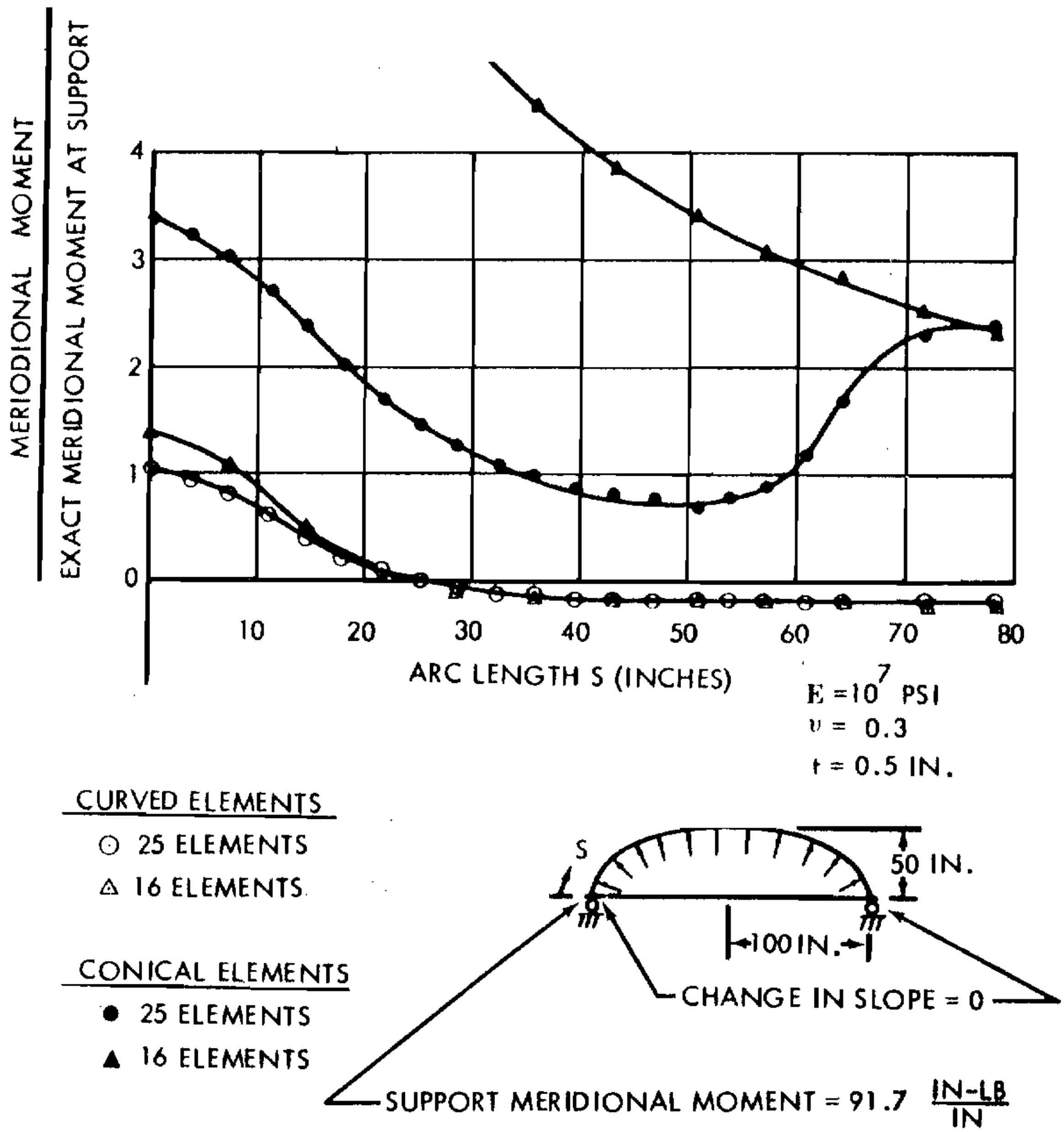


Figure 13. Meridional Moments in Pressurized Ellipsoidal Shells – Conical and Curved Elements

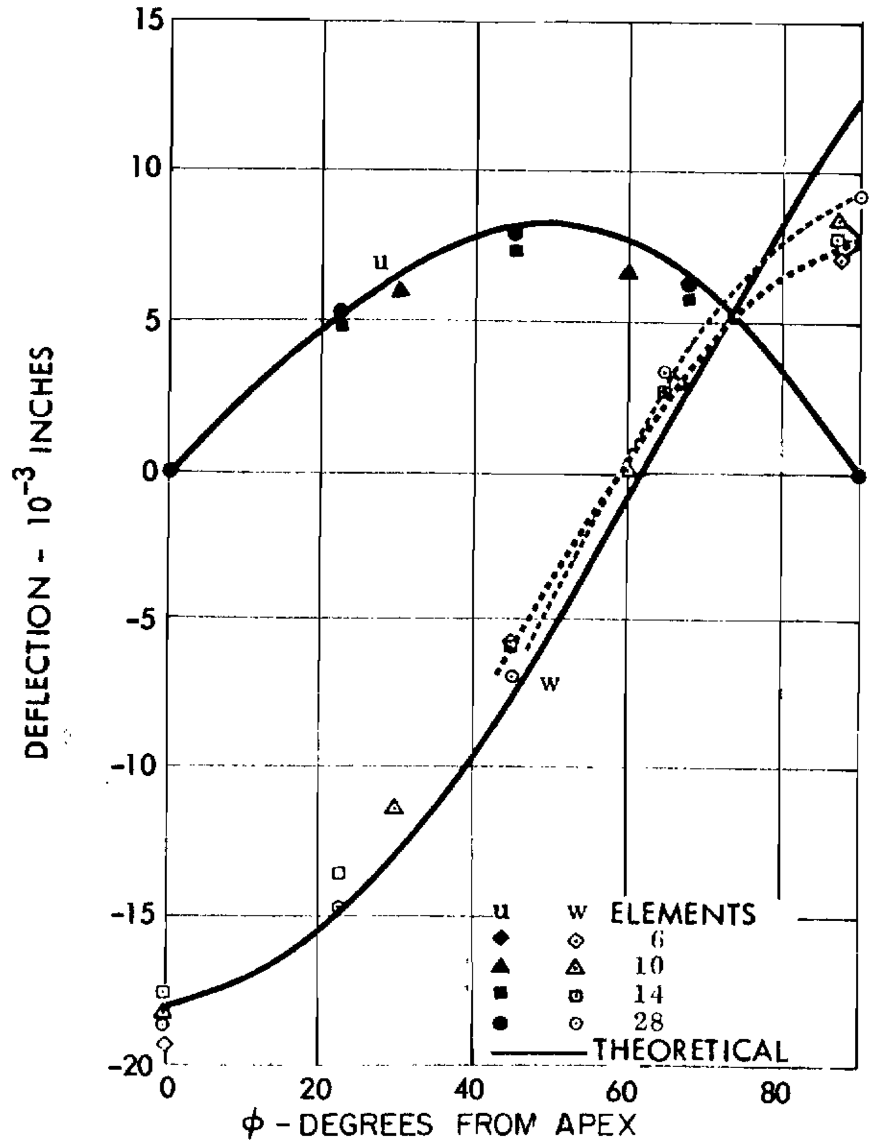


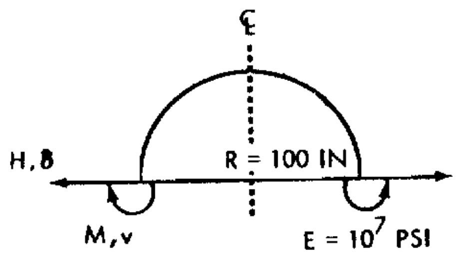
Figure 14. Deflection of Hemispherical Shell Under Dead Load

Table 1: Per Cent Error*-Influence Coefficients

		NON-SHALLOW REGION		
		H = 1		M = 1
		α_{11}	$\alpha_{12} = \alpha_{21}$	α_{22}
GECKELER SOLUTION	R/t = 250 = 25	0 0	0 -0.35	+0.36 0
CONICAL ELEMENT a	R/t = 250 = 25	-1.23 -0.30	-1.44 -0.31	-1.02 -0.22
CONICAL ELEMENT b	R/t = 250 = 25	-5.13 -0.56	-3.76 -0.67	-2.20 -0.48

$$\delta = \alpha_{11} H + \alpha_{12} M$$

$$v = \alpha_{21} H + \alpha_{22} M$$



$E = 10^7$ PSI

$\nu = 0.3$

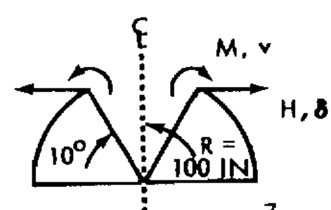
a- 2 @ 2.5°, 3 @ 5°, 6 @ 10°
b- 4 @ 5°, 6 @ 10°

Table 2: Per Cent Error*-Influence Coefficients

		SHALLOW REGION		
		H = 1		M = 1
		α_{11}	$\alpha_{12} = \alpha_{12}$	α_{22}
GECKELER SOLUTION	R/t = 250 = 25	-6.88 -19.9	+7.85 +38.8	+7.4 +29.8
CONICAL ELEMENT a	R/t = 250 = 25	-0.04 -0.40	+0.28 +0.23	-1.77 -0.73
CONICAL ELEMENT b	R/t = 250 = 25	+0.40 -1.15	+1.72 +1.75	-6.65 -0.91

$$\delta = \alpha_{11} H + \alpha_{12} M$$

$$v = \alpha_{21} H + \alpha_{22} M$$



$E = 10^7$ PSI

$\nu = 0.3$

a- 2 @ 2.5°, 3 @ 5°, 6 @ 10°
b- 4 @ 5°, 6 @ 10°

*Note - Error Based on Solutions Given by Galletly