COMPUTATIONAL METHODS IN OPTIMAL CONTROL PROBLEMS

JAMES A. PAYNE

DEPARTMENT OF ENGINEERING (REPORT No. 65-8, PART I)
UNIVERSITY OF CALIFORNIA

FOREWORD

This report was prepared by the Department of Engineering at the University of California, Los Angeles, on Air Force Contract AF33(615)-1794 under Task No. 822501 of Project No. 8225, Research in Advanced Applications of Sampled Data and Other Nonlinear Control Systems Theory. The work was under the direction of AF Flight Dynamics Laboratory, Research and Technology Division. Charles Harmon was project engineer for the laboratory.

The studies presented represent the efforts from May 1964 to May 1965. Professor C. T. Leondes and Professor A. R. Stubberud served as principal investigators.

This report is based on a dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Engineering at the University of California, Los Angeles.

This is the final report and it concludes the work on AF33(615)-1794. The contractor's report number is 65-8.

This report was submitted by the author, July 1965.

This technical report has been reviewed and is approved.

W. A. SLOAN, JR.
Colonel USAF
Chief, Flight Control Division
AF Flight Dynamics Laboratory

ABSTRACT

This report considers the application of optimization techniques to the development of methods for the control of engineering systems. The systems considered are those physical processes which are subject to independent control forces and in which the dynamics of the process are of central importance. It is assumed that the process can be described by a system of ordinary nonlinear differential equations.

The optimization, with respect to a general criterion function, of such systems is considered. The conditions and equations which specify the optimal system behavior are derived by means of the Maximum Principle. System trajectories which satisfy the optimal conditions, i.e., optimal trajectories, can only be obtained by numerical computation. Various approaches to this computational problem are reviewed and their primary limitations are discussed.

In order to provide a realistic evaluation of certain computational methods, the optimization of a particular engineering system is considered in detail. This system is a variable lift aerodynamic vehicle during the atmospheric reentry phase. A mathematical model for this system is developed and the optimization of this model is considered. The criterion function is a linear combination of the heating and acceleration effects which are experienced by the vehicle during the reentry phase.

The computational requirements to generate optimal trajectories and the procedures used to meet these requirements are presented. The special features which affect the selection of a numerical integration method for optimal trajectories are discussed. A special integration technique, which has given significant reductions in the amount of computation required, is presented.

By varying the relative weighting of the two components of the criterion function, a family of optimal trajectories is obtained. These results specify the optimal trade-off function between the competitive effects of heating and acceleration for the reentry maneuver.

The optimum linear feedback control system for the vehicle about a particular optimum trajectory is derived. A simulation study of this neighboring optimum control system is carried out. It is shown that in order to obtain terminal accuracy it is necessary to compensate for the difference in time between the simulated trajectory and the reference trajectory. A method to accomplish this compensation is presented. The results of this simulation indicate that neighboring optimum control gives excellent performance with respect to both optimizing the criterion function and satisfying terminal conditions.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont.)

LIST OF ILLUSTRATIONS

LIST OF TABLES

CHAPTER 1

INTRODUCTION

1.1 Developments in Control Systems Theory

Automatic control is now being applied to systems of increasing complexity and importance. These applications have led to requirements for extreme precision, which have, in some cases, been met by intuitively conceived control schemes of an unconventional nature. The availability of very reliable digital computers capable of performing extensive computations at great speed suggests the possibility of controlling increasingly larger and more complex systems by increasingly sophisticated control schemes. But this introduces an additional requirement, that of a theory capable of guiding the search for new control schemes and providing a means of evaluating performance.

Automatic control theory in its present state is inadequate for this purpose. But the need for developing a new theory is recognized and much research is directed toward this goal. It has been stated[1] that the following requirements and corresponding research areas will form the basis of this new theory;
(1) the necessity to devise and to construct a comprehensive performance index, implying a detailed knowledge of the physical structure of the system (state vector analysis),
(2) the need for statistical analysis of variables (stochastic methods),
(3) the necessity to determine a dynamic control strategy, utilizing the available control variables from an application of a minimization principle (optimization theory).

Each of these areas is important in itself but it should be remembered that an adequate theory of automatic control must combine them all as interdependent elements.

The work of this dissertation is concerned with area (3). In particular it is an attempt to extend the applicability of techniques from the Calculus of Variations in control system problems.

1.2 Calculus of Variations and Control Problems

When a dynamic control system is described by a set of state variables which are related by ordinary differential equations, the problem of optimizing the control with respect to some explicit function of these state variables can be formulated mathematically. This problem is, in fact, a classical problem in the Calculus of Variations;[2] specifically it is the problem of finding a minimum of a functional subject to subsidiary conditions. In particular, the subsidiary conditions are that all admissible curves from which the minimum is selected must satisfy the given differential equations and specified end conditions. Also, in realistic engineering systems there are bounds, or inequality constraints, on the control and state variables.

One of the earliest formulations of an optimal control problem as a problem in the Calculus of Variations was given by Hestenes,[3] who considered the flight path of an aircraft subject to aerodynamic forces. Extensive work in this area has been done by L. S. Pontryagin and his co-workers.[4] This group has shown that Weierstrass' necessary condition is applicable to the case of bounded control variables and refer to this extended condition as the "Maximum Principle". This group has also developed a concise notation and formulation of the optimal control problem which gives the pertinent results from the theory of the Calculus of Variations in a form more accessible to those interested in engineering applications.

Although it is possible to formulate the optimal control problem and derive a set of necessary conditions, it is not easy to obtain the solution to this optimization problem. The essential difficulty is that the application of the theory results in a description of the optimal solution as being the solution of a set of nonlinear differential equations which satisfy a set of boundary conditions specified partially at the initial value and partially at the final value of the independent variable. Only in exceptionally simple cases is it possible to obtain analytic solutions to the optimization problem. So the usual procedure is to attempt to find numerical solutions. But these two point boundary value problems are extremely difficult to solve by numerical methods. It has been indicated that this is due to the fundamental nature of these problems.[5,6] But some problems have been solved,[7,8] and it would seem that improved numerical techniques will make it possible to solve many more. It is the objective of this dissertation to discuss the value of certain techniques for use in solving two point boundary value problems of a particular class, and also to develop methods for using these techniques in synthesizing feedback control systems.

Another approach to optimizing control systems is to use the methods of mathematical programming,[9] especially, dynamic programming.[10,11] These methods may have significant advantage when inequality constraints are essential features of the control problem.[12] But computational difficulties exclude their use when the number of system variables or the duration of system operation exceed certain very restrictive limitations.

1.3 Realization of an Optimal Control System

Any method of solving the optimal control problem can, in theory, be used as a control system. In practice, this means that the equipment used to implement the optimization method must be able to obtain the solution with sufficient speed and accuracy to meet the particular system specifications. It is usually assumed that such an approach is not practical,[13, 14] and apparently it has not been used in operating systems because of the excessive amount of equipment required to obtain optimal solutions. However, it should be remembered that this is due to the low efficiency of present methods and improved methods may well make such an approach quite reasonable in some cases.

Because of the difficulty in obtaining solutions, the present use of optimization theory for control systems engineering is limited to determining the nature of the optimal control law. This information is useful in evaluating alternate control schemes and in indicating approximations to the optimal control which may be more easily implemented. Of particular importance is the scheme for obtaining the linear feedback approximation developed by Kelley[15] and Breakwell, Speyer and Bryson.[16] This scheme yields the optimum linear, time varying, feedback control system in the neighborhood of an optimized nominal trajectory. The resulting control system does not require a great deal of equipment for implementation. This method has recently been extended to include quadratic and higher order approximations to the optimum control.[17, 18]

It would appear that for increasingly complex systems there is an increasing need for methods which yield easily implemented approximate optimal feedback control in a formal manner, without relying on intuitively conceived control structures. In view of this need, it is worthwhile to make a comparative study of the region of convergence and rate of convergence for several optimization techniques in a given problem, since these two properties determine the control capabilities of the technique.

1.4 Practical Limitations in Applying Optimization Techniques

In the simulation and evaluation of any control scheme, all quantities which are physically measured in the actual system should be represented in the machine only to an accuracy consistent with that of the measurement. This means that accurate simulation results can only be obtained by adding random noise to the computer representation of measured quantities. Also, for realistic simulation, disturbances should be added which represent the deviation of the system from its mathematical model.

Despite the many limitations of computer simulation, it has been shown in practice that this method is of great value in the design, development and operation of complex engineering systems. It seems evident that dynamic optimization can extend the usefulness of such studies.

1.5 Scope of the Dissertation

The objective of the research upon which this dissertation is based was to investigate the capabilities and limitations of certain computational methods as applied to optimal control problems. The approach used was essentially to consider a single specific example. The justification for such an approach is due primarily to the nature of the presently available published results in this area. All of the examples discussed in the literature are fairly simple and use a very simple criterion function. It is, of course, quite reasonable to use such examples to demonstrate computational methods, and this often is the best way to avoid extraneous computational difficulties. However, in the application of optimization techniques to more practical, and hence more complex, problems, it is clear that the consideration of complex criterion functions is necessary. This introduces computational difficulties which are not present in the simple examples. For this and similar reasons, it appeared to be more desirable to develop methods for solving more complex problems than to determine the best methods for the simple examples.

The purpose of this first chapter is to indicate the relationship between the work of this dissertation and the general field of control systems engineering.

In Chapter 2, the basic results of the mathematical theory of optimal control are presented. Also, the presently available methods for obtaining solutions of the optimal control problem are reviewed. The material of Chapter 2 is basic to all optimal control problems, but the emphasis is on the results which are applied to the main problem considered in this dissertation.

Chapter 3 includes descriptions of the computational methods necessary to obtain the numerical solutions presented in Chapter 5. These are general methods for obtaining solutions of an optimal control problem. However, their use is not justified for every optimal control problem.

An optimal control problem involving the atmospheric reentry of a variable lift aerodynamic vehicle is formulated in Chapter 4. The importance of this problem and the reasons for carrying out certain specific investigations are discussed.

The solutions of this optimal control problem are presented in Chapter 5. Also, certain characteristics of the computational procedures used to obtain these solutions are discussed.

Chapter 6 considers the optimal linear feedback control system for the reentry problem. The optimal linear control system is the best linear approximation to the true, nonlinear, optimal control system. Included are the results of a simulation study of the control scheme and a discussion of the time distortion problem in neighboring optimum control. Also, some of the computational problems associated with this type of simulation and control are discussed.

The final chapter summarizes the results obtained for the reentry problem and presents the conclusions arrived at from this investigation. Areas requiring further investigations and certain extensions to the work presented in this dissertation are also covered.

CHAPTER 2

THE OPTIMAL CONTROL PROBLEM

2.1 State Space Description of Control Systems

This study is concerned with physical processes which are subject to independent control and in which the dynamics of the process are of central importance. It is assumed that the process can be described by a system of ordinary differential equations:

$$\frac{dx_i}{dt} = \dot{x}_i = f_i(x_1, \ldots, x_n, u_1, \ldots, u_r), \ i = 1, 2, \ldots, n \tag{2.1}$$

where $x_1, \ldots, x_n$ are the state variables which characterize the process, i.e., define its state at each instant of time t, and which can be observed; $u_1, \ldots, u_r$ are the control parameters which may be varied independent of the state variables in order to determine the course of the process. If a set of initial conditions is given

$$x_i(t^o) = x_i^o \ , \quad i = 1, \ldots, n$$

and the control parameters are specified in a certain time interval

$$u_j = u_j(t) \qquad , \qquad j = 1, \ldots, r \qquad , \qquad t^o \leqq t \leqq t'$$

then the solution of system (2.1):

$$x_i = x_i(t) \qquad , \qquad i = 1, \ldots, n \qquad , \qquad t^o \leqq t \leqq t' \tag{2.2}$$

is uniquely determined. And this solution describes the motion of the physical process during the time interval $t^o \leqq t \leqq t'$.

From the mathematical viewpoint,* each state of the process is considered to be a point in the vector space X of the vector variable $x = (x_1, \ldots, x_n)$. We shall call X the state space. Thus the motion of the process (2.2) corresponds to a trajectory in X from the point $x^o = x(t^o)$ to $x' = x(t')$. The control variables are characterized by points $u = (u_1, \ldots, u_r)$ of a certain control region U, which may be a closed and bounded set in some r-dimensional Euclidean space E_r. The physical meaning of considering a closed and bounded control region U, is that the control

*In the following development, the mathematical terms correspond to general usage and in particular to the definitions given by Taylor.[19]

parameters cannot take on arbitrarily large values. In practical problems the limitations on the control function is a major consideration.

Every vector function

$$u(t) = \Big(u_1(t),\ u_2(t), \dots, u_r(t) \Big)$$

defined on some time interval $t^o \leqq t \leqq t'$, with range in U, shall be called a control. Admissible controls are considered to be arbitrary piecewise continuous controls, i.e., u(t) with range in U, which are continuous for all t under consideration, with the exception of only a finite number of t, at which u(t) may have discontinuities of the first kind. Piecewise continuous controls correspond to the assumption of "inertialess" controllers, since the values of the function u(t) may jump (at an instant of discontinuity) instantaneously from one point of the control region to another. This class of admissible controls seems to be the most interesting for practical applications (Pontryagin,[4] p. 11).

The functions f_i of (2.1) are defined for all $x \epsilon X$ and for all $u \epsilon U$. They are assumed to be continuous in the variables $x_1, \dots, x_n$, $u_1, \dots, u_r$ and continuously differentiable with respect to $x_1, \dots, x_n$. That is, the functions

$$f_i(x_1, \dots, x_n,\ u_1, \dots, u_r)$$

and

$$\frac{\partial f_i(x_1, \dots, x_n,\ u_1, \dots, u_r)}{\partial x_j}\ ,\ i, j = 1, 2, \dots, n$$

are defined and continuous on the direct product space $X \times U$.

If u(t) is an admissible control, given for $t^o \leqq t \leqq t'$, and $\theta_1, \theta_2, \dots, \theta_k$ are its points of discontinuity, where $t^o < \theta_1 < \theta_2 < \dots < \theta_k < t'$, first consider Equation (2.1) on the interval $t^o \leqq t \leqq \theta_1$, where its right hand side is continuous. Denote the solution of (2.1), with initial condition x^o by x(t). If this solution is defined on the entire interval $t^o \leqq t \leqq \theta_1$, and has the value $x(\theta_1)$ at the point θ_1, we can consider Equation (2.1) on the interval $\theta_1 \leqq t \leqq \theta_2$, using $x(\theta_1)$ as the initial value. This solution will also be denoted by x(t). Thus the x(t) so constructed is continuous at all points at which it is defined, and, in particular, at the "junction point" θ_1. In a similar manner the solution x(t) for $t^o \leqq t \leqq t'$ is obtained, providing x(t) is defined over the entire interval, i.e., it does not go to infinity. And this solution is continuous and piecewise differentiable, specifically, it is continuously differentiable at all points except $\theta_1, \theta_2, \dots, \theta_k$. We shall call x(t) the solution of the system (2.1) corresponding to the control u(t) for the initial condition x^o.

An admissible control $u(t)$, $t^o \leqq t \leqq t'$ is said to transfer the system from state x^o to state x' if the corresponding solution $x(t)$ of (2.1), satisfying the initial condition $x(t^o) = x^o$, is defined for all t, $t^o \leqq t \leqq t'$ and also satisfies the boundary condition $x(t') = x'$.

2.2 Statement of the Optimal Control Problem

An optimal control system is one which performs its specified control function in such a manner that the absolute minimum value of some performance index is obtained. As indicated in Section 1.1, determining the proper performance index for a complex control system is a difficult task. In this thesis we consider only performance indices which are functions of the state variables and the control variables over the control period. That is, we assume that the criterion function is of the form

$$J = \int_{t^o}^{t'} f_o \left(x(t), u(t)\right) dt \qquad (2.3)$$

Further, it is assumed that f_o is defined and is continuous together with its partial derivatives $\frac{\partial f_o}{\partial x_i}$, $i = 1, 2, \ldots, n$ on all of the product space $X \times U$. It should be noted that this form of criterion function is not adequate for all control systems. In particular, there are systems in which more than one consideration enters into the assessment of performance, and these considerations cannot always be subsumed under a single scalar valued criterion.[20] However, this criterion function is adequate for a large class of control systems, which includes most of the practical problems within the reach of present day technology.

If it is assumed that the control is required to transfer the system, defined by (2.1), from state x^o to state x', then the optimal control problem may be stated as follows:

> Among all admissible controls $u(t)$ which transfer the system from the position x^o to the position x' in X (if such controls exist), find one for which the functional J takes on the least possible value.

In this problem the time is not fixed. That is, in the integral of (2.3) the upper limit t' is not a fixed number, but depends on the control $u(t)$. For fixed time problems, the procedure is to add to the system of Equation (2.1), one more equation

$$\dot{x}_{n+1} = 1 \;;\; x_{n+1}(t^o) = t^o \qquad (2.4)$$

Then $x_{n+1} \equiv t$ and if $x_{n+1}(t') = t'$ the fixed time condition is met.

For many problems the goal of the control process is not to have the system reach a definite position in the state space, but rather to satisfy given terminal values for some of the state variables with the rest of the state variables free to assume arbitrary values. If there are $k (k < n)$ fixed terminal state variables then the terminal state x' of the process must lie in a (n-k) dimensional manifold S' in X. For this case the optimal control problem becomes:

> Find an admissible control u(t) which transfers the system from the state x^o in X to some state x' (not previously given) in S', and which in so doing imparts a minimum value to the functional J.

This is called the optimal problem with variable right-hand end points.

More generally, the terminal state may be determined by some relations among the terminal state rather than by given values of the terminal states. That is, S' may be defined as the intersection of the hypersurfaces given in the space X by the equations

$$\begin{aligned} &g_1(x_1, x_2, \ldots, x_n) = 0 \\ &g_2(x_1, x_2, \ldots, x_n) = 0 \\ &\vdots \\ &g_k(x_1, x_2, \ldots, x_n) = 0 \end{aligned} \tag{2.5}$$

It is assumed that the hypersurfaces are smooth. That is, all g_i are continuously differentiable and the hypersurface contains no singular points, i.e., points $\bar{x}$ at which

$$\frac{\partial g_i(\bar{x})}{\partial x_1} = \frac{\partial g_i(\bar{x})}{\partial x_2} = \cdots = \frac{\partial g_i(\bar{x})}{\partial x_n} = 0$$

In formulating the necessary optimality conditions, it will be convenient to adjoin a new variable x_o to the state variables $x_1, x_2, \ldots, x_n$. Let x_o vary according to

$$\dot{x}_o = f_o(x_1, x_2, \ldots, x_n, u_1, \ldots, u_r)$$

where f_o is the function which appears in the definition of J, (2.3). Thus we consider the system of differential equations

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, u_1, \ldots, u_r), \quad i = 0, 1, 2, \ldots, n \tag{2.6}$$

or in vector form

$$\dot{x} = f(x, u)$$

where x and f are vectors in the (n+1)-dimensional space X. Note that $f(x, u)$ does not depend on the coordinate x_o of the vector x.

For the initial conditions $x(t^o) = (0, x_1^o, \ldots, x_n^o)$, the solution of (2.6) which satisfies the terminal conditions

$$x_1(t') = x_1'$$
$$x_2(t') = x_2'$$
$$\vdots$$
$$x_n(t') = x_n'$$

will also satisfy the terminal condition

$$x_o(t') = J$$

2.3 The Maximum Principle

The necessary conditions which must be satisfied by a solution of the optimal control problem, are presented in this section. The formulation is due to Pontryagin[4] and is used because of its more concise notation, although the resulting equations are entirely equivalent to those obtained from the classical formulation of the Calculus of Variations.

In addition to the fundamental system of state equations (2.6), we shall consider another system of equations in the auxiliary variables $p_o, p_1, \ldots, p_n$:

$$\dot{p}_i = -\sum_{j=0}^{n} \left[\frac{\partial f_j\big(x(t),\ u(t)\big)}{\partial x_i} \right] p_j \qquad (2.7)$$

Where u(t) is an admissible control and x(t) the corresponding solution of (2.6) with initial condition x^o. This system is linear and homogeneous in p. Therefore, for any initial conditions, the solution $p = (p_o, p_1, \ldots, p_n)$ (which is defined on the entire interval $t^o \leqq t \leqq t'$, on which u(t) and x(t) are defined) is unique. Also the solution of (2.7) consists of continuous functions p_i which have everywhere, except at a finite number of points (namely, at the points of discontinuity of u(t)), continuous derivatives with respect to t. We shall say that each solution of (2.7) corresponds to the chosen control u(t) and state trajectory x(t).

The Hamiltonian function, H, is defined by

$$H(x, p, u) = \sum_{j=0}^{n} p_j f_j \tag{2.8}$$

It can easily be verified that the systems of Equations (2.6) and (2.7) may be rewritten as:

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \tag{2.9}$$

$$, \quad i = 0, 1, \ldots, n$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} \tag{2.10}$$

Thus, using an admissible control u(t), $t^o \leqq t \leqq t'$, and the initial condition x^o, we can find the corresponding trajectory $x(t) = \left(x_o(t), \ldots, x_n(t)\right)$ by solving (2.9). Then the solution of (2.10) $p(t) = \left(p_o(t), \ldots, p_n(t)\right)$ corresponding to the functions u(t) and x(t), can be found.

For fixed (constant) values of p and x, the function H becomes a function of the parameter u in U. We denote the least upper bound of the values of this function by M(p, x):

$$M(p, x) = \sup_{u \in U} H(p, x, u)$$

If the continuous function H achieves its upper bound on U, then M(p, x) is the maximum of the values of H, for fixed p and x.

The necessary conditions for optimality of an autonomous, free time, variable right-hand end point control problem are given in the following theorem.

The Maximum Principle (Pontryagin)

Let u(t), $t^o \leqq t \leqq t'$, be an admissible control such that the corresponding trajectory x(t) (i.e., the solution of (2.9)) which begins at the point x^o at the time t^o reaches, at some time t', some point $x' \in S'$, the terminal manifold. In order that u(t) and x(t) be optimal it is necessary that there exists a non-zero continuous vector function $p(t) = \left(p_o(t), p_1(t), \ldots, p_n(t)\right)$ corresponding to u(t) and x(t) (i.e., solution of (2.10)), such that:

1. For every t, $t^o \leqq t \leqq t'$, the function $H\left(x(t), p(t), u\right)$ of the variable $u \in U$ attains its maximum at the point u = u(t):

$$H\left(x(t),\ p(t),\ u(t)\right) = M\left(x(t),\ p(t)\right) \tag{2.11}$$

2. At the terminal time t' the relations

$$p_o(t') \leqq 0 \quad , \quad M\left(x(t'),\ p(t')\right) = 0 \tag{2.12}$$

are satisfied. Furthermore, it turns out that if x(t), p(t), and u(t) satisfy Equations (2.9), (2.10) and condition 1, the time functions $p_o(t)$ and $M\left(x(t),\ p(t)\right)$ are constants. Thus, (2.12) may be verified at any time t, $t^o \leqq t \leqq t'$, and not just at t'.

3. A transversality condition is satisfied at x(t') on S'. That is, the vector $p(t') = \left(p_1(t'), \ldots, p_n(t')\right)$ is orthogonal to the tangent plane of S' at the point x'.

For a proof of this theorem, see Reference 4.

The Maximum Principle provides an adequate number of conditions to completely specify the optimal trajectory and control. Since there are $2n + 2 + r$ relations (2.9), (2.10), and (2.11) for $2n+2+r$ variables (x_i, p_i, and u_i) there is a complete system of relations for determining the variables. Furthermore, since relation (2.11) can be solved, at least implicitly, for the r control variables, and the number of differential equations equals $2n+2$ (2.9) and (2.10), the solution of the system of Equations (2.9), (2.10) and (2.11), in general, depend on $2n+2$ parameters (the initial conditions). However, since H is homogeneous in the functions p_i, these functions are determined only up to a common multiple. Thus in view of the fact that $p_o(t)$ is a constant and less than zero, we may arbitrarily select

$$p_o(t) \equiv -1 \tag{2.13}$$

In addition, one of the parameters is determined by the condition that

$$\max_{u \in U} H\left(x(t^o),\ p(t^o),\ u\right) = 0 \tag{2.14}$$

Thus the solution of the system of Equations (2.9), (2.10) and (2.11) depends on 2n parameters. These parameters must be chosen in such a way that the trajectory passes through x^o at the given time t^o and through some point $x' \epsilon S'$ at some time t'. The number $(t' - t^o)$ is also a parameter, so that we have altogether $2n+1$ parameters. The condition that $x_i(t^o) = x_i^o$, $i = 1, 2, \ldots, n$ determines n of these parameters. The conditions that define S' and the transversality condition 3 of the theorem also determine n of the parameters. Finally, the condition $x_o(t^o) = 0$ completes the $2n+1$ conditions for the 2n + 1 parameters. Hence, one can expect that there exist only separate, isolated trajectories joining the points x^o and x' and satisfying the conditions of the Maximum Principle. And, since the conditions of the theorem are necessary for optimality, only these trajectories can be optimal.

If one, and only one, trajectory can be found which satisfies the conditions of the Maximum Principle and if from physical arguments it is clear that an optimal trajectory must exist, then it may reasonably be assumed that this trajectory is indeed optimal. However, it should be noted that the mathematical question of the existence of optimal trajectories is a very important and difficult one.[21, 22]

2.4 Optimal Solutions by the Neighboring Optimum Method

As discussed in the previous section, the Maximum Principle provides a sufficient number of conditions to determine the optimal trajectory. However, the 2n parameters, upon which depends the solution of the system of differential equations (2.9) and (2.10), are given at the two ends of the trajectory. That is, there are n initial conditions given at t^o, and n terminal conditions at t'. Therefore, in applying the Maximum Principle, we are faced with solving what is referred to in mathematical literature as two point boundary value problems (TPBVP).

In general, control system problems involve a system of nonlinear differential state equations. So, except for special cases, it is not possible to obtain analytic solutions. Thus it is necessary to resort to numerical methods to obtain solutions. The essential difficulty in solving TPBVP numerically is that a complete set of initial conditions for the system of differential equations is required to start the integration procedure.

One approach to solving TPBVP is to arbitrarily choose a set of values for the undetermined initial conditions, then, using the resulting complete set of initial conditions, integrate the system of equations and observe the error between the resulting values and the specified values of the given terminal conditions. Next, using some type of correction procedure, the original guesses on the initial conditions are modified in such a manner that the errors in the terminal values are reduced. The procedure is then successively repeated until agreement with the complete set of boundary conditions is achieved.

This is not the only approach to solving TPBVP and in many cases is not the most effective method. Two alternate approaches will be discussed in the following section. However, for the work of this thesis, the method that uses variations of the undetermined initial conditions is of particular importance.

The method for solving TPBVP which we shall consider in this section will use the variation of undefined initial conditions, as described above, and will use the Newton-Raphson method for the correction procedure. The derivative to be used in the Newton-Raphson method will be determined by solving a set of perturbation equations about the optimum trajectory. This procedure is referred to as the "Neighboring Optimum" method. It is similar to that described by Breakwell, Speyer and Bryson.[16] The technique is also discussed by Scharmack.[8]

In developing the method of solving TPBVP it is convenient to assume that the relation (2.11) may be solved for the u vector,

$$u(t) = f\left(x(t),\ p(t)\right)$$

This may be done, at least implicitly, if the determinant

$$\begin{vmatrix} \dfrac{\partial^2 H}{\partial u_1^2} & \cdots & \dfrac{\partial^2 H}{\partial u_1 \partial u_r} \\ \vdots & & \vdots \\ \dfrac{\partial^2 H}{\partial u_r \partial u_1} & \cdots & \dfrac{\partial^2 H}{\partial u_r^2} \end{vmatrix}$$

is different from zero. This result is used to eliminate u from the system of Equations (2.9) and (2.10). Also we define the vector $y = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ and rewrite (2.9) and (2.10) as

$$\dot{y}(t) = F\left(y(t)\right) \tag{2.15}$$

where F is a 2n dimensional vector. It may be assumed, without loss of generality, that the given boundary conditions are

$$\begin{aligned} y_1(t^o) &= c_1 \\ &\vdots \\ y_n(t^o) &= c_n \\ y_1(t') &= c_{n+1} \\ &\vdots \\ y_n(t') &= c_{2n} \end{aligned} \tag{2.16}$$

where the c_i are fixed constants.

The system of differential equations (2.15) has a unique solution for a given set of initial conditions, i.e., a complete set of 2n initial conditions. Thus if a set of initial conditions for $y_{n+1}, \ldots, y_{2n}$ is arbitrarily selected, e.g.,

$$
\begin{aligned}
y_{n+1}(t^o) &= d_1 \\
&\vdots \\
y_{2n}(t^o) &= d_n
\end{aligned}
\tag{2.17}
$$

and if we define the vector $d^o = (d_1, \ldots, d_n)$, then the solution of (2.15) which satisfies the first n conditions of (2.16) has the form

$$
y = y(d^o, t) \tag{2.18}
$$

That is, the solution is a function of time t and the arbitrary initial conditions d^o. Moreover, from the definition of F and from the theory of differential equations, it is known that the solution (2.18) has continuous partial derivatives in the variables t and d^o of at least second order.

Thus the TPBVP reduces to finding the vector d^o and the time t' such that the solution (2.18) satisfies the terminal conditions (i.e., the second half) of the relations (2.16). Since the relation (2.14) must be satisfied, one of the d_i is determined. If it is assumed that (2.14) is met by a particular choice of d_n, we may re-define d^o as $d^o = (d_1, \ldots, d_{2n-1}, t)$. Then the problem of satisfying the terminal conditions may be written as

$$
y(d^o) - c' = 0 \tag{2.19}
$$

where $c' = (c_{n+1}, \ldots, c_{2n})$, and $y(d^o)$ is the solution (2.18).

Thus, the problem is to determine the n roots (i.e., the components of d^o) of the system (2.19) of n algebraic equations. This is a familiar problem in numerical analysis and many approaches to the solution have been developed. Probably the most widely used and most successful method is the Newton-Raphson iterative technique. Its most important feature is rapid convergence to the solution. However, convergence is guaranteed only if the initial starting point is sufficiently close to the final solution.

The Newton-Raphson successive approximation procedure for Equation (2.19) may be represented by

$$
d^{o(n+1)} = d^{o(n)} - \left[\frac{\partial y\left(d^{o(n)}, t'\right)}{\partial d^o} \right]^{-1} \left(y\left(d^{o(n)}, t'\right) - c' \right) \tag{2.20}
$$

where $\left[\partial y/\partial d^o\right]^{-1}$ is the inverse of

$$\begin{bmatrix} \frac{\partial y_1}{\partial d_1^o} & \frac{\partial y_1}{\partial d_2^o} & \cdots & \frac{\partial y_1}{\partial d_n^o} \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial d_1^o} & \cdots & \cdots & \frac{\partial y_n}{\partial d_n^o} \end{bmatrix}$$

evaluated at t'. The superscripts in parentheses denote the index of the approximation, e.g., $d^{o(o)}$ is the initial starting point, $d^{o(1)}$ the first approximation, etc.

The conditions sufficient to ensure convergence with the Newton-Raphson method, as well as an estimate of the rate of convergence, have been established by L. V. Kantorovich.[23] To state this theorem, we require the concept of the uniform norm. For a vector $x = (x_1, \ldots, x_n)$ we have

$$|x| = \max_{1 \leqq i \leqq n} |x_i| \tag{2.21}$$

The vertical brackets on the scalar x_i denote absolute value, while the vertical brackets on the vector x denote the uniform norm. Also, for an n x n matrix $A = (a_{ij})$,

$$|A| = \max_{1 \leqq i \leqq m} \left[\sum_{j=1}^{n} |a_{ij}|\right] \tag{2.22}$$

where the significance of the brackets are as in (2.21). For problem (2.19) the theorem has the following form:[24]

<u>Theorem on Convergence</u> (Kantorovich)

Assume that the following conditions are satisfied:

1. For $d^{o(o)}$, the initial approximation, the matrix $A = \left[\frac{\partial y}{\partial d^o}\right]$ has an inverse $G_o = A\left(d^{o(o)}\right)^{-1}$, and an estimate for its norm is known:

$$|G_o| \leqq B_o \tag{2.23}$$

2. The vector $d^{o(o)}$ approximately satisfies the system of Equations (2.19) in the sense that

$$|G_o|\left[y\left(d^{o(o)}\right)-c'\right]| \leq n_o \tag{2.24}$$

3. In the region defined by inequality (2.27) below, the components of the vector $y(d^o)$ are twice continuously differentiable with respect to the components of d^o and satisfy

$$\sum_{j,k=1}^{n}\left[\frac{\partial^2 y_i}{\partial d_j^o \, \partial d_k^o}\right] \leq K, \qquad i = 1, 2, \ldots, n \tag{2.25}$$

4. The constants B_o, n_o, and K introduced above satisfy the inequality

$$h_o \equiv B_o n_o K \leq \frac{1}{2} \tag{2.26}$$

Then the system of Equations (2.19) has a solution $\overline{d^o}$ which is located in the cube

$$|\overline{d^o} - d^{o(o)}| \leq \frac{1 - \sqrt{1-2h_o}}{h_o} n_o \tag{2.27}$$

Moreover, the successive approximation $d^{o(\ell)}$ defined by (2.20) exist and converge to $\overline{d^o}$, and the speed of convergence may be estimated by the inequality

$$|d^{o(\ell)} - \overline{d^o}| \leq \frac{1}{2^{\ell-1}} (2h_o)^{2^{\ell}-1} n_o \tag{2.28}$$

In order to use this theorem to determine if an initial approximation is sufficiently accurate, would require, in any significant problem, an excessive amount of computation. However, since assumption (3) is satisfied for some K, the theorem shows that in some region about the solution $\overline{d^o}$ (i.e., for sufficiently small B_o and n_o) the method (2.20) will converge. And the rate of convergence, as estimated by (2.28), is very rapid.

Thus the theorem delineates the advantages and disadvantages of the method. That is, the rapid convergence property and the requirement of a good initial approximation.

It should be noted that the conditions of the theorem are sufficient but not necessary for convergence. In practice, the region of convergence may be much larger than the theorem indicates, and, correspondingly, in this extended region the rate of convergence may be less rapid.

Since convergence is guaranteed in a neighborhood of the solution, it is possible to converge to the correct solution by use of a series of intermediate steps. To illustrate this, assume an initial approximation $d^{o(o)}$ satisfies the equation $y\left(d^{o(o)}\right) = c^n$. Then select a sequence of c^i, which differ in norm by a small amount, and which converge to c'. By using $d^{o(o)}$ as the initial approximation, converge to the solution of $y(d^o) - c^{n-1} = 0$. This solution is then used as the initial approximation for the equation $y(d^o) - c^{n-2} = 0$. And in this manner it is possible to converge to the solution of $y(d^o) - c' = 0$ providing $\left| c^j - c^{j-1} \right|$ is sufficiently small.

In order to use the Newton-Raphson procedure (2.20) we require the matrix $\left[\partial y(d^o)/\partial d^o\right]$. Now y as a function of d^o exists only implicitly, so it is necessary to determine this matrix numerically. One possible approach to this is to solve (2.15) for a nominal d^o and find $y^o(t')$, then alter the first component of d^o by a small amount, i.e., set $d_1^o = d_1^o + \Delta d_1^o$, and resolve (2.15) for $y'(t')$. Then find

$$\Delta y_1 = y_1^o(t') - y_1'(t')$$

$$\vdots$$

$$\Delta y_n = y_n^o(t') - y_n'(t')$$

and assume that $\left[\partial y_i/\partial d_1^o\right] = \left[\Delta y_i/\Delta d_1^o\right]$. In a similar manner all the components of $\left[\partial y(d^o)/\partial d^o\right]$ could be determined. The difficulty with this approach is that the Δd_i^o must be arbitrarily small in order to provide a good approximation to $\left[\partial y_j/\partial d_i^o\right]$ and this will cause the differences Δy_j to be small. The unavoidable build-up in integration errors in the $y^i(t')$ solutions will cause an error in these differences.

A more accurate estimate of $\left[\partial y/\partial d^o\right]$ may be obtained by solving the perturbation equations of the system (2.15). Consider a nominal value of $\bar{y}(t^o)$ and the corresponding solution $\bar{y}(t)$ of (2.15). Now consider a small perturbation of $y(t^o) = \bar{y}(t^o) + \Delta y(t^o)$ and the corresponding solution $y(t)$ of (2.15). Let the deviation between these two solutions be

$$\delta y(t) = y(t) - \bar{y}(t) \qquad (2.29)$$

Then this deviation satisfies the differential equation

$$\delta\dot{y}(t) = \left[\frac{\partial F\left(y(t)\right)}{\partial y}\right] \delta y(t) \tag{2.30}$$

with initial conditions $\delta y(t^o) = \Delta y(t^o)$. Note that (2.30) is linear in δy. Now for the initial conditions

$$\delta y(t^o) = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

$$\qquad\qquad\qquad\qquad \llcorner (n+j)^{th} \text{ component}$$

the solution of (2.30) is

$$\delta y(t) = \left(\frac{\partial y_1}{\partial d_j^o}, \frac{\partial y_2}{\partial d_j^o}, \ldots, \frac{\partial y_n}{\partial d_j^o}\right)$$

So by solving (2.30) n times the matrix $\left[\partial y/\partial d^o\right]$ can be determined. Note that $[\partial F/\partial y]$ in Equation (2.30) is evaluated along the nominal trajectory $\bar{y}(t)$.

For the optimal control problem, the perturbation equations (i.e., the equations satisfied by the deviation variables corresponding to a perturbation of the initial or final conditions) of Equations (2.9) and (2.10) are, respectively,

$$\delta\dot{x} = \left(\frac{\partial^2 H}{\partial p \partial x}\right) \delta x + \left(\frac{\partial^2 H}{\partial p \partial u}\right) \delta u \tag{2.31}$$

$$\delta\dot{p} = \left(\frac{\partial^2 H}{\partial x^2}\right) \delta x - \left(\frac{\partial^2 H}{\partial x \partial p}\right) \delta p - \left(\frac{\partial^2 H}{\partial x \partial u}\right) \delta u \tag{2.32}$$

On the boundary of U the deviation variable δu is zero; off the boundary we may write (2.11) as

$$\frac{\partial H(x, p, u)}{\partial u} = 0$$

and the corresponding perturbation equation is

$$\left(\frac{\partial^2 H}{\partial u \partial x}\right) \delta x + \left(\frac{\partial^2 H}{\partial u \partial p}\right) \delta p + \left(\frac{\partial^2 H}{\partial u^2}\right) \delta u = 0 \tag{2.33}$$

In addition, the deviation variables must satisfy the perturbation equations of the boundary conditions. For the initial conditions $x(t^o) = x^o$ where x^o is a given fixed vector, this means that

$$\{\delta x_i - f_i \, \delta t\}_{t=t^o} = 0 \quad , \qquad i = 1, \ldots, n \tag{2.34}$$

And for terminal conditions

$$x_j(t') = x_j' \quad , \qquad j = 1, \ldots, q \tag{2.35}$$

$$p_j(t') = c_j \quad , \qquad j = q+1, \ldots, n \tag{2.36}$$

where x_j' and c_j are fixed constants, we have

$$\{\delta x_j + f_j \, \delta t\}_{t=t'} = 0 \tag{2.37}$$

$$\{\delta p_j + \dot{p}_j \, \delta t\}_{t=t'} = 0 \tag{2.38}$$

Since for optimal control, $H(x, p) = 0$ for all t, at each end point the relation

$$\left(\frac{\partial H}{\partial x}\right) \delta x + \left(\frac{\partial H}{\partial p}\right) \delta p = -\dot{p} \, \delta x + f \, \delta p = 0 \tag{2.39}$$

must be satisfied.

Thus we have 2n differential Equations (2.31) and (2.32) involving $2n + r + 1$ deviation variables

$$(\delta x_1, \ldots, \delta x_n, \ \delta p_1, \ldots, \delta p_n, \ \delta u_1, \ldots, \delta u_r, \ \delta t)$$

The r relations of Equation (2.33) may be used to solve for the δu_i, or, when on the boundary, the δu_i are set equal to zero. And at either end point there are n+1 relations among the deviation variables, given by (2.39) and either (2.34) or (2.37) and (2.38). And since the differential equations are linear in the perturbation variables, there exist n independent solutions of these equations. These solutions may be obtained in the same manner as was indicated for Equation (2.30). Similarly, the combination of these solutions provides the matrix to be used in the Newton-Raphson procedure.

To summarize, the solution to the optimal control problem, by the "neighboring optimum" method requires the following steps:

1. An initial estimate is made of the unspecified initial (or terminal) conditions.

2. The differential Equations (2.9) and (2.10) are integrated using the specified and estimated initial (or final) conditions. The values of the control variables are chosen such that relation (2.11) is met at all points along the trajectory.

3. The perturbation Equations (2.31) and (2.32) are integrated n times, for n linearly independent initial (or final) conditions which also satisfy (2.34) (or 2.37 and 2.38) and (2.39).

4. The Newton-Raphson procedure is used to provide a new estimate for unspecified initial (or final) conditions. For integration in the forward direction, the Newton-Raphson method is represented by

$$p(t^o)^{(n+1)} = p(t^o)^{(n)} - A^{-1} \begin{bmatrix} x_1(t') - x'_1 \\ \cdot \\ \cdot \\ \cdot \\ x_q(t') - x'_q \\ p_{q+1}(t') - c_{q+1} \\ \cdot \\ \cdot \\ \cdot \\ p_n(t') - c_n \end{bmatrix} \quad (2.40)$$

where $p(t^o)$ is the vector of initial values on p and the superscript n and n+1 represent the successive estimates. The matrix A is obtained from the results of Step 3.

5. Steps 2, 3 and 4 are repeated until the terminal (or initial) conditions given by (2.35) and (2.36) are satisfied to within sufficient accuracy.

2.5 Other Methods of Solution

The method presented in the previous section is typical of the classical approach to solving two point boundary value problems. There are several disadvantages to that approach; these include the large amount of computer programming required, the severe computational accuracy required and, most importantly, the requirement for a very good initial approximation to the undefined boundary conditions. Because of this, other approaches to solving optimal control problems are needed.

There are many special methods for solving particular optimal control problems and many variations to each of the general methods. Here, we shall consider only a brief comparison between the three basic general approaches.

The Maximum Principle gives essentially three conditions which an optimal trajectory must satisfy. These are: (a) the system of differential equations, (2.9) and (2.10); (b) the relation which determines the optimal control, (2.11); and (c) the boundary conditions determined by the end points x^o and x', and condition (3) of the Maximum Principle.

In the "neighboring optimum" method the procedure is to generate solutions which satisfy conditions (a) and (b), and then iteratively modify the parameters of these solutions until condition (c) is satisfied.

An alternate procedure could be to find solutions which satisfy conditions (a) and (c), then modify these solutions, by some iterative technique, such that condition (b) is satisfied. Such a procedure has been developed;[25, 26] it is called the "gradient" method. In practice the "gradient" technique does not use the same problem formulation as is given by the Maximum Principle, but conceptually the procedure is as stated. The actual procedure is that some control function is chosen to give a solution of the system equations which approximately satisfies the boundary conditions. Then the control function is modified at each iteration in such a manner that the boundary conditions are approached and the criterion function is increased. Thus the control function approaches the optimal control function.

Before considering the gradient method further, we will indicate the third general approach. This third procedure is to find solutions which satisfy (b) and (c), then modify the parameters of these solutions in order to satisfy condition (a). This method is referred to as "quasilinearization",[27] and is closely related to the "neighboring optimum" method.

The reason for considering different methods of solution is that the usefulness of each method differs for different types of problems. It may even be necessary to use more than one method to solve a given problem. This may be the case just to converge to any solution, but also, different solutions may be needed to determine the degree of accuracy of these solutions.

The gradient method was originally developed in order to avoid the TPBVP that arise when the classical variational techniques are applied to optimal control problems. And the problem formulation and resulting computational requirements are not as complex as those of the classical approach. Also, it appears that the gradient technique has been successfully applied to more practical problems than has been the classical formulation. However, the computational requirements have a great similarity for the two methods and as yet there has not been an adequate comparison between the two methods with respect to the amount of computer time required to obtain solutions. The apparent advantage of the gradient technique is that convergence to an optimal solution can be obtained over a much larger region of the state space and that a reasonable initial approximation can be obtained based on knowledge of the physical process.

To describe the gradient method we again consider the system state equations

$$\dot{x} = f(x, u) \tag{2.41}$$

and the corresponding perturbation equation

$$\delta\dot{x} = \left(\frac{\partial f}{\partial x}\right) \delta x + \left(\frac{\partial f}{\partial u}\right) \delta u \tag{2.42}$$

where the partial derivatives are evaluated along the nominal trajectory. The adjoint equation to (2.42) is defined to be

$$\dot{S} = - \left[\frac{\partial f}{\partial x}\right]^T S \tag{2.43}$$

where the superscript T denotes the transpose of the matrix and S is an n-vector $(S_1, S_2, \ldots, S_n)$ whose components $S_i(t)$ are said to be the influence functions corresponding to $x_i(t)$.

If Equation (2.42) is multiplied by S and Equation (2.43) by δx, and the sum is integrated over the interval $t^o \leqq t \leqq t'$, the following is obtained:

$$\left[S^T \delta x\right]_{t^o}^{t'} = \int_{t^o}^{t'} G(t)\ \delta u(t)\ dt \tag{2.44}$$

where

$$G = S^T \left(\frac{\partial f}{\partial u}\right) \tag{2.45}$$

Let

$$S(t') = \left.\frac{\partial J}{\partial x}\right|_{t=t'} \tag{2.46}$$

where J is the criterion function, which depends on the final values of the state variables. Then $S^T \delta x \Big|_{t=t'} = \delta J\Big|_{t=t'}$ so (2.44) becomes

$$\delta J = \int_{t^o}^{t'} G(t)\ \delta u(t)\ dt + \left[S^T\ \delta x\right]_{t=t^o} \tag{2.47}$$

and G(t) is the desired influence function (or Green's function) that gives the effect of small changes in the control function $\delta u(t)$, on the criterion function J. Note that the influence functions S_i are determined by Equation (2.43) and boundary conditions (2.46).

If no constraints are placed on $x(t')$, the greatest change, δJ, in J for a given value of

$$\int_{t^o}^{t'} (\delta u)^2 \, dt$$

is obtained when

$$\delta u = K \, G(t) \tag{2.48}$$

where K is a constant. This is the "steepest descent (or ascent)" direction to the minimum (or maximum) J. The value of K is somewhat of an experience factor, i.e., it is best chosen by observing trial solutions. However, procedures for deriving reasonably good values for K have been developed.[28]

Next, suppose there is one constraint on x(t'), i.e.,

$$L \left(x(t') \right) = 0 \tag{2.49}$$

Then the procedure is to calculate a second set of influence functions using the same Equation (2.43) but different boundary conditions, namely

$$S^L(t') = \left. \frac{\partial L}{\partial x} \right|_{t=t'} \tag{2.50}$$

where the superscript L is used to distinguish these influence functions from the previous ones (which shall now be denoted by S^J). As in (2.47) we then have

$$\delta L = \int_{t^o}^{t'} G^L(t) \, \delta u(t) \, dt + \left[(S^L)^T \delta x \right]_{t=t^o} \tag{2.51}$$

Now if the nominal trajectory approximately satisfies the constraint (2.49), then by setting δL equal to the negative of L(t'), the next trajectory should satisfy L(t') = 0. The "steepest descent" direction to minimize J and satisfy a given δL is

$$\delta u(t) = K_J G^J + K_L G^L \tag{2.52}$$

where K_J and K_L are constants. Substituting (2.52) into (2.47) and (2.51), and neglecting the influence of initial conditions, yields the simultaneous linear equations

$$\begin{aligned} \delta J &= K_J I_{JJ} + K_L I_{LJ} \\ \delta L &= K_J I_{LJ} + K_L I_{LL} \end{aligned} \tag{2.53}$$

where $I_{JL} = \int_{t^o}^{t'} G^J G^L \, dt$. By estimating a reasonable value for δJ, Equations (2.53) may be solved for K_J and K_L. As the optimum trajectory satisfying the constraint (2.49) is approached, this solution will become singular, but it may be possible to obtain sufficient accuracy before this happens.

A similar extension to the case of more than one terminal condition and to the case of unspecified terminal time t', can be carried out. The resulting computational requirements are similar to those of the previous paragraph.

Thus the basic steps of the "gradient" method are:

1. Estimate a control function u(t) that gives a nominal trajectory (i.e., a solution of Equation (2.41) using the given initial conditions) which approximately satisfies the desired terminal conditions.

2. Using the terminal conditions (2.46) and (2.50), integrate the adjoint Equation (2.43) backward in time $\left(\text{using the nominal trajectory to obtain } \left[\frac{\partial f}{\partial x}\right]\right)$ to obtain the influence functions.

3. Recompute the nominal trajectory using Equation (2.52) to give a new estimate for the control function.

4. Repeat Steps 2 and 3 until the actual change in the criterion function is well below the estimated δJ and the terminal conditions are satisfied.

It is obvious that the gradient method is somewhat dependent on judgment and intuition, but it has been used successfully with practical problems. The most serious limitation of the method is that it does not converge in the immediate neighborhood of the optimum trajectory.

Quasilinearization, the third approach, solves the TPBVP by replacing the nonlinear boundary value problem with a sequence of linear boundary value problems. These linear boundary value problems involve the following equation:

$$\delta\dot{y} = \left(\frac{\partial F(y)}{\partial x}\right) \delta y - \left[F(y + \delta y) - F(y)\right] \tag{2.54}$$

where δy is again a perturbation variable and y is evaluated along the corresponding nominal trajectory. It is evident that this equation is the same as the perturbation equation (2.30) but with a forcing function which depends upon the distance between the nominal trajectory and the perturbation trajectory. As before, the components of y are $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ and $F = (f_1, \ldots, f_2, \dot{p}_1, \ldots, \dot{p}_n)$. The procedure is this: first select a nominal trajectory for the state variables x_i and the auxiliary variables

p_i which satisfy the given boundary conditions at t^o and t'; it is not required that this trajectory satisfy the system equations (2.9) and (2.10). Next, obtain n solutions to the perturbation equations (2.54) by successively using the initial conditions obtained by setting one of the n independent perturbation variables to unity with the others equal to zero, as was discussed in Section 2.4. At the terminal point, the perturbation variables must satisfy n conditions, e.g.,

$$\begin{aligned} \delta x_i &= 0 \quad , \quad i = 1, \ldots, q \\ \delta p_i &= 0 \quad , \quad i = q+1, \ldots, n \end{aligned} \tag{2.55}$$

if the terminal conditions are given by (2.35) and (2.36). The proper initial condition corresponding to the constraint (2.55) can be solved for. Using these initial conditions, a new solution of the perturbation equations is obtained, and this solution is added to the nominal trajectory. Using this new nominal trajectory, the procedure is repeated until the nominal trajectory is unchanging. A convergence proof for this procedure has been given by McGill and Kenneth.[29] The conditions and results of the proof are similar to those of Kantorovich's theorem.

The apparent advantage of quasilinearization is that convergence can be obtained over a larger region of state space than with the neighboring optimum method. Very little has been reported concerning computational experience with the method.

2.6 State Variable Inequality Constraints

In actual physical systems, often not only the control function, but also the state variables, must be subject to certain restrictions. For example, any acceptable flight path for an aircraft must satisfy the restriction that the altitude above the ground be greater than or equal to zero. Such restrictions may be expressed as inequality constraints,

$$G_i(x, u) \geqq 0 \quad , \quad i = 1, 2, \ldots, m \tag{2.56}$$

In general the inequality constraints may not explicitly include the control variable u. When this is the case, the procedure becomes much more complex but the basic approach remains the same.[30, 31] Here we consider only those with the form (2.56).

When the inequalities $G = (G_1, G_2, \ldots, G_m)$ are introduced into the optimum control problem, the procedure is to define an augmented Hamiltonian by

$$H' = H + v\,G \tag{2.57}$$

where H is as given by (2.8) and $v = (v_1, v_2, \ldots, v_m)$ is an m vector of multipliers which satisfy the relations

$$v_i G_i = 0 \qquad (2.58)$$

$$, \quad i = 1, 2, \ldots, m$$

$$v_i \leqq 0 \qquad (2.59)$$

Now Equation (2.10) becomes

$$\dot{p}_i = -\frac{\partial H'}{\partial x_i} \qquad (2.60)$$

and the optimal control is determined from the equation

$$\frac{\partial H'}{\partial u} = 0 \qquad (2.61)$$

provided this u(t) is within the control space U.

From (2.58) it is seen that $v_i = 0$ if $G_i > 0$, i.e., the multipliers v_i differ from zero only when the corresponding constraint G_i equals zero. It has been shown[32] that the multipliers v_i are continuous functions of time. Thus, if over a portion of the optimal trajectory, all $G_i > 0$, then the equations describing the optimal trajectory are the same as if no constraints of type (2.56) existed. Now consider what happens when one of the constraints, say G_1, becomes equal to zero. Then the optimal trajectory equations contain an additional variable v_1 (which is equal to zero at the time at which G_1 first becomes equal to zero). But there is also an additional equation which must be satisfied, namely

$$G_1(x, u) = 0 \qquad (2.62)$$

and the optimal control u and also v_1 can be determined, provided that the determinant

$$\begin{vmatrix} \frac{\partial^2 H'}{\partial u^2} & \left(\frac{\partial G_1}{\partial u}\right)^T \\ \frac{\partial G_1}{\partial u} & 0 \end{vmatrix} \qquad (2.63)$$

is not equal to zero. This situation will hold over a segment of the optimal trajectory until v_1 increases to the value zero. At this point Equation (2.62) ceases to hold and the optimal trajectory equations are again the same as when no inequality constraints exist.

So the optimal trajectory is made up of a series of subarcs, along which either $G_i > 0$ or $G_i = 0$. It can be shown that the trajectory is continuous at the junction points between these subarcs.[33]

To illustrate this method, consider Figure 2.1 which represents a system with two variables. In this drawing the given initial condition at t^o is x^o and the given final condition at t' is x'. Two trajectories, which represent the system solution for the initial conditions (x^o, p_a) and (x^o, p_b), are shown; the corresponding final values for x are x_a and x_b, respectively. A possible correction procedure for the initial value of p could make use of linear interpolation to give

$$p_c = \frac{(x_a - x')\, p_b - (x_b - x')\, p_a}{(x_a - x_b)}$$

where p_c represents the next guess for the undetermined initial condition.

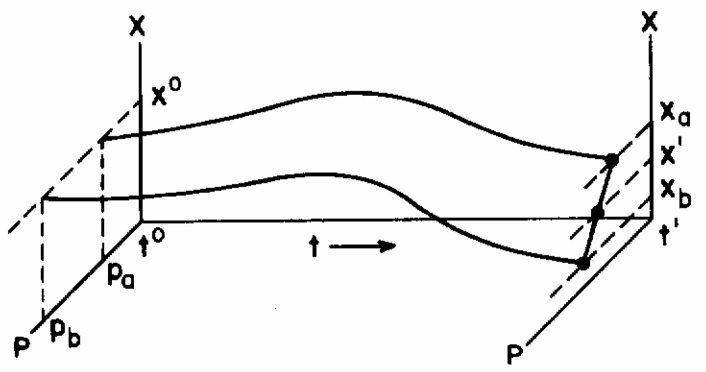

Figure 2.1. Representation of a Two Point Boundary Value Problem

CHAPTER 3

COMPUTATIONAL METHODS AND INTEGRATION OF OPTIMAL TRAJECTORIES

3.1 Computational Requirements for Obtaining Optimal Trajectories

The formulation of the optimal control problem given in Chapter 2 is quite general and can be applied to a great many practical engineering systems. As was indicated, except for a limited number of special cases, only numerical solutions can be obtained. It is the difficulty in obtaining these solutions, that is, the lack of effective computation methods, that has prevented the practical application of optimal control techniques to complex problems.

In this chapter we shall consider the numerical procedures that are required in order to generate solutions of optimal control problems, i.e., optimal trajectories. We shall restrict our consideration to those methods necessary to obtain solutions by the neighboring optimum method, which was discussed in Section 2.4.

Numerical computation methods are, in general, only approximations. So all numerical solutions, at least for large scale computations, are to some extent in error. The nature of this error is dependent on the problem being solved, the method of solution and the computing machine that is used.* There is not, at present, an adequate theory of errors for modern machine computation. And it is usually necessary to provide some form of experimental verification for error bounds.

The problem of accuracy in machine computation is closely related to the amount of computing or number of operations performed. For example, two of the basic causes of error are due to approximating functions by a truncated series expansion and the use of finite length or rounded-off numbers in the computations. Each of these errors may be reduced by performing additional computation, that is, by using more terms in the truncated series and using numbers of greater length. Thus there is a trade-off between the accuracy and length of computation. Various methods for computing numerical solutions may be compared by evaluating the ratio of solution error to the length of machine time required to obtain the solution. This ratio shall be called the computational efficiency of the method.

The most essential program operations, required in the neighboring optimum approach to solving the optimal control problem, are the following:

---

*It is assumed that the class of problems being considered is of such complexity that a modern digital computer, designed for scientific computation, is necessary to perform the calculations.

(a) An integration method for the state and auxiliary variables' differential equations (2.9) and (2.10)

(b) An integration method for the perturbation variables' differential equations (2.31) and (2.32)

(c) An iterative procedure for determining the control values which satisfy the maximum principle (2.11)

(d) A matrix inversion routine for determining the coefficient of the correction term in the Newton-Raphson procedure (2.20).

There are a number of routine logical and arithmetic operations required in addition to these. But the four operations listed above require most of the machine time used.

The integration procedures introduce the greatest problem insofar as accuracy is concerned. There are many methods for performing integration with a computer and there is a considerable amount of theory concerning the propagation of error in these methods.[34] However, there is no satisfactory method for determining which integration procedure is best for a given system of differential equations. But for the type of problem considered here the choice of the integration routine may well determine success or failure in obtaining useful programs. For this reason we place special emphasis on the question of choosing an accurate and efficient integration method for optimal control problems.

The neighboring optimum method for solving the TPBVP essentially consists of using integration to convert the problem into one of solving for the root of a system of algebraic equations. This root is then found by means of the Newton-Raphson procedure.

There exist methods for increasing the computational efficiency, for accelerating the rate of convergence and for increasing the region of convergence by modification of the Newton-Raphson technique.

The value of these alternate methods in machine computation depends entirely on the nature of the problem being solved. In the general case, the effect of these procedures can only be determined experimentally. Therefore, the standard Newton-Raphson method is recommended for obtaining initial solutions to a given problem. However, for many problems there is great potential advantage in using suitable modifications.

The primary methods for accelerating the convergence rate and increasing the computational efficiency are based on the idea of using, at each iteration, data computed at previous iterations. Many procedures of this type have been developed. Recently, J. F. Traub[35] has presented a general theory of iteration algorithms for the solution of systems of equations. His book provides a suitable basis for devising alternatives to the Newton-Raphson procedure.

Methods for increasing the region of convergence of the iteration procedure depend upon using additional data points at each iteration. An example of this approach is the method described by Kizner.[36] It seems clear that, at least in the initial stages of the process, these methods will reduce the computational efficiency.

3.2 Computing the Optimal Control Values

When the optimal control variables do not exceed their bounds, (i.e., $u \epsilon U$) the maximum condition (2.11) can be replaced by the equations

$$\frac{\partial H}{\partial u_j} = 0 \quad , \quad j = 1, 2, \ldots, r \tag{3.1}$$

and these equations may be of a form such that it is possible to solve for the u_j explicitly, that is, find $u_j = u_j(x, p)$. If so, these expressions can be used to eliminate the control variables from the differential equations (2.9) and (2.10). If this is done, then the side condition (3.1) is satisfied by any solution of the differential equations. Similarly it may be possible to eliminate u from Equations (2.9) and (2.10) when it takes on values at the boundary of U.

But in certain cases, as when the u_j enter into the state equations (2.9) in high order or transcendental terms, it may not be possible to solve Equations (3.1) explicitly for u_j. This will occur in problems for which the criterion function consists of several complex combinations of the original state variables, as it may in practical engineering studies. In these cases it is necessary to use some iterative procedure to solve Equation (3.1), or it may be more convenient to use the maximum condition (2.11) to find u.

When iterative procedures are necessary to solve for the optimal values of u, they require a great deal of additional computation as compared with an explicit equation. For example, if the maximum operation is used, the procedure is to compute and compare a sequence of values of H for different values of u until the correct solution is determined to within a suitable accuracy.

Using the maximum operation has two advantages over solving Equation (3.1) (assuming that (3.1) cannot be solved explicitly). One advantage is that no new equations are introduced to compute H and this reduces the amount of effort required to prepare the computer program. Another advantage is that no alternate procedure is required when u assumes values on the boundary of U. For these reasons we shall use the maximum operation to compute the optimal control values.

There are many methods of searching for the maximum of a function. However, for the type of problems to be considered in this thesis, the control variable is a continuous function of time and hence a sophisticated search procedure will not be required because a good estimate of the optimal control value will be available at each increment of time.

The accuracy to which the optimal control variables must be computed (i.e., the size of the final increment of u_j in the search routine) can only be determined experimentally.

Note that each computation of H requires the computation of the state variable derivatives, and this represents a significant fraction of the computation required to integrate the state variables one step. Therefore it is possible that the amount of computation necessary to find the optimal values of u_j may be several times as great as that necessary for all other operations required in integrating the differential equations.

So for efficient computational procedures it is desirable to minimize the number of computations of the optimum control values in the integration method. In Section 3.3 it is shown that for one-step integration procedures it is not practical to attempt to reduce the frequency of computation of u. In Section 3.4 it is shown that such a reduction is possible for the predictor-corrector type of integration schemes.

3.3 One-Step Integration Methods

Any computational algorithm for solving a differential equation such as

$$\frac{dx}{dt} = f(x,t) \tag{3.2}$$

in which the approximation to the solution at $t + \Delta t$ can be calculated if only $x(t)$, t and Δt are known, is called a one-step method. Such methods have important advantages, primarily their stability properties and the fact that the step size Δt can be readily varied. The primary disadvantage of such methods is their low efficiency.

The most commonly used one-step methods are those of the Runge-Kutta type which do not require higher derivatives than those appearing in the differential equations. The standard, or four point Runge-Kutta method is represented by the following equation:

$$x(t + \Delta t) = x(t) + \Delta t \left(\frac{1}{6} \right) \left[K_1 + 2K_2 + 2K_3 + K_4 \right] \tag{3.3}$$

where

$$K_1 = f(x,t)$$

$$K_2 = f\left(x + \Delta t\left(\frac{1}{2}\right)K_1,\ t + \Delta t\left(\frac{1}{2}\right)\right)$$

$$K_3 = f\left(x + \Delta t\left(\frac{1}{2}\right)K_2,\ t + \Delta t\left(\frac{1}{2}\right)\right)$$

$$K_4 = f(x + \Delta t\, K_3,\ t + \Delta t)$$

Note that this method requires the evaluation of the derivative at four points per interval. It can be shown[24] that under suitable conditions on the differential equation the solution generated by this algorithm converges to the correct solution as $\Delta t \rightarrow 0$. It can also be shown that this method is of the fourth order. That is, if the solution $x(t+\Delta t)$ is expressed by its Taylor series about $x(t)$, then the solution generated by the algorithm agrees with the terms of the Taylor series which are of third order or less in Δt, and the dominant error term is a function of $(\Delta t)^4$.

For a control problem where the differential equation is of the form

$$\frac{dx}{dt} = f(x,t,u)$$

the expressions for the predicted points of Equation (3.3) become

$$\begin{aligned}
K_1 &= f\Big(x, t, u(x,t)\Big)\\
K_2 &= f\left(x+\Delta t\left(\tfrac{1}{2}\right)K_1,\ t+\Delta t\left(\tfrac{1}{2}\right),\ u\left(x+\Delta t\left(\tfrac{1}{2}\right)K_1, t+\Delta t\left(\tfrac{1}{2}\right)\right)\right)\\
K_3 &= f\left(x+\Delta t\left(\tfrac{1}{2}\right)K_2,\ t+\Delta t\left(\tfrac{1}{2}\right),\ u\left(x+\Delta t\left(\tfrac{1}{2}\right)K_2, t+\Delta t\left(\tfrac{1}{2}\right)\right)\right)\\
K_4 &= f\Big(x+\Delta t\, K_3,\ t+\Delta t,\ u(x+\Delta t\, K_3,\ t+\Delta t)\Big)
\end{aligned} \qquad (3.4)$$

Thus it is necessary to compute the value of the control variable four times in each interval.

It may be desired to modify this method so that the computation of u is less frequent, even at the expense of additional integration steps. This could be done by using the same value of u at each predicted point, that is, evaluate u only after each integration step. The method would still be convergent as $\Delta t \rightarrow 0$, but it would become only a first order method. This means that the error would become a function of (Δt) rather than $(\Delta t)^4$, and in general would greatly reduce the accuracy of the integration procedure.

Example: To illustrate the effect of integration errors, we consider the classical Brachistochone problem formulated with a control variable. In this problem a body acted upon only by gravity falls from an initial height along some path such that the time to achieve a given horizontal range is minimized. The equations for this system may be written as

$$\dot{x}_1 = (2\ g\ x_2)^{\frac{1}{2}} \cos u$$
$$\dot{x}_2 = (2\ g\ x_2)^{\frac{1}{2}} \sin u \tag{3.5}$$

where x_1 represents the height of the particle, x_2 represents the horizontal range, t is time, g is the gravity constant, and u is the direction of the path in the $x_1 - x_2$ plane and is considered to be the control variable. For convenience in obtaining an analytic solution, the boundary conditions are chosen as:

$$x_1(o) = x_2(o) = x_3(o) = 0 \quad , \quad x_1(T) = \ell$$

The criterion function is

$$J = \int_o^T dt = T$$

Applying the procedure of the Maximum Principle, we obtain the auxiliary differential equations:

$$\dot{p}_1 = 0$$
$$\dot{p}_2 = -g(2\ g\ x_2)^{\frac{1}{2}}[p_1 \cos u + p_3 \sin u] \tag{3.6}$$

with the boundary conditions

$$p_2(T) = 0$$

The Hamiltonian function is

$$H = (2\ g\ x_2)^{\frac{1}{2}}\ [p_1 \cos u + p_2 \sin u] + 1$$

and from the optimal control conditions we find

$$(2\ g\ x_2)^{\frac{1}{2}}\ [-p_1 \sin u + p_2 \cos u] = 0$$

This equation may be solved for u,

$$u = \arc\tan\left(\frac{p_2}{p_1}\right) \tag{3.7}$$

This example was chosen so that an analytic solution could be obtained to compare with numerical solutions. The analytic solution is

$$x_1 = \frac{2\ell}{\pi}\left(\omega t - \frac{1}{2}\sin 2\,\omega t\right)$$

$$x_2 = \frac{2\ell}{\pi}\quad \sin^2 \omega t$$

$$p_1 = -\omega/g$$

$$p_2 = -\frac{\omega}{g}\cot \omega t$$

$$u = \frac{\pi}{2} - \omega t$$

$$T = (\pi\,\ell/g)^{\frac{1}{2}} \tag{3.8}$$

where

$$\omega = (\pi\, g/4\ell)^{\frac{1}{2}}$$

The results of several integration procedures, using the standard Runge-Kutta method on the system of differential equations (3.5) and (3.6), are shown in Table 3.1. The integration was run backward in time because the initial condition on p_2 could not be obtained numerically. The first column gives the correct values, taken from the analytic solution. The first integration method used (3.7) to recompute u before each computation of the derivative, as indicated in Equations (3.4). The second integration only computed u at the end of each step. For both the first two integrations the step size was 0.5 seconds. The third integration used the same procedure as the second but the step size was reduced to 0.125 seconds. Note that even though the third integration recomputed u the same number of times as the first, it did not achieve as good accuracy.

It can be shown that the results demonstrated above for the standard Runge-Kutta method are obtained for any one-step integration method. That is, any reduction in the number of computations of the control variable will result in reducing the order and accuracy of the integration method. In general there is no advantage in following such a procedure.

TABLE 3.1

INTEGRATION RESULTS, ONE-STEP METHODS

Variable	Time	True Solution	Integration Method 1	2	3
x_1	9	8.0163681	8.0163689	8.0061353	8.0134250
	7	4.4248188	4.4248200	4.3241427	4.3985117
	5	1.8169011	1.8168991	1.5528159	1.7491255
	3	0.4248189	0.4248090	-0.2764395	0.3098504
	1	0.0163684	0.0163677	-0.5760696	-0.1325418
x_2	9	6.2104057	6.2104062	6.2881398	6.2298086
	7	5.0540773	5.0540819	5.2651793	5.1065552
	5	3.1830987	3.1831116	3.4614787	3.2517722
	3	1.3121201	1.3121497	1.5579479	1.3716586
	1	0.1557922	0.1560099	0.2655347	0.1803593
p_2	9	-.00791922	-.00791923	-.00789045	-.00791276
	7	-.02547627	-.02547630	-.02519244	-.02540702
	5	-.05000000	-.05000012	-.04880073	-.04969792
	3	-.09813053	-.09813155	-.09272533	-.09671404
	1	-.31568733	-.31594490	-.25327157	-.29697341
u	9	.15707962	.15707979	.15651817	.15695363
	7	.47123889	.47124427	.46672195	.47013869
	5	.78539813	.78536067	.77326047	.78236818
	3	1.0995575	1.0993623	1.0762618	1.0936508
	1	1.4137165	1.4120620	1.3758861	1.4039954

3.4 Multi-step Methods

Multi-step integration methods are those algorithms which use previously computed values of the approximate solution to calculate each new value of the approximate solution. The general linear k-step method for Equation (3.2) is given by the formula

$$x^{n+k} = \alpha_{k-1} x^{n+k-1} + \alpha_{k-2} x^{n+k-2} + \ldots + \alpha_o x^n + \Delta t \left[\beta_k f^{n+k} + \ldots + \beta_o f^n \right] \quad (3.9)$$

where

$$x^j = x(t^o + j\,\Delta\, t)$$

$$f^j = f(x^j,\ t^o + j\,\Delta t)$$

and α_i, β_i are constants.

Methods such as these offer the possibility of much greater efficiency than one-step methods. However they may introduce additional stability problems. Also, they require special procedures to start the integration and to vary the step size.

The multi-step methods most widely used in practice are of the predictor-corrector type. These methods use an equation of type (3.9) but with $\beta_k = 0$ to "predict" a new value of x^{n+k}, then use this value to calculate f^{n+k}. With this estimate of f^{n+k}, another formula of type (3.9) with $\beta_k \neq 0$ is used to obtain the "corrected" value of x^{n+k}. Predictor-corrector methods appear to be the most efficient integration method for general first order differential equations. Also they have the advantage that an estimate of the integration error at each step is immediately available in the difference between the predicted and the corrected values of x^{n+k}. For a detailed discussion of predictor-corrector methods see Reference 37.

Predictor-corrector methods require the computation of the derivatives two times in each integration step, as compared to four times for the standard Runge-Kutta method. Also, the order of the method is a function of k, not of the number of times the derivatives are computed. So any order predictor-corrector method will require two computations of u per step in optimal control problems, one each for the predicted and corrected values of x^{n+k}.

In order to reduce the number of computations of u by the Maximum Principle (2.11), the following equation can be used to obtain a predicted value of u^{n+k}

$$u^{n+k} = \gamma_{k-1} u^{n+k-1} + \gamma_2 u^{n+k-2} + \ldots + \gamma_o u^n \quad (3.10)$$

This value may then be used with the predicted value of x^{n+k} to find the estimated value of f, i.e.,

$$f^{n+k} = f\left(x^{n+k},\ t^{o} + (n+k)\,\Delta t,\ u^{n+k} \right)$$

for use in the corrector formula. The use of this extrapolation technique will require some starting procedure and special routines for use when the integration step size is changed. But these can be similar to what is used in the predictor-corrector program. As in the predictor-corrector method, the difference between the value of u^{n+k} obtained from (3.10) and that obtained from the Maximum Principle will give an estimate of the error at each step.

The choice of values for the constants (γ_i) in (3.10) will depend on the behavior of u(t) in any particular problem. If u(t) can be approximated by a low order polynomial, then the γ_i can be chosen as functions of the coefficients of this polynomial. A method for choosing these constants is discussed in Section 3.7.

Example: To illustrate the predictor-corrector method and the extrapolation approximation, they are applied to the integration of the example of Section 3.3. The predictor formula used was Milne's three-point method,

$$(x^{n+k})_p = x^{n+k-4} + \left(\frac{4}{3}\right)\Delta t \left[\ 2f^{n+k-1} - f^{n+k-2} + 2f^{n+k-3}\ \right] \tag{3.11}$$

Using this value of x^{n+k}, u^{n+k} was calculated from (3.7) and then $f(x^{n+k}, u^{n+k})$ was calculated from (3.5) and (3.6). The corrector formula chosen was

$$(x^{n+k})_c = \left(\frac{1}{8}\right)\left[9\ x^{n+k-1} - x^{n+k-3} + 3\ \Delta t(f^{n+k} + 2f^{n+k-1} - f^{n+k-2})\right] \tag{3.12}$$

The correct value was assumed to be

$$x^{n+k} = \left(\frac{9}{121}\right)(x^{n+k})_p + \left(\frac{112}{121}\right)(x^{n+k})_c \tag{3.13}$$

The results of this integration are listed under method 1 of Table 3.2. The step size is the same as was used in method 1 of Table 3.1. The accuracies of these two methods are very nearly the same. Similarly, if the computation of u^{n+k} following the evaluation of $(x^{n+k})_p$ were omitted, the results would be similar to those of method 2 in Table 3.1.

Next, the procedure was modified such that the computation of u^{n+k} following the evaluation of $(x^{n+k})_p$ was obtained using

TABLE 3.2

INTEGRATION RESULTS, MULTI-STEP METHODS

Variable	Time	True Solution	Integration Method 1	Integration Method 2
x_1	9	8.0163681	8.0163689	8.0163689
	7	4.4248188	4.4248196	4.4248203
	5	1.8169011	1.8168924	1.8169098
	3	0.4248189	0.4247554	0.4248582
	1	0.0163684	0.0159342	0.0165080
x_2	9	6.2104057	6.2104062	6.2104062
	7	5.0540773	5.0540811	5.0540793
	5	3.1830987	3.1831100	3.1830930
	3	1.3121201	1.3121552	1.3121045
	1	0.1557922	0.1558612	0.1557764
p_2	9	-.00791922	-.00791923	-.00791923
	7	-.02547627	-.02547632	-.02547632
	5	-.05000000	-.05000048	-.05000051
	3	-.09813053	-.09813691	-.09813740
	1	-.31568733	-.31619550	-.31623075
u	9	.15707962	.15707979	.15707979
	7	.47123889	.47123965	.47123968
	5	.78539813	.78540297	.78540324
	3	1.0995575	1.0995837	1.0995859
	1	1.4137165	1.4139649	1.4139820

$$u^{n+k} = (9/4)u^{n+k-1} - (3/4)u^{n+k-2} - (5/4)u^{n+k-3} + (3/4)u^{n+k-4} \quad (3.14)$$

The results are listed under method 2 in Table 3.2. There is no significant loss of accuracy.

Note that x^1, x^2 and x^3 can not be calculated from Equation (3.11) and must be found by some alternate procedure. This is usually done by employing some one-step integration procedure. For this example, these starting values were obtained by using a four-point Runge-Kutta integration routine.

This same problem occurs when the integration step size is varied. A one-step integration could also be used for this situation. However, more efficient procedures are available. For example, if the step size is to be doubled, the best method is to store the present data for x^{n+k}, $x^{n+k-1}, \ldots, x^n$ then compute the next k steps using the same step size. At that point the step size can be doubled and the new data points in the integration formulas would be the old x^{n+k}, $x^{n+k-2}, \ldots, x^{n-k}$. And for cutting the step size in half, interpolation formulas can be used to obtain the new data points which are intermediate between the old data points.

The best choice of an integration method for a large system of differential equations is not easily determined. For optimal control problems it appears that predictor-corrector methods offer significant advantages. There are pre-coded, standard predictor-corrector routines available at most computation centers which are entirely adequate for most systems. But for general optimal control problems, these routines must be modified to include the computation of the control variable u.

The development of a highly efficient integration routine for a particular optimal control problem requires considerable effort. But for many problems this effort is justified and in certain cases, necessary.

3.5 Integration of the Perturbation Equations

The deviation variables, δx, δp and δu satisfy the differential equations (2.31) and (2.32) and the Equation (2.33). If the determinant of the matrix

$$\left[\frac{\partial^2 H}{\partial u^2} \right]$$

is not zero, Equation (2.33) may be solved for δu

$$\delta u = - \left(\frac{\partial^2 H}{\partial u^2} \right)^{-1} \left[\left(\frac{\partial^2 H}{\partial u \partial x} \right) \delta x + \left(\frac{\partial^2 H}{\partial u \partial p} \right) \delta p \right] \quad (3.15)$$

Substituting this expression into (2.31) and (2.32) gives

$$\delta\dot{x} = \left[\left(\frac{\partial^2 H}{\partial p \partial x} \right) - \frac{\partial^2 H}{\partial p \partial u} \left(\frac{\partial^2 H}{\partial u^2} \right)^{-1} \frac{\partial^2 H}{\partial u \partial x} \right] \delta x - \left[\left(\frac{\partial^2 H}{\partial u^2} \right)^{-1} \frac{\partial^2 H}{\partial u \partial p} \right] \delta p \tag{3.16}$$

$$\delta\dot{p} = \left[\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial x \partial u} \left(\frac{\partial^2 H}{\partial u^2} \right)^{-1} \frac{\partial^2 H}{\partial u \partial x} \right] \delta x - \left[\frac{\partial^2 H}{\partial x \partial p} - \frac{\partial^2 H}{\partial x \partial u} \left(\frac{\partial^2 H}{\partial u^2} \right)^{-1} \frac{\partial^2 H}{\partial u \partial p} \right] \delta p \tag{3.17}$$

If we define the vector

$$\delta z^T = \{\delta x_1, \ \delta x_2, \dots, \delta x_n, \ \delta p_1, \dots, \delta p_n\} \tag{3.18}$$

then Equations (3.16) and (3.17) may be written as

$$\delta\dot{z} = g\Big(x(t), \ p(t), \ u(t) \Big) \ \delta z \tag{3.19}$$

Equation (3.19) is linear and is not explicitly a function of the deviation control variable δu. As indicated, g is a function of the state vector x, the auxiliary vector p and the control vector u, which are to be evaluated along the corresponding nominal trajectory.

Due to the fact that Equation (3.19) is linear and does not involve the deviation control variable, the best integration procedure for the perturbation differential equations will probably not be the same as for the system of state and auxiliary differential equations. But in the neighboring optimum method for solving the optimization problem the integration procedure for the perturbation differential equations is to some extent dependent on the procedure used to integrate the optimal trajectory differential equations. The reason for this is that at each data point used in the integration equations for the perturbation differential equations, the values of the variables x, p and u must be available in order to compute the function g of Equation (3.19).

In order to reduce data storage and transmission in the computer operations, it is desirable to integrate the trajectory equations and the perturbation equations simultaneously.

For these reasons and also to reduce the programming effort, it seems reasonable to use the same integration procedure for the perturbation equations as is used for the trajectory equations. However, it should be noted that it may be possible to gain increased computational efficiency by choosing some alternate integration procedure.

In order to simultaneously integrate the trajectory and perturbation equations, it is necessary to perform the computational operations in a particular order. For the predictor-corrector integration method the proper sequence is as follows: assume that x, p, u, and δz, and also their derivatives have been computed at points which correspond to the values $n\,\Delta t$, $(n-1)\,\Delta t, \ldots, (n-k)\Delta t$ of the independent variable; the first step is to compute x^{n+1}, p^{n+1} and u^{n+1} by the procedure discussed in Section 3.4, then the predicted values for the variables δz^{n+1} are computed using previously available data; next the value of $g(x^{n+1}, p^{n+1}, u^{n+1})$ is computed and used to find the corrected value of δz^{n+1}; then the value of $\left[d(\delta z^{n+1})/dt\right]$ must be computed using the corrected value of δz^{n+1} and $g(x^{n+1}, p^{n+1}, u^{n+1})$. It is important to note that, even though $\left[d(\delta z^{n+1})/dt\right]$ is only to be used to compute δz^{n+2}, it must be computed before x^{n+2}, p^{n+2} and u^{n+2} are computed.

The perturbation equation (3.19) must be solved a total of n times, corresponding to n sets of linearly independent initial values for δz. In order to avoid recomputing or storing the time varying coefficients (g of 3.19), all n linearly independent solutions should be integrated simultaneously.

3.6 Matrix Inversion

The inversion of square, non-singular matrices of fairly small size (e.g., a 5x5 matrix) is best done by a direct method. For computing machines the best direct methods appear to be those of the Gaussian elimination type. A particular computational process of this type[38] may be represented as the generation of a succession of matrix products:

$$\begin{aligned}[A,I] = I[A,I] &= V^{(o)}W^{(o)}\\ &= V^{(1)}W^{(1)}\\ &= V^{(2)}W^{(2)}\\ &\;\;\vdots\\ &= V^{(n)}W^{(n)}\end{aligned} \tag{3.20}$$

where A is an nxn matrix, I is the identity matrix and $[A,I]$ denotes that A and I adjoined are to be treated as a single matrix. $V^{(k)}$ is a sequence of square matrices, initially the identity matrix I, and at the final stage

$V^{(n)} = A$. The $W^{(k)}$ are rectangular matrices (nx2n) which satisfy the matrix product expression. Computationally only the matrix W is operated on, since the left side of (3.20) is always the same and the successive matrices $V^{(k)}$ merely require a bookkeeping process to keep track of which columns of A are in $V^{(k)}$. It can be shown[38] that the elements $w_{ij}^{(k)}$ of $W^{(k)}$ can be successively obtained by the recursion formulas:

$$c_i^{(k)} = w_{ik}^{(k-1)}/w_{kk}^{(k-1)} \quad ; \quad i = 1, \ldots, n;\ i \neq k;\ j=1, \ldots, 2n$$

$$w_{ij}^{(k)} = w_{ij}^{(k-1)} - w_{kj}^{(k-1)} c_i^{(k)} \quad ; \quad i = 1, \ldots, n;\ i \neq k;\ j=1, \ldots, 2n$$

$$w_{kj}^{(k)} = w_{kj}^{(k-1)}/w_{kk}^{(k-1)} \quad ; \quad j = 1, \ldots, 2n \qquad (3.21)$$

If A is singular, the procedure terminates at the stage k for which $V^{(k)}$ is non-singular.

3.7 Extrapolation by the Method of Least Squares

In the modified integration routine described in Section 3.4 and in the system simulation it is desired to estimate or "predict" the value of the control variable at the next step of the integration, based on values at previously computed points of the integration, as indicated by Equation (3.10). It is known that the control function is continuous, so over any sub-arc in which the control variable does not enter or leave the boundary of the control region U, the function u(t) may be approximated by a polynomial. Such a polynomial may be determined from the previous values, i.e., $u^{n+k-1}, \ldots, u^n$, and then used to estimate the value of u^{n+k}.

Because of the errors introduced in computing u^j it is clear that some method for "smoothing" the data is desirable to increase the accuracy of the predicted values. The most widely used approach in situations like this is to use "least-squares" approximation to do the smoothing.

Suppose that n values of u have been obtained at the times t^j,

$$u^j = u(t^j) \quad ; \quad j = 1, \ldots, n$$

and we wish to approximate u(t) by a polynomial of degree $m < n$,

$$\hat{u}(t) = a_o + a_1 t + a_2 t^2 + \ldots + a_m t^m$$

The least squares criterion determines the constants a_i in such a manner as to minimize

$$\sum_{j=1}^{n} \left[u^j - \hat{u}(t^j) \right]^2$$

It is easily shown (Reference 37, Section 17.5) that this leads to the following system of "normal equations" for determining the a_i:

$$\sum_{j=1}^{n} u^j \cdot [t^j]^k = \sum_{i=0}^{m} a_i \sum_{j=1}^{n} [t^j]^{k+j} \; ; \; k=0, 1, \ldots, m \qquad (3.22)$$

As an example of this procedure, assume that u(t) can be adequately approximated by a second order polynomial,

$$\hat{u}(t) = a_o + a_1 t + a_2 t^2$$

and that we wish to predict u^{n+1} based on the values u^n, u^{n-1}, u^{n-2} and u^{n-3}. Since these are equally spaced data points with respect to t, we may arbitrarily assume that

$$t^{n-3} = 0, \; t^{n-2} = 1, \; t^{n-1} = 2 \; \text{ and } \; t^n = 3$$

Then the solution of system (3.22) gives the relations

$$a_o = (1/20)u^n - (3/20)u^{n-1} + (3/20)u^{n-2} + (19/20)u^{n-3}$$

$$a_1 = -(9/20)u^n + (17/20)u^{n-1} + (13/20)u^{n-2} - (21/20)u^{n-3}$$

$$a_2 = (1/4)u^n - (1/4)u^{n-1} - (1/4)u^{n-2} + (1/4)u^{n-3}$$

And the best least squares estimate of u^{n+1} is

$$u^{n+1} = \hat{u}(t^{n+1}) = \hat{u}(4) = a_o + 4a_1 + 16a_2$$
$$= (9/4)u^n - (3/4)u^{n-1} - (5/4)u^{n-2} + (3/4)u^{n-3}$$

There are many other approaches to the problem of approximating a continuous function (see, for example, Reference 39), and in some problems these may be more effective. However, this usually can only be determined by experience. The least squares method is adequate in most cases and is easy to apply, so it is a good choice for the first attempt to solve these problems.

CHAPTER 4

A REENTRY PROBLEM

4.1 Reentry Trajectories

One of the primary goals of present day engineering is the development of the means for manned interplanetary travel. Of all the many difficult problems associated with this task, perhaps the most critical is that of returning the vehicle from outer space to the earth's surface without exceeding the various structural tolerances of the spacecraft or its human passengers. The terminal portion of this return flight is usually referred to as the reentry trajectory of the vehicle, since it involves reentering the earth's atmosphere.

Reentry trajectories are commonly divided into three operational regions or phases.[40] The initial phase occurs at extremely high altitudes and is known as the Approach Phase. In this phase, the primary effects which influence the path and motion of the vehicle are due to gravitational and centrifugal forces. In this region, the gas-dynamic effects are considered to be negligibly small.

The second phase is referred to as the Atmospheric Reentry Phase. It is in this region that the spacecraft is first affected by the atmosphere. As the vehicle altitude decreases, the gas-dynamic effects become of comparable magnitude with the gravitational and centrifugal forces in affecting the shape of the trajectory, and eventually they become dominant. During this phase it is necessary to reduce the kinetic energy of the vehicle by a large amount. This causes the vehicle to experience large heating and acceleration forces. These are the most critical problems for manned reentry vehicles.

The third phase is designated the Final Descent Phase. The major problems in this region are aerodynamic stability and control of the vehicle for landing on the earth's surface. Heating and deceleration are not considered to be serious problems in this stage.

For any particular reentry mission these three phases are interdependent. For example, the method used in the Approach Phase determines the initial conditions for the Atmospheric Reentry Phase, and the method used for the Final Descent Phase affects the terminal conditions for the Atmospheric Reentry Phase. Therefore a truly optimal reentry system must consider the complete reentry trajectory. However, since the objectives of the various phases are different it is reasonable to to choose representative initial and terminal conditions for the Atmospheric Reentry Phase, and consider optimal procedures for this phase independently. This is the approach which we shall follow here.

The Atmospheric Reentry Phase is considered to begin at approximately 400,000 feet, which is the point at which atmospheric effects begin to be significant. The velocity of spacecraft returning from lunar missions is expected to be 36,000 feet per second. The kinetic energy of such a vehicle is easily calculated to be 25,000 BTU per pound. Since 10,000 BTU per pound is more than enough energy to completely vaporize any known practical material,[41] the central problem of the Atmospheric Reentry Phase is clear. It is to convert this energy into some other form in a manner that will not endanger the structural integrity of the vehicle or the safety of its passengers.

There are presently two basic techniques available for energy removal during reentry. One is through the use of retrorocket braking. The other is to use the properties of the atmosphere to achieve aerodynamic braking.

A major disadvantage of retrorocket braking is that the rocket and fuel for carrying out this operation must be carried by the vehicle throughout its entire mission. Because of the critical weight limitations on present-day spacecraft, this method seems to be unacceptable for earth reentry, although in some cases it may be necessary, as, for example, when landing on the moon, which has no atmosphere.

The chief problems with aerodynamic braking are associated with the severe heating and deceleration effects experienced by the vehicle. This heating is the result of the vehicle moving through the atmosphere and being slowed down by aerodynamic drag which causes the air near the vehicle to be heated by compression and viscous effects. Part of this energy transferred from the vehicle to the surrounding air is converted to heat and radiated back to the spacecraft.

If the vehicle is allowed to make an uncontrolled steep dive into the atmosphere, the deceleration load on a reentry vehicle could be as high as several thousand times the force of natural gravity (g).[42] Since an acceptable tolerance for a manned vehicle is around 10 g's, it is evident that the reentry flight path must be carefully controlled.

4.2 Mathematical Model for Reentry of Variable Lift Vehicle

During the Atmospheric Reentry Phase, the primary forces acting on a spacecraft are those due to gravity and aerodynamic effects. By assuming a non-rotating planar inertial coordinate system, these forces may be represented as in Figure 4.1. The force due to the earth's gravity (denoted by G in Figure 4.1) is directed toward the center of the earth. The aerodynamic force which represents the resistance of the atmosphere is referred to as "drag" (D). This force acts opposite to the instantaneous direction of motion of the vehicle. The aerodynamic forces which tend to deflect the vehicle from its direction of velocity are called "lift" (L).

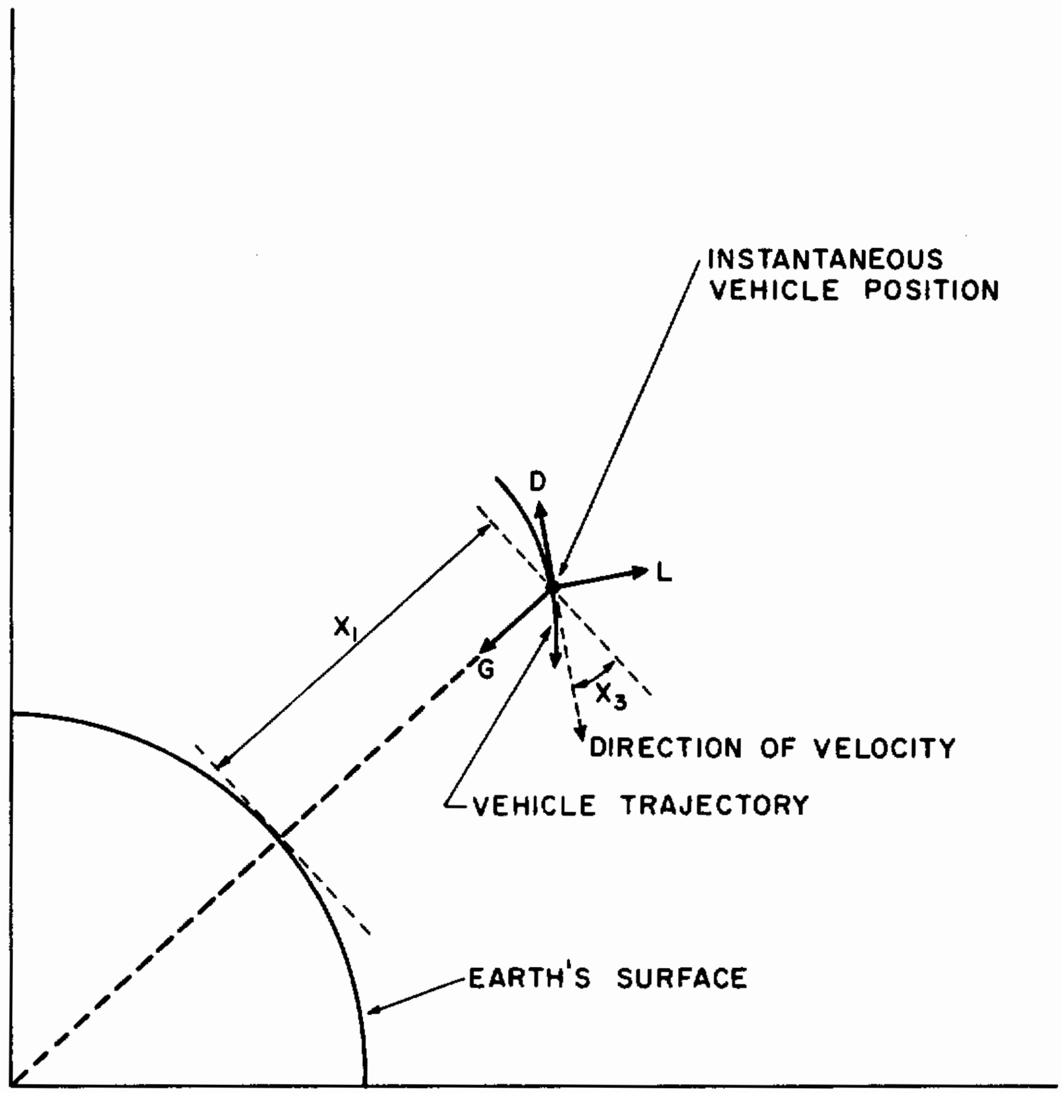

Figure 4.1. Reentry Coordinate System

This force acts in a direction perpendicular to the direction of motion of the vehicle.

For notational convenience we shall denote the altitude of the vehicle, measured from the earth's surface by x_1, the velocity of the vehicle by x_2, the angle between the velocity direction and the local horizontal by x_3, the radius of the earth by K_1, the gravity constant by K_2, and the mass of the vehicle by K_3.

Then, by summing the forces on the vehicle, the following equations are obtained:

$$\dot{x}_1 = -x_2 \sin x_3 \tag{4.1}$$

$$\dot{x}_2 = K_2 \sin x_3 - (D/K_3) \tag{4.2}$$

$$\dot{x}_3 = (K_2 \cos x_3/x_2) - \Big(x_2 \cos x_3/(K_1 + x_1)\Big) - (L/K_3 x_2) \tag{4.3}$$

The aerodynamic forces are dependent on the atmospheric density, the velocity of the vehicle relative to the air, and the physical characteristics of the vehicle. This dependency can be expressed as

$$L = (1/2)\, \rho\, x_2^2\, C_L\, K_{10} \tag{4.4}$$

$$D = (1/2)\, \rho\, x_2^2\, C_D\, K_{10} \tag{4.5}$$

where ρ is the density of the air, K_{10} is the wing plan-form area, and C_L and C_D are the lift and drag coefficients.

In the Atmospheric Reentry Phase, the density of the air (ρ) is primarily a function of the altitude (x_1). However, it is also a function of the season, time of day, latitude, longitude, magnetic storms and other atmospheric effects. Often these effects are significant, but if the dynamic effects are neglected and only a static density model is used, the exponential approximation

$$\rho = K_5^2 \exp\,[K_6 x_2] \tag{4.6}$$

is often sufficiently accurate. K_5^2 is the air density at sea level.

The lift and drag coefficients are functions of such factors as vehicle shape, the Mach number of the velocity and the angle of attack (u). The angle of attack is the angle between the direction of velocity and the direction of the zero lift axis of the vehicle. For a variable lift vehicle the angle of attack is considered to be the control variable. For an approximate analysis these coefficients may be assumed to be functions only of the angle of attack. The particular lift-drag polar which shall be used in the problem considered in this thesis is represented by*

$$C_L = K_{11} \sin u \cos u \tag{4.7}$$

$$C_D = K_{12} + K_{13} \sin^2 u \tag{4.8}$$

As discussed in Section 4.1, the primary problems during the Atmospheric Reentry Phase are the heating and deceleration effects. The rate of heating, due to aerodynamic friction, may be expressed approximately by

$$K_4(\rho)^{\frac{1}{2}} x_2^3 \tag{4.9}$$

where K_4 is a heating constant. The expression (4.9) represents the heating rate per unit surface area, so for a particular vehicle, K_4 is a function of the surface area of the vehicle nose region. In addition to the convective heating, given by (4.9), which is absorbed by the vehicle, there will also be a radiative heating component representing heat lost by the vehicle. But above 250,000 feet altitude this radiative component is negligible, and will not be included in the problem formulation.

In a reentry vehicle the acceleration which the crew senses is due only to the aerodynamic forces. This acceleration force a is given by

$$a = (1/K_3)\,(L^2 + D^2)^{\frac{1}{2}} \tag{4.10}$$

The limit of human endurance to acceleration is a function of the acceleration magnitude and the length of time it is applied. This limit is also a function of the direction of acceleration forces, but it may be assumed that the crew can be positioned to the direction of greatest endurance. The endurance limits as given by Bryson[25] are plotted in Figure 4.2. Within the range of 5 to 10 g's the endurance time limit is roughly a linear function of the acceleration squared. This suggests that a reasonable measure of the crew comfort during a trajectory over the time interval t^o to t' may be given by the integral.

$$\int_{t^o}^{t'} a^2 dt \tag{4.11}$$

*These expressions for C_L and C_D were chosen in order to have approximate agreement to the results given by Breakwell, Speyer and Bryson.[16]

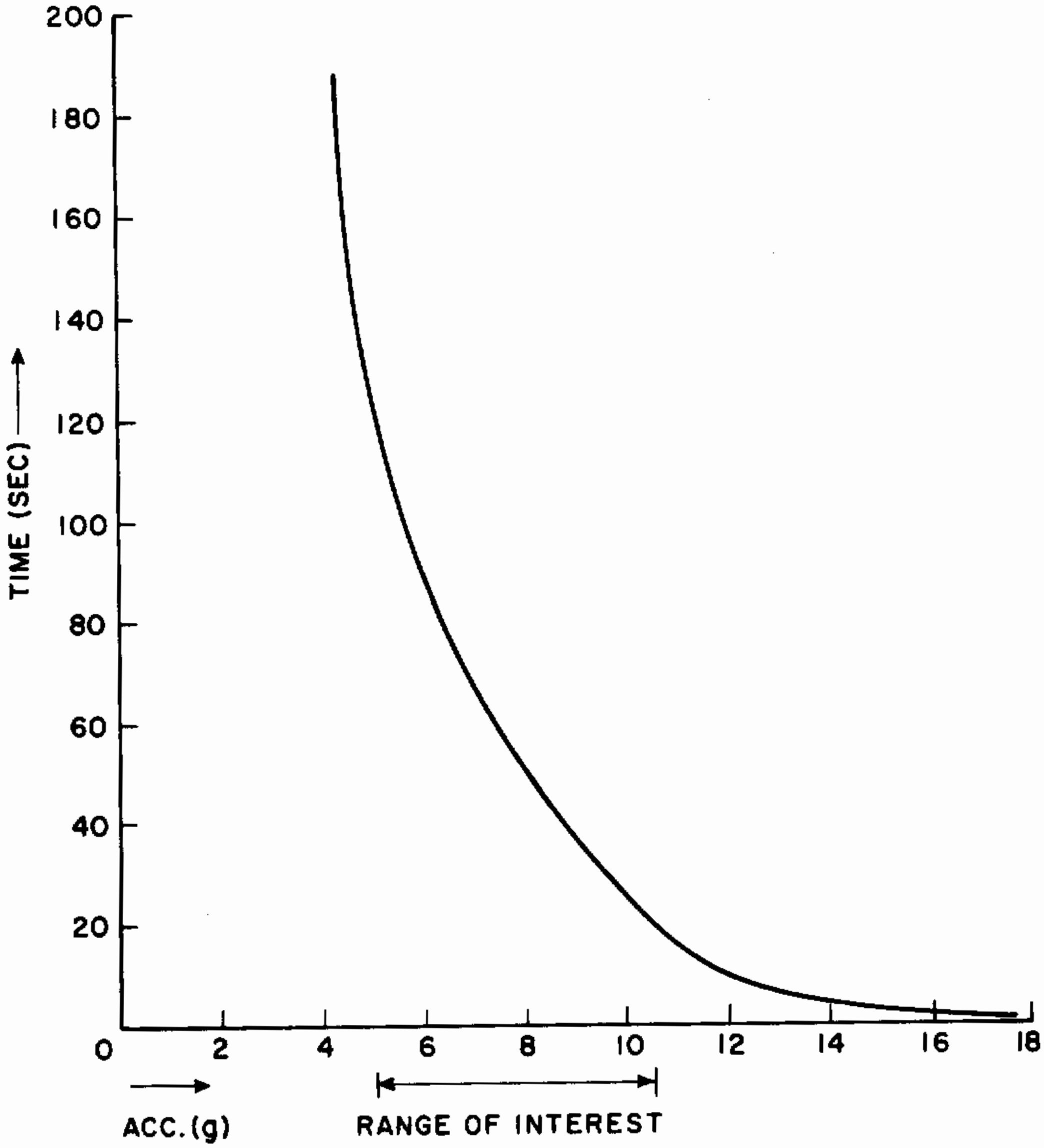

Figure 4.2. Endurance Limit for Useful Consciousness

Making use of all the above relations, we may write the following system of state equation

$$
\begin{aligned}
\dot{x}_1 &= f_1 = -x_2 \sin x_3 \\
\dot{x}_2 &= f_2 = K_2 \sin x_3 - (K_5^2\, K_{10}/K_3) \exp\,[K_6 x_1] x_2^2\, (K_{12} + K_{13} \sin^2 u) \\
\dot{x}_3 &= f_3 = K_2(\cos x_3/x_2) - x_2 \left(\cos x_3/(K_1 + x_1)\right) \\
&\quad - (K_5^2\, K_{10} K_{11}/K_3) \exp\,[K_6 x_1]\, x_2 \sin u \cos u \\
\dot{x}_4 &= f_4 = K_4\, K_5 \exp\,[K_6 x_1/2]\, x_2^3 \qquad (4.12) \\
\dot{x}_5 &= f_5 = (K_7 K_5^2 K_{10}^2/K_3^2) \exp\,[2K_6 x_1]\, x_2^4 \{K_{11}^2 \sin^2 u \cos^2 u \\
&\quad + K_{12}^2 + 2K_{12} K_{13} \sin^2 u + K_{13}^2 \sin^4 u\}
\end{aligned}
$$

The new state variables, x_4 and x_5, represent, respectively, the total heat absorbed by the vehicle and a measure of crew comfort.

In deriving the state equations (4.12) several approximations were made, and this will necessarily limit the value of any corresponding solutions. But these approximations are assumed to be sufficient to describe the primary characteristics of reentry dynamics. The alternative to making these approximations would be to vastly complicate the state equations and this would quite probably lead to excessive computational requirements.

It appears that in optimization problems the best plan is to start with a simple mathematical model and obtain the corresponding solutions first. These solutions would then serve as a starting point for solutions to the successively more complicated (and realistic) mathematical models. This approach would also help to show which simplifying approximations are acceptable in the final model.

Successful engineering analysis often depends upon making the proper simplifying approximations. This is particularly true in optimization problems.

4.3 An Optimization Problem

Since heating and acceleration effects are the most critical problems in the atmospheric phase of reentry, it is natural to seek trajectories which minimize these quantities. This problem has received wide attention.[43, 44] Early attempts to apply the methods of the Calculus of Variation to problems of this type were not successful.[43] This was due to difficulties involved in the computational stage. Because of this the alternate approach, referred to as the "gradient" technique, was developed. Using this method, Bryson, Denham, and their associates solved some reentry problems which minimized the heating effects.[25] The difficulty with these optimum heating trajectories is that acceleration forces exceed acceptable tolerances. The method used to overcome this objection was to impose certain constraints which limited the acceleration effects. However, these constraints led to control function[30] which would be very hard to achieve in a physical system. Later, Speyer[7] solved a reentry problem that approximated the minimum heat case, using Calculus of Variation methods. The computational method Speyer used was the neighboring optimum approach described in Section 3.4. A similar problem was considered by Scharmack,[8] who also used the Calculus of Variation methods.

The essential feature of a minimum heating reentry trajectory is shown in Figure 4.3. The initial maneuver is to execute a fairly steep dive into the atmosphere, represented by the period $t^o \leqq t \leqq t^p$ in Figure 4.3. During this maneuver the vehicle velocity is increased. Then a gradual climb to the specified terminal height (at t') is made. During this period $t^p \leqq t \leqq t'$ the vehicle velocity is reduced to its specified terminal value. The largest acceleration forces are experienced during the initial dive and pull up (at time t^p).

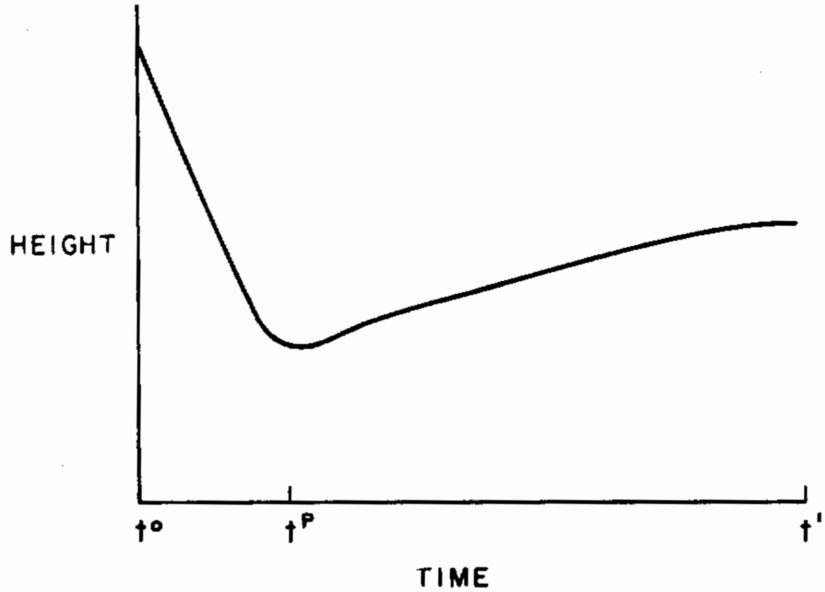

Figure 4.3. Typical Reentry Height Profile

Intuitively it seems that by considering an optimization criterion of the form

$$J = x_4(t^f) + K_7\, x_5(t^f) \tag{4.13}$$

where K_7 is a relative weighting constant between the heating and acceleration effects, a trajectory similar to that of Figure 4.3, but with a less steep dive, could be obtained. And by varying the constant K_7, a family of such trajectories could be generated.

Such a family of trajectories would specify the amount of heat which a space vehicle must absorb in order to meet a given acceleration tolerance. This would be useful information for the preliminary design of reentry vehicles.

Considering the state equations (4.12) and the criterion function (4.13) as an optimal control problem, we may apply the Maximum Principle to determine the conditions for an optimal trajectory. This leads to the following system of differential equations for the auxiliary variables:

$$
\begin{aligned}
\dot{p}_1 = {} & p_2 \left\{ (K_5^2 K_6 K_{10}/K_3) \exp[K_6 x_1]\, x_2^2 \,(K_{12} + K_{13} \sin^2 u) \right\} \\
& - p_3 \left\{ \left(x_2 \cos x_3/(K_1 + x_1)^2\right) - (K_5^2 K_6 K_{10} K_{11}/K_3) \right. \\
& \qquad \left. \exp[K_6 x_1]\, x_2 \sin u \cos u \right\} \\
& - p_4 \left\{ (K_4 K_5 K_6/2) \exp[K_6 x_1/2]\, x_2^3 \right\} \\
& - p_5 \left\{ (2K_5^4 K_6 K_7 K_{10}^2/K_3) \exp[2 K_6 x_1]\, x_2^4 \right. \\
& \qquad \left. (K_{11}^2 \sin^2 u \cos^2 u + K_{12}^2 + 2K_{12} K_{13} \sin^2 u + K_{13}^2 \sin^4 u) \right\} \\
\dot{p}_2 = {} & p_1 \{\sin x_3\} + p_2 \left\{ (2K_5^2 K_{10}/K_3) \exp[K_6 x_1]\, x_2 (K_{12} + K_{13} \sin^2 u) \right\} \\
& + p_3 \left\{ (K_2 \cos x_3/x_2^2) + \left(\cos x_3/(K_1 + x_1)\right) \right. \\
& \qquad \left. + (K_5^2 K_{10} K_{11}/K_3) \exp[K_6 x_1] \sin u \cos u \right\} \\
& - p_4 \left\{ (3 K_4 K_5) \exp[K_6 x_1/2]\, x_2^2 \right\} \\
& - p_5 \left\{ (4 K_5^4 K_7 K_{10}^2/K_3^2) \exp[2 K_6 x_1]\, x_2^3 \right. \\
& \qquad \left. (K_{11}^2 \sin^2 u \cos^2 u + K_{12}^2 + 2 K_{12} K_{13} \sin^2 u + K_{13}^2 \sin^4 u) \right\} \\
\dot{p}_3 = {} & p_1 \{x_2 \cos x_3\} - p_2 \{K_2 \cos x_3\} \\
& + p_3 \left\{ (K_2 \sin x_3/x_2) - \left(x_2 \sin x_3/(K_1 + x_1)\right) \right\} \\
\dot{p}_4 = {} & 0 \\
\dot{p}_5 = {} & 0
\end{aligned}
\tag{4.14}
$$

In the discussion of Section 2.3, the criterion function was considered to be x_o, whereas in this section the criterion function is a linear combination of x_4 and x_5, as given by (4.13). Similar to the result in Section 2.3 that $\dot{p}_o = 0$, we have here that p_4 and p_5 are constants. And, as in Equation (2.13), we may arbitrarily choose the constants to be

$$\begin{aligned} p_4 &= -1 \\ p_5 &= -1 \end{aligned} \tag{4.15}$$

By the Maximum Principle we have that the optimal control condition is

$$u(t) = \text{argument}\left\{\max_{u \epsilon U} H\Big(x(t),\ p(t),\ u\Big)\right\} \tag{4.16}$$

where

$$H = \sum_{i=1}^{5} p_i f_i \tag{4.17}$$

When the optimum control u is in the interior of U, condition (4.16) may be replaced by the equation

$$\frac{\partial H(x, p, u)}{\partial u} = 0 \tag{4.18}$$

By considering Equations (4.12) it is clear that Equation (4.18) can not be explicitly solved for the optimal u. Hence this is an example of a situation in which the computational difficulty discussed in Section 3.2 is encountered.

To completely specify an optimal trajectory, it remains to determine the boundary conditions. In the formulation of the optimal control problem discussed in Chapter 2, it was assumed that all the state variables were specified at the initial point. (This is not a necessary condition, but is convenient for the class of problems considered here.) Thus it is required to specify the initial conditions.

$$x_i(t^o) = x_i^o \quad , \qquad i = 1, 2, 3 \tag{4.19}$$

The initial conditions on x_4 and x_5 are zero, and the final conditions of p_4 and p_5 are given by (4.15). There are three additional final conditions required. There may be specified final values on the state variables,

$$x_i(t') = x_i' \quad , \qquad i = 1, 2, 3 \tag{4.20}$$

Or, alternatively, specified final values on the auxiliary variables, as determined by the transversality conditions. For example, if any x_j is unspecified, then the terminal boundary condition is

$$p_j(t') = 0 \qquad (4.21)$$

Thus we have a system of ten differential equations, (4.12) and (4.13), with eleven variables, $(x_1, x_2, \ldots, x_5, p_1, \ldots, p_5, u)$, one optimal control condition (4.16), and a total of ten boundary conditions. Five of these boundary conditions are given at t^o and the other five are given at t'. This completely determines the optimal trajectory.

In order to obtain this optimal trajectory we must generate a solution to the system of differential equations which satisfies the optimal control condition and the boundary conditions. We shall consider the solution of this two-point boundary value problem, with an appropriate set of numerical values for the constants, in Chapter 5. The solution will be obtained by the neighboring optimum method, as described in Section 2.4.

4.4 A Control Problem

After an optimal trajectory has been determined, the optimal control law is available as $u(t)$. The simplest method of instrumenting this control law would be to program the control surfaces such that the vehicle angle of attack, during the reentry interval ($t^o \leqq t \leqq t'$), is identical with the function $u(t)$. This approach is referred to as open loop control.

It is well known that open loop control is inadequate for reentry vehicles. The reason for this is that disturbances and deviations from the optimal trajectory would cause an intolerable amount of error in the terminal position of the vehicle. There are several factors which would introduce terminal error with open loop control, but the most critical factor, in the sense that it will introduce the greatest expected error, is initial position deviations, that is, the difference in the position (in state space) of the actual vehicle and that assumed for the optimal trajectories.

In order to correct for these initial errors, a servo control system, using measurement of the state variables as feedback, could be employed. This we shall call path-following control. A properly designed path-following control system could compensate for initial state error and random disturbances along the vehicle trajectory such that the terminal conditions would be satisfied. The block diagram of this type of control system is shown in Figure 4.4, where K is a vector of scalars which multiply the errors in the vehicle state (δx) to produce a control correction (δu) that tends to cancel the error δx. Figure 4.5 shows two possible

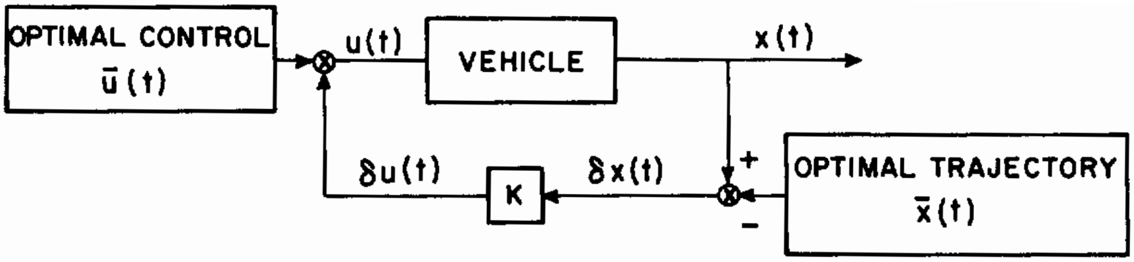

Figure 4.4. Block Diagram of Path-following Control System

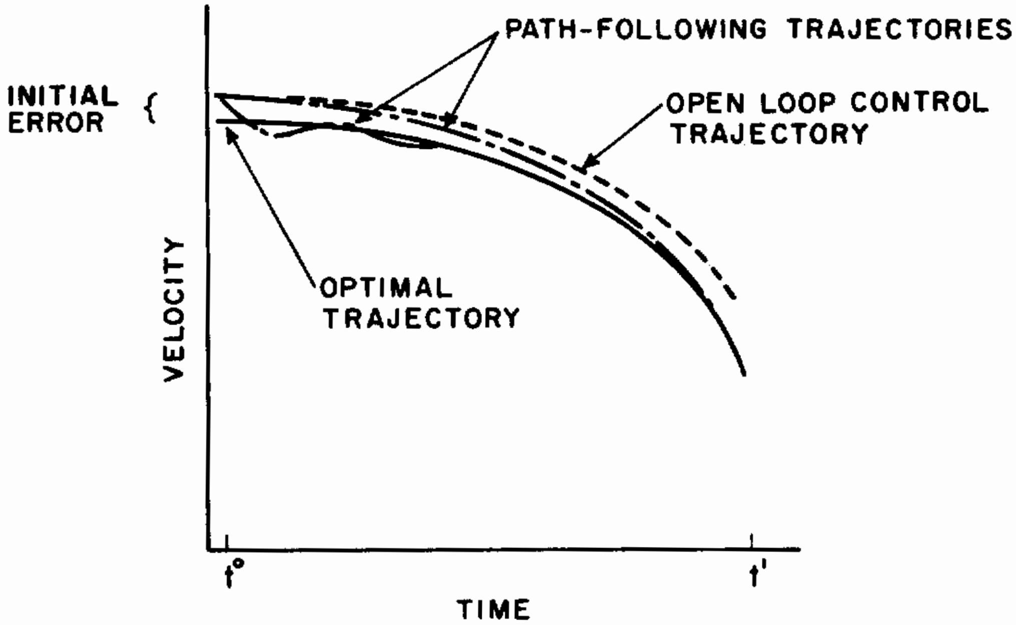

Figure 4.5. Possible Velocity Profiles for Path-following Control Schemes

types of path-following trajectories corresponding to an initial error in velocity. A possible open loop control trajectory is also shown. The difference between the path-following trajectories is due to different values of K.

Even though a path-following control system could satisfy the terminal conditions, it will necessarily introduce an increase in the optimal value of the criterion function. To illustrate this, Figure 4.6 shows two possible optimal paths in the velocity-time plane. Note that the two paths do not converge toward each other during the entire trajectory. In this case, a path-following trajectory from $(V^o + \delta V)$ to V' would not be the optimal trajectory.

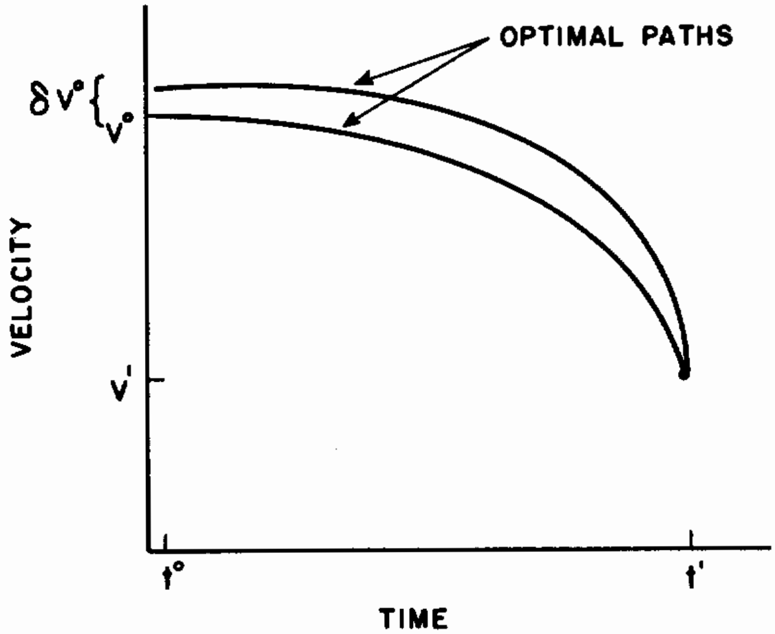

Figure 4.6. Diverging Neighboring Optimal Paths

Of course, the best control system would be one which measures the present vehicle position and instantaneously solves the complete optimization problem to provide the instantaneous optimal control. Obviously, the optimization problem cannot be solved instantaneously and, in fact, the computing time and equipment required to solve the optimization problem makes this, the truly optimal control scheme, impractical.

For these reasons, it is desired to find a control system which will approximate the optimal control system and yet can be instrumented in a reasonably simple manner. One method for such a control system has been developed by Breakwell, Speyer and Bryson[16] and by Kelley.[15] In order to understand the concept of this method it should be noted that the action of a control system in correcting for initial errors is identical with the action of the neighboring optimum method in converging to the correct boundary conditions. For example, with reference to Figure 4.6, assume that the boundary conditions for the optimal trajectory are $V^o + \delta V$ and V', and that the optimization program has generated a trajectory from V^o to V'. Then the correction which the optimization program computes for use in the next iteration will be the change necessary to give the optimal trajectory from $V^o + \delta V$ to V' (provided δV is sufficiently small). So one method of optimal control is to compute this correction and use it to solve for the optimal value of K(t) in the relation

$$\delta u(t) = K(t)\ \delta x(t) \tag{4.22}$$

This function, K(t), is then used in a control system such as is represented in Figure 4.4. It has been shown[16] that this is the optimal linear control system, i.e., the best linear approximation to the optimal control system. The essential difference between this control system and the path-following system is that the feedback gains K(t) are based on a family of optimum trajectories rather than on a single optimum trajectory.

The derivation of this optimal control system and its application to the reentry problem will be considered in Chapter 6.

CHAPTER 5

OPTIMAL REENTRY TRAJECTORIES

5.1 The Optimization Method

In this chapter we consider solutions of the optimization problem posed in Section 4.3. These solutions are obtained by the neighboring optimum method which was described in Section 2.4.

The particular optimal trajectories are determined by the choice of numerical values for the boundary conditions and constants of the state equations (4.12). The choice of these values is discussed in Section 5.2.

The gradient used in the Newton-Raphson procedure

$$\frac{\partial x(\gamma, t')}{\partial \gamma(t^o)}$$

is obtained by solving the perturbation differential equations (2.31) and (2.32). For the reentry problem of Section 4.3, these equations become

$$\delta \dot{x}_i = \sum_{j=1}^{3} \left(\frac{\partial f_i}{\partial x_j} \right) \delta x_j + \left(\frac{\partial f_i}{\partial u} \right) \delta u \tag{5.1}$$

$$\delta \dot{p}_i = -\sum_{j=1}^{3} \left[\sum_{\ell=1}^{5} \left(\frac{\partial^2 f_\ell}{\partial x_j \partial x_i} \right) p_\ell \, \delta x_j + \left(\frac{\partial f_i}{\partial x_j} \right) \delta p_j \right] - \sum_{\ell=1}^{5} \left(\frac{\partial^2 f_\ell}{\partial u \partial x_i} \right) p_\ell \, \delta u \tag{5.2}$$

where the functions f_i are defined by Equation (4.12). The detailed expressions for the partial derivatives of f are given in Appendix A.

The control variable deviation δu satisfies the equation

$$\sum_{j=1}^{3} \left[- \sum_{\ell=1}^{5} \left(\frac{\partial^2 f_\ell}{\partial u \partial x_j} \right) p_\ell \, \delta x_j + \left(\frac{\partial f_j}{\partial u} \right) \delta p_j \right] \left[+ \sum_{j=1}^{5} \left(\frac{\partial^2 f_j}{\partial u^2} \right) p_j \quad \delta u = 0 \right] \tag{5.3}$$

If u is not on the boundary of the admissible control set U, or if u is on the boundary, then

$$\delta u = 0 \tag{5.4}$$

The deviation variables must also satisfy the perturbed boundary conditions (4.19) and (4.20), i.e.,

$$\left[f_j \, \delta t + \delta x_j \right]_{t=t^o} = d\, x_j^o = 0 \quad , \qquad j = 1, 2, 3 \tag{5.5}$$

$$\left[f_j \, \delta t + \delta x_j \right]_{t=t'} = d\, x_j' = 0 \quad , \qquad j = 1, 2, 3 \tag{5.6}$$

Also they must satisfy the perturbed transversality condition,

$$\left[\sum_{j=1}^{3} \left[f_j \, \delta p_j - \sum_{\ell=1}^{5} \left(\frac{\partial f_\ell}{\partial x_j} \right) p_\ell \, \delta x_j \right] \right]_{t=t'} = 0 \tag{5.7}$$

Thus we have a total of six differential equations, (5.1) and (5.2), and one equation (5.3) involving eight deviation variables: $[\delta x_1, \delta x_2, \delta x_3, \delta p_1, \delta p_2, \delta p_3, \delta u, \delta t]$. This is a system of linear differential equations with time varying coefficients which are functions of the nominal trajectory. There are four boundary conditions given at both t^o and t', by (5.7) and (5.5) or (5.6). Hence there are three independent solutions of the system of equations which satisfy the boundary conditions at either end point.

Due to condition (5.7) the three auxiliary variable deviations, δp_1, δp_2, and δp_3, are not independent. So it is necessary to use nonzero initial conditions for the state variables deviations, δx_1, δx_2 and δx_3, by considering changes in the time interval δt. The particular components which are chosen as the independent deviation variables are

$$\delta \gamma = \left[\delta p_1, \ \delta t, \ \delta p_3 \right]^T$$

And the corresponding undetermined initial (or final) conditions are $\gamma = [p_1, \ t' - t^o, \ p_3]^T$.

For the reentry problem considered here, the steps of the optimization procedure are as follows:

1. Specify an estimate of the initial values of p_1 and p_3 and of the time interval t' (t^o may be considered to be zero). The value of p_2 is chosen such that the transversality condition $H = 0$ is satisfied.

2. Using the estimated values of p and t' (i.e., γ) and the specified values of x, the system of differential equations is integrated over the time interval. This solution is referred to as the nominal trajectory. The optimal control condition (4.16) is satisfied at all points along the trajectory, by the method discussed in Section 3.2.

3. Three independent solutions of the deviation variables differential equation are obtained by using three sets of linearly independent initial conditions such as

$$\delta\gamma_1 = [1, 0, 0]^T$$
$$\delta\gamma_2 = [0, 1, 0]^T$$
$$\delta\gamma_3 = [0, 0, 1]^T$$

The other initial conditions for the deviation variables are found by solving the boundary condition equations (5.7) and (5.5) (or 5.6). The integrations of the perturbation equations and state equations are performed simultaneously as discussed in Section 3.5.

4. The estimated initial conditions $\gamma^{(n)}$ are corrected by Newton-Raphson procedure,

$$\gamma^{(n+1)} = \gamma^{(n)} - \left[\frac{\partial x}{\partial \gamma} \bigg|_{t=t'} \right]^{-1} [x(t') - x']$$

where $\gamma^{(n+1)}$ represents the new estimate, $x(t') - x'$ represents the error between the terminal values of the nominal trajectory and the specified terminal values, and

$$\frac{\partial x}{\partial \gamma} = \begin{bmatrix} \frac{\partial x_1}{\partial \gamma_1} & \frac{\partial x_2}{\partial \gamma_1} & \frac{\partial x_3}{\partial \gamma_1} \\ \frac{\partial x_1}{\partial \gamma_2} & \frac{\partial x_2}{\partial \gamma_2} & \frac{\partial x_3}{\partial \gamma_2} \\ \frac{\partial x_1}{\partial \gamma_3} & \frac{\partial x_2}{\partial \gamma_3} & \frac{\partial x_3}{\partial \gamma_3} \end{bmatrix}$$

where $\partial x_i / \partial \gamma_j$ represents the final value of δx_i in the solution of the perturbation differential equations corresponding to the initial conditions $\delta \gamma_j$.

5. Steps 2, 3 and 4 are repeated until the terminal values of nominal trajectory are sufficiently close to the specified terminal values.

This optimization procedure can also be run in the opposite direction, that is, integrating from t' to t^o and converging to the specified initial conditions. In practice, the integration is usually run backward in time using given final conditions. The reason for this is discussed in Section 5.4.

5.2 Specification of the Numerical Parameters

Before numerical solutions can be obtained it is necessary to assign values to the constants in the system equations and to the boundary conditions.

It is assumed that the reentry is into the earth's atmosphere, so the following constants are known:

The earth's radius,

$$K_1 = 2.09 \times 10^7 \quad \text{ft.}$$

The gravity constant,

$$K_2 = 32. \quad \text{ft./sec.}^2$$

The square root of air density at sea level,

$$K_5 = .052 \quad (\text{lb.})^{\frac{1}{2}}\text{-sec./ft.}^2$$

The air density gradient,

$$K_6 = -4.26 \times 10^{-5} \quad (\text{ft.})^{-1}$$

The parameters of the vehicle model are chosen to be approximately the same as were used in Reference 16. They are:

The mass of the vehicle,

$$K_3 = 250. \quad \text{lb.-sec.}^2/\text{ft.}$$

The heating rate factor,

$$K_4 = 1.0 \times 10^{-4} \quad (\text{lb.})^{\frac{1}{2}}\text{-sec.}$$

The wing plan form area

$$K_{10} = 66.5 \quad \text{ft.}^2$$

And the lift-drag polar parameters are:

$K_{11} = 1.2$

$K_{12} = 0.274$

$K_{13} = 1.8$

The initial conditions are assumed to be

$x_1(t^o)$	$= 400,000$	ft.
$x_2(t^o)$	$= 36,000$	ft./sec.
$x_3(t^o)$	$= 8.09$	degrees

The value for $x_1(t^o)$ is that height at which atmospheric effects become significant. The initial reentry velocity $x_2(t^o)$ is approximately the value which a vehicle returning from a lunar mission would have.[40] The initial reentry angle must necessarily lie within the range of 6 to 10 degrees in order to avoid either skip-out or an excessively steep dive into the atmosphere. An arbitrary intermediate value was selected for $x_3(t^o)$.

The final conditions are chosen as:

$x_1(t')$	$= 250,000$	ft.
$x_2(t')$	$= 27,000$	ft./sec.
$x_3(t')$	$= 0$	degrees

These values would specify a satellite orbit except for the fact that $x_2(t')$ is slightly greater than satellite velocity. For lower values of $x_2(t')$ the shape of the optimal trajectory is not changed significantly but the length of the trajectory is greatly increased. For this reason, the higher value of $x_2(t')$ is used.

5.3 A Family of Optimal Trajectories

In this section, solutions of the optimum control problem are presented. That is, the optimal trajectories for the previously specified system are given. The solutions listed represent a family of optimum trajectories with the criterion function weighting factor K_7 as the parameter.

Tables 5.1 through 5.6 give the values for the system variables along six optimal trajectories. The numerical values were printed out at five-second intervals and at the terminal points. The column headings correspond to the system variables as follows

TABLE 5.1

OPTIMAL TRAJECTORY FOR $K_7 = 0$

HEIGHT FT	VELOCITY FT/SEC	FL ANG DEG	CONTROL DEG	HEAT RT	ACCEL G	TIME SEC	AUXILIARY VARIABLES			HEAT	TACC
400000.	36000.	8.09	34.28	484.	0.001	0.	0.9086E 03	0.4037E 05	-0.5644E 10	-0.	-0.
374962.	36022.	7.87	34.41	826.	0.003	5.	0.9047E 03	0.3905E 05	-0.5486E 10	32.	0.
350621.	36043.	7.64	34.42	1390.	0.010	10.	0.8974E 03	0.3778E 05	-0.5328E 10	86.	0.
326988.	36061.	7.40	34.44	2303.	0.026	15.	0.8858E 03	0.3658E 05	-0.5172E 10	177.	0.
304075.	36076.	7.17	34.46	3757.	0.070	20.	0.8707E 03	0.3544E 05	-0.5018E 10	326.	0.
281912.	36080.	6.92	34.46	6025.	0.180	25.	0.8584E 03	0.3440E 05	-0.4866E 10	566.	0.
260564.	36059.	6.65	34.43	9478.	0.447	30.	0.8690E 03	0.3350E 05	-0.4715E 10	948.	1.
240177.	35981.	6.31	34.32	14537.	1.058	35.	0.9548E 03	0.3280E 05	-0.4557E 10	1541.	3.
221073.	35780.	5.85	34.09	21472.	2.345	40.	0.1228E 04	0.3241E 05	-0.4369E 10	2433.	17.
203910.	35337.	5.14	33.56	29815.	4.680	45.	0.1871E 04	0.3249E 05	-0.4105E 10	3712.	78.
189819.	34517.	4.00	32.45	37512.	7.877	50.	0.3028E 04	0.3323E 05	-0.3690E 10	5405.	278.
180239.	33304.	2.37	30.19	41324.	10.295	55.	0.4462E 04	0.3470E 05	-0.3061E 10	7398.	709.
176144.	31969.	0.48	25.96	39882.	9.806	60.	0.5426E 04	0.3662E 05	-0.2249E 10	9447.	1243.
177189.	30907.	-1.17	18.94	35244.	6.667	65.	0.5402E 04	0.3850E 05	-0.1392E 10	11332.	1597.
181792.	30267.	-2.16	9.45	30010.	3.339	70.	0.4683E 04	0.4003E 05	-0.6232E 09	12963.	1723.
188034.	29900.	-2.50	-0.15	25329.	1.817	75.	0.3837E 04	0.4133E 05	0.9321E 07	14343.	1753.
194522.	29630.	-2.45	-7.67	21467.	1.704	80.	0.3155E 04	0.4257E 05	0.5198E 09	15509.	1767.
200602.	29392.	-2.25	-12.95	18408.	1.686	85.	0.2676E 04	0.4384E 05	0.9406E 09	16503.	1782.
206082.	29176.	-2.03	-16.64	16022.	1.568	90.	0.2355E 04	0.4516E 05	0.1300E 10	17361.	1795.
210962.	28982.	-1.82	-19.30	14155.	1.413	95.	0.2144E 04	0.4652E 05	0.1618E 10	18114.	1806.
215300.	28809.	-1.62	-21.31	12676.	1.261	100.	0.2011E 04	0.4792E 05	0.1910E 10	18783.	1815.
219167.	28654.	-1.46	-22.87	11486.	1.125	105.	0.1931E 04	0.4938E 05	0.2185E 10	19386.	1822.
222625.	28516.	-1.31	-24.15	10516.	1.009	110.	0.1890E 04	0.5091E 05	0.2450E 10	19936.	1828.
225732.	28390.	-1.19	-25.19	9714.	0.910	115.	0.1877E 04	0.5250E 05	0.2709E 10	20441.	1833.
228532.	28276.	-1.08	-26.08	9041.	0.827	120.	0.1887E 04	0.5416E 05	0.2967E 10	20909.	1836.
231064.	28172.	-0.98	-26.85	8472.	0.757	125.	0.1914E 04	0.5591E 05	0.3226E 10	21347.	1839.
233360.	28075.	-0.89	-27.52	7985.	0.698	130.	0.1955E 04	0.5774E 05	0.3489E 10	21758.	1842.
235444.	27986.	-0.81	-28.10	7566.	0.647	135.	0.2009E 04	0.5967E 05	0.3757E 10	22146.	1844.
237338.	27903.	-0.74	-28.63	7202.	0.604	140.	0.2074E 04	0.6170E 05	0.4033E 10	22515.	1846.
239061.	27825.	-0.67	-29.11	6885.	0.567	145.	0.2149E 04	0.6384E 05	0.4317E 10	22867.	1848.
240626.	27752.	-0.61	-29.54	6606.	0.535	150.	0.2233E 04	0.6610E 05	0.4611E 10	23204.	1850.
242045.	27682.	-0.56	-29.93	6362.	0.507	155.	0.2326E 04	0.6847E 05	0.4916E 10	23528.	1851.
243329.	27616.	-0.51	-30.28	6146.	0.483	160.	0.2428E 04	0.7098E 05	0.5233E 10	23841.	1852.
244486.	27552.	-0.46	-30.61	5955.	0.463	165.	0.2539E 04	0.7363E 05	0.5564E 10	24143.	1853.
245523.	27492.	-0.41	-30.91	5786.	0.445	170.	0.2658E 04	0.7643E 05	0.5909E 10	24437.	1854.
246444.	27433.	-0.36	-31.19	5638.	0.430	175.	0.2787E 04	0.7938E 05	0.6271E 10	24722.	1855.
247255.	27376.	-0.32	-31.45	5507.	0.416	180.	0.2925E 04	0.8250E 05	0.6649E 10	25001.	1856.
247958.	27321.	-0.27	-31.69	5392.	0.405	185.	0.3074E 04	0.8579E 05	0.7046E 10	25273.	1857.
248556.	27267.	-0.23	-31.91	5293.	0.396	190.	0.3233E 04	0.8927E 05	0.7462E 10	25540.	1858.
249050.	27215.	-0.19	-32.12	5207.	0.389	195.	0.3405E 04	0.9294E 05	0.7899E 10	25803.	1858.
249441.	27163.	-0.14	-32.31	5134.	0.383	200.	0.3589E 04	0.9683E 05	0.8359E 10	26061.	1859.
249729.	27112.	-0.10	-32.50	5074.	0.379	205.	0.3787E 04	0.1009E 06	0.8844E 10	26316.	1860.
249912.	27062.	-0.06	-32.67	5026.	0.377	210.	0.4001E 04	0.1053E 06	0.9355E 10	26569.	1861.
249990.	27012.	-0.01	-32.83	4990.	0.376	215.	0.4233E 04	0.1099E 06	0.9894E 10	26819.	1861.
249993.	27000.	0.00	-32.87	4984.	0.376	216.	0.4289E 04	0.1110E 06	0.1002E 11	26877.	1862.

TABLE 5.2

OPTIMAL TRAJECTORY FOR K_7 = 50

HEIGHT FT	VELOCITY FT/SEC	FL ANG DEG	CONTROL DEG	HEAT RT	ACCEL G	TIME SEC	AUXILIARY VARIABLES			HEAT	IACC
400000.	36000.	8.09	39.27	484.	0.001	0.	0.1078E 04	0.2277E 05	-0.6827E 10	-0.	-0.
374962.	36022.	7.87	39.39	826.	0.004	5.	0.1075E 04	0.2116E 05	-0.6635E 10	32.	0.
350621.	36043.	7.64	40.08	1390.	0.011	10.	0.1070E 04	0.1961E 05	-0.6443E 10	86.	0.
326988.	36061.	7.40	40.41	2303.	0.031	15.	0.1062E 04	0.1812E 05	-0.6252E 10	177.	0.
304078.	36074.	7.17	40.64	3756.	0.082	20.	0.1058E 04	0.1671E 05	-0.6062E 10	326.	0.
281920.	36074.	6.92	40.73	6021.	0.212	25.	0.1073E 04	0.1541E 05	-0.5871E 10	566.	0.
260584.	36045.	6.64	40.65	9462.	0.525	30.	0.1149E 04	0.1424E 05	-0.5673E 10	947.	1.
240228.	35947.	6.31	40.18	14479.	1.228	35.	0.1379E 04	0.1327E 05	-0.5450E 10	1538.	4.
221195.	35705.	5.84	38.91	21283.	2.642	40.	0.1924E 04	0.1263E 05	-0.5162E 10	2425.	23.
204163.	35208.	5.11	36.31	29330.	4.961	45.	0.2931E 04	0.1259E 05	-0.4741E 10	3688.	95.
190234.	34372.	3.98	32.17	36718.	7.611	50.	0.4251E 04	0.1360E 05	-0.4120E 10	5348.	297.
180668.	33270.	2.43	27.06	40821.	9.102	55.	0.5283E 04	0.1587E 05	-0.3308E 10	7304.	661.
176158.	32131.	0.72	21.92	40480.	8.511	60.	0.5524E 04	0.1880E 05	-0.2416E 10	9354.	1064.
176322.	31180.	-0.78	16.84	36860.	6.417	65.	0.5068E 04	0.2140E 05	-0.1575E 10	11297.	1350.
179952.	30505.	-1.83	11.12	31951.	3.991	70.	0.4308E 04	0.2329E 05	-0.8543E 09	13019.	1487.
185597.	30072.	-2.36	4.08	27140.	2.200	75.	0.3542E 04	0.2464E 05	-0.2654E 09	14495.	1534.
191991.	29777.	-2.48	-3.53	22995.	1.614	80.	0.2904E 04	0.2580E 05	0.2103E 09	15745.	1550.
198290.	29536.	-2.36	-10.16	19624.	1.633	85.	0.2425E 04	0.2693E 05	0.5989E 09	16807.	1563.
204096.	29316.	-2.15	-15.19	16955.	1.611	90.	0.2090E 04	0.2810E 05	0.9249E 09	17719.	1577.
209304.	29112.	-1.93	-18.87	14861.	1.502	95.	0.1867E 04	0.2930E 05	0.1208E 10	18512.	1589.
213934.	28926.	-1.72	-21.57	13210.	1.362	100.	0.1723E 04	0.3052E 05	0.1463E 10	19212.	1599.
218047.	28759.	-1.54	-23.60	11892.	1.222	105.	0.1636E 04	0.3178E 05	0.1700E 10	19839.	1607.
221712.	28607.	-1.38	-25.18	10827.	1.097	110.	0.1589E 04	0.3308E 05	0.1925E 10	20406.	1614.
224988.	28471.	-1.25	-26.43	9953.	0.987	115.	0.1572E 04	0.3443E 05	0.2145E 10	20924.	1620.
227929.	28346.	-1.13	-27.44	9227.	0.894	120.	0.1576E 04	0.3583E 05	0.2363E 10	21403.	1624.
230577.	28233.	-1.02	-28.29	8616.	0.815	125.	0.1597E 04	0.3729E 05	0.2581E 10	21849.	1628.
232969.	28129.	-0.92	-29.00	8098.	0.748	130.	0.1632E 04	0.3882E 05	0.2802E 10	22267.	1631.
235134.	28033.	-0.84	-29.61	7654.	0.691	135.	0.1678E 04	0.4043E 05	0.3028E 10	22660.	1633.
237095.	27944.	-0.76	-30.13	7271.	0.642	140.	0.1734E 04	0.4211E 05	0.3260E 10	23033.	1635.
238873.	27860.	-0.69	-30.59	6938.	0.600	145.	0.1799E 04	0.4389E 05	0.3500E 10	23388.	1637.
240485.	27782.	-0.63	-30.99	6648.	0.564	150.	0.1872E 04	0.4576E 05	0.3748E 10	23728.	1639.
241943.	27708.	-0.57	-31.36	6393.	0.534	155.	0.1952E 04	0.4774E 05	0.4005E 10	24053.	1641.
243259.	27638.	-0.52	-31.68	6169.	0.507	160.	0.2040E 04	0.4982E 05	0.4274E 10	24367.	1642.
244442.	27571.	-0.46	-31.97	5972.	0.484	165.	0.2136E 04	0.5202E 05	0.4554E 10	24671.	1643.
245500.	27507.	-0.41	-32.24	5799.	0.464	170.	0.2240E 04	0.5434E 05	0.4846E 10	24965.	1644.
246438.	27445.	-0.37	-32.47	5646.	0.447	175.	0.2351E 04	0.5679E 05	0.5153E 10	25251.	1645.
247261.	27386.	-0.32	-32.68	5512.	0.432	180.	0.2471E 04	0.5938E 05	0.5474E 10	25530.	1646.
247973.	27328.	-0.28	-32.89	5395.	0.420	185.	0.2599E 04	0.6212E 05	0.5811E 10	25802.	1647.
248577.	27272.	-0.23	-33.07	5293.	0.410	190.	0.2737E 04	0.6502E 05	0.6165E 10	26070.	1648.
249074.	27218.	-0.19	-33.24	5206.	0.402	195.	0.2885E 04	0.6809E 05	0.6537E 10	26332.	1649.
249464.	27164.	-0.14	-33.38	5132.	0.395	200.	0.3044E 04	0.7133E 05	0.6929E 10	26590.	1650.
249749.	27111.	-0.10	-33.53	5072.	0.391	205.	0.3216E 04	0.7476E 05	0.7341E 10	26845.	1650.
249927.	27059.	-0.05	-33.66	5023.	0.388	210.	0.3401E 04	0.7840E 05	0.7777E 10	27098.	1651.
249996.	27007.	-0.01	-33.78	4987.	0.386	215.	0.3601E 04	0.8224E 05	0.8237E 10	27348.	1652.
249997.	27000.	0.00	-33.80	4983.	0.386	216.	0.3629E 04	0.8279E 05	0.8302E 10	27382.	1652.

TABLE 5.3

OPTIMAL TRAJECTORY FOR K_7 = 100

HEIGHT FT	VELOCITY FT/SEC	FL ANG DEG	CONTROL DEG	HEAT RT	ACCEL G	TIME SEC	AUXILIARY VARIABLES			HEAT	TACC
400000.	36000.	8.09	41.36	484.	0.001	0.	0.1409E 04	0.1821E 05	-0.8971E 10	-0.	-0.
374962.	36022.	7.87	41.62	826.	0.004	5.	0.1408E 04	0.1608E 05	-0.8717E 10	32.	0.
350622.	36042.	7.64	42.20	1390.	0.012	10.	0.1405E 04	0.1402E 05	-0.8464E 10	86.	0.
326989.	36061.	7.40	42.61	2303.	0.032	15.	0.1402E 04	0.1205E 05	-0.8211E 10	177.	0.
304078.	36073.	7.17	42.86	3756.	0.087	20.	0.1411E 04	0.1017E 05	-0.7957E 10	326.	0.
281922.	36072.	6.92	42.98	6020.	0.223	25.	0.1456E 04	0.8400E 04	-0.7699E 10	566.	0.
260589.	36039.	6.64	42.84	9457.	0.552	30.	0.1603E 04	0.6784E 04	-0.7425E 10	947.	1.
240241.	35934.	6.30	42.14	14460.	1.284	35.	0.1989E 04	0.5386E 04	-0.7107E 10	1538.	5.
221223.	35680.	5.83	40.26	21225.	2.723	40.	0.2825E 04	0.4376E 04	-0.6684E 10	2423.	24.
204217.	35172.	5.10	36.55	29207.	4.970	45.	0.4222E 04	0.4180E 04	-0.6068E 10	3681.	99.
190301.	34356.	3.98	30.95	36614.	7.308	50.	0.5803E 04	0.5492E 04	-0.5196E 10	5334.	292.
180638.	33330.	2.50	24.73	41069.	8.421	55.	0.6744E 04	0.8518E 04	-0.4122E 10	7293.	614.
175779.	32291.	0.89	19.25	41422.	7.831	60.	0.6643E 04	0.1229E 05	-0.3013E 10	9372.	955.
175303.	31397.	-0.51	14.76	38463.	6.168	65.	0.5835E 04	0.1560E 05	-0.2016E 10	11379.	1204.
178151.	30712.	-1.53	10.60	33880.	4.255	70.	0.4831E 04	0.1798E 05	-0.1191E 10	13191.	1342.
183116.	30227.	-2.14	5.84	29058.	2.648	75.	0.3919E 04	0.1961E 05	-0.5303E 09	14764.	1401.
189129.	29890.	-2.39	0.03	24720.	1.733	80.	0.3185E 04	0.2087E 05	-0.2399E 07	16105.	1424.
195371.	29635.	-2.38	-6.24	21092.	1.538	85.	0.2625E 04	0.2202E 05	0.4241E 09	17248.	1436.
201333.	29413.	-2.23	-11.85	18164.	1.550	90.	0.2215E 04	0.2319E 05	0.7758E 09	18226.	1448.
206785.	29210.	-2.03	-16.32	15839.	1.503	95.	0.1929E 04	0.2440E 05	0.1074E 10	19074.	1460.
211676.	29021.	-1.82	-19.73	13996.	1.399	100.	0.1737E 04	0.2563E 05	0.1336E 10	19818.	1471.
216037.	28847.	-1.63	-22.31	12527.	1.275	105.	0.1614E 04	0.2689E 05	0.1573E 10	20480.	1480.
219924.	28689.	-1.46	-24.29	11344.	1.152	110.	0.1542E 04	0.2818E 05	0.1795E 10	21076.	1487.
223397.	28546.	-1.32	-25.84	10378.	1.040	115.	0.1506E 04	0.2950E 05	0.2008E 10	21618.	1493.
226509.	28415.	-1.19	-27.07	9579.	0.943	120.	0.1496E 04	0.3087E 05	0.2217E 10	22116.	1498.
229307.	28296.	-1.07	-28.07	8912.	0.858	125.	0.1507E 04	0.3229E 05	0.2425E 10	22578.	1502.
231831.	28187.	-0.97	-28.89	8348.	0.786	130.	0.1534E 04	0.3377E 05	0.2634E 10	23009.	1505.
234111.	28086.	-0.88	-29.58	7867.	0.724	135.	0.1573E 04	0.3531E 05	0.2847E 10	23414.	1508.
236176.	27992.	-0.80	-30.17	7454.	0.671	140.	0.1622E 04	0.3693E 05	0.3065E 10	23797.	1510.
238047.	27905.	-0.73	-30.67	7096.	0.625	145.	0.1681E 04	0.3863E 05	0.3290E 10	24160.	1513.
239743.	27824.	-0.66	-31.11	6784.	0.586	150.	0.1749E 04	0.4041E 05	0.3522E 10	24507.	1514.
241277.	27747.	-0.60	-31.49	6512.	0.553	155.	0.1824E 04	0.4228E 05	0.3764E 10	24839.	1516.
242664.	27674.	-0.54	-31.83	6273.	0.524	160.	0.1906E 04	0.4426E 05	0.4015E 10	25159.	1517.
243912.	27605.	-0.49	-32.13	6063.	0.498	165.	0.1996E 04	0.4634E 05	0.4277E 10	25467.	1519.
245031.	27539.	-0.44	-32.40	5878.	0.477	170.	0.2093E 04	0.4854E 05	0.4551E 10	25766.	1520.
246027.	27476.	-0.39	-32.64	5715.	0.458	175.	0.2197E 04	0.5086E 05	0.4838E 10	26055.	1521.
246905.	27415.	-0.34	-32.85	5572.	0.442	180.	0.2309E 04	0.5330E 05	0.5139E 10	26337.	1522.
247670.	27356.	-0.30	-33.05	5446.	0.429	185.	0.2430E 04	0.5589E 05	0.5455E 10	26613.	1523.
248324.	27299.	-0.25	-33.22	5337.	0.417	190.	0.2559E 04	0.5862E 05	0.5786E 10	26882.	1524.
248871.	27243.	-0.21	-33.38	5243.	0.408	195.	0.2698E 04	0.6151E 05	0.6135E 10	27147.	1525.
249310.	27189.	-0.16	-33.53	5163.	0.400	200.	0.2847E 04	0.6456E 05	0.6502E 10	27407.	1526.
249643.	27135.	-0.12	-33.67	5097.	0.395	205.	0.3007E 04	0.6779E 05	0.6889E 10	27663.	1526.
249869.	27082.	-0.07	-33.79	5043.	0.391	210.	0.3179E 04	0.7121E 05	0.7297E 10	27917.	1527.
249988.	27030.	-0.03	-33.91	5001.	0.389	215.	0.3366E 04	0.7483E 05	0.7728E 10	28168.	1528.
250006.	27000.	0.00	-33.97	4982.	0.388	218.	0.3481E 04	0.7702E 05	0.7989E 10	28312.	1528.

TABLE 5.4

OPTIMAL TRAJECTORY FOR K_7 = 150

HEIGHT FT	VELOCITY FT/SEC	FL ANG DEG	CONTROL DEG	HEAT RT	ACCEL G	TIME SEC	AUXILIARY VARIABLES			HEAT	TACC
400000.	36000.	8.09	41.59	484.	0.001	0.	0.1835E 04	0.2020E 05	-0.1168E 11	-0.	-0.
374962.	36022.	7.87	41.83	826.	0.004	5.	0.1835E 04	0.1742E 05	-0.1135E 11	32.	0.
350622.	36042.	7.64	42.63	1390.	0.012	10.	0.1835E 04	0.1473E 05	-0.1102E 11	86.	0.
326989.	36060.	7.40	43.09	2303.	0.033	15.	0.1838E 04	0.1214E 05	-0.1069E 11	177.	0.
304078.	36073.	7.17	43.35	3756.	0.088	20.	0.1860E 04	0.9659E 04	-0.1036E 11	326.	0.
281922.	36072.	6.92	43.49	6020.	0.226	25.	0.1936E 04	0.7317E 04	-0.1001E 11	566.	0.
260590.	36038.	6.64	43.33	9456.	0.558	30.	0.2159E 04	0.5147E 04	-0.9646E 10	947.	1.
240243.	35931.	6.30	42.49	14457.	1.294	35.	0.2710E 04	0.3228E 04	-0.9214E 10	1538.	5.
221228.	35675.	5.83	40.34	21215.	2.728	40.	0.3854E 04	0.1781E 04	-0.8637E 10	2422.	25.
204223.	35171.	5.10	36.15	29200.	4.916	45.	0.5687E 04	0.1390E 04	-0.7801E 10	3680.	98.
190291.	34375.	4.00	30.00	36681.	7.103	50.	0.7633E 04	0.2978E 04	-0.6639E 10	5334.	284.
180544.	33392.	2.54	23.38	41381.	8.069	55.	0.8642E 04	0.6740E 04	-0.5243E 10	7301.	583.
175461.	32400.	1.00	17.79	42127.	7.501	60.	0.8318E 04	0.1138E 05	-0.3834E 10	9405.	894.
174603.	31531.	-0.34	13.50	39543.	6.030	65.	0.7162E 04	0.1543E 05	-0.2593E 10	11458.	1127.
176965.	30840.	-1.33	9.88	35182.	4.352	70.	0.5823E 04	0.1835E 05	-0.1585E 10	13330.	1263.
181447.	30326.	-1.96	6.11	30408.	2.896	75.	0.4652E 04	0.2034E 05	-0.7904E 09	14970.	1323.
187086.	29957.	-2.27	1.57	25993.	1.922	80.	0.3733E 04	0.2180E 05	-0.1657E 09	16377.	1357.
193131.	29682.	-2.33	-3.69	22229.	1.536	85.	0.3040E 04	0.2306E 05	0.3327E 09	17580.	1371.
199062.	29455.	-2.24	-8.98	19145.	1.481	90.	0.2528E 04	0.2430E 05	0.7382E 09	18612.	1382.
204595.	29252.	-2.07	-13.60	16668.	1.454	95.	0.2160E 04	0.2558E 05	0.1076E 10	19505.	1393.
209621.	29065.	-1.88	-17.32	14690.	1.380	100.	0.1905E 04	0.2690E 05	0.1367E 10	20287.	1403.
214136.	28893.	-1.69	-20.24	13107.	1.276	105.	0.1736E 04	0.2825E 05	0.1626E 10	20980.	1412.
218176.	28734.	-1.52	-22.51	11830.	1.164	110.	0.1629E 04	0.2962E 05	0.1863E 10	21602.	1419.
221794.	28590.	-1.37	-24.31	10788.	1.057	115.	0.1568E 04	0.3103E 05	0.2087E 10	22167.	1426.
225040.	28458.	-1.24	-25.74	9928.	0.961	120.	0.1541E 04	0.3248E 05	0.2303E 10	22684.	1431.
227960.	28337.	-1.12	-26.89	9211.	0.876	125.	0.1539E 04	0.3397E 05	0.2516E 10	23162.	1435.
230595.	28226.	-1.01	-27.84	8606.	0.802	130.	0.1556E 04	0.3553E 05	0.2729E 10	23607.	1438.
232977.	28123.	-0.92	-28.63	8092.	0.739	135.	0.1588E 04	0.3714E 05	0.2944E 10	24024.	1441.
235135.	28028.	-0.84	-29.31	7650.	0.684	140.	0.1632E 04	0.3882E 05	0.3164E 10	24418.	1444.
237092.	27940.	-0.76	-29.88	7269.	0.637	145.	0.1688E 04	0.4058E 05	0.3390E 10	24790.	1446.
238867.	27857.	-0.69	-30.37	6937.	0.596	150.	0.1752E 04	0.4243E 05	0.3623E 10	25145.	1448.
240476.	27779.	-0.63	-30.81	6648.	0.561	155.	0.1825E 04	0.4436E 05	0.3865E 10	25485.	1450.
241932.	27706.	-0.57	-31.18	6393.	0.531	160.	0.1905E 04	0.4640E 05	0.4116E 10	25811.	1451.
243247.	27636.	-0.52	-31.52	6170.	0.505	165.	0.1994E 04	0.4854E 05	0.4378E 10	26125.	1452.
244430.	27570.	-0.46	-31.82	5973.	0.482	170.	0.2089E 04	0.5079E 05	0.4652E 10	26428.	1454.
245488.	27506.	-0.41	-32.09	5800.	0.462	175.	0.2192E 04	0.5316E 05	0.4939E 10	26722.	1455.
246426.	27445.	-0.37	-32.33	5647.	0.445	180.	0.2303E 04	0.5566E 05	0.5239E 10	27008.	1456.
247250.	27386.	-0.32	-32.54	5513.	0.431	185.	0.2423E 04	0.5829E 05	0.5553E 10	27287.	1457.
247963.	27328.	-0.28	-32.75	5396.	0.419	190.	0.2550E 04	0.6107E 05	0.5884E 10	27560.	1458.
248568.	27273.	-0.23	-32.93	5294.	0.409	195.	0.2687E 04	0.6401E 05	0.6231E 10	27827.	1458.
249067.	27218.	-0.19	-33.10	5207.	0.400	200.	0.2834E 04	0.6711E 05	0.6597E 10	28090.	1459.
249460.	27165.	-0.14	-33.24	5133.	0.394	205.	0.2992E 04	0.7038E 05	0.6981E 10	28348.	1460.
249747.	27112.	-0.10	-33.38	5072.	0.389	210.	0.3161E 04	0.7384E 05	0.7387E 10	28603.	1461.
249927.	27060.	-0.05	-33.51	5024.	0.386	215.	0.3344E 04	0.7750E 05	0.7815E 10	28855.	1462.
250000.	27009.	-0.01	-33.63	4988.	0.385	220.	0.3541E 04	0.8137E 05	0.8268E 10	29106.	1462.
250002.	27000.	-0.00	-33.65	4983.	0.385	221.	0.3574E 04	0.8201E 05	0.8343E 10	29146.	1462.

TABLE 5.5

OPTIMAL TRAJECTORY FOR K_7 = 200

HEIGHT FT	VELOCITY FT/SEC	FL ANG DEG	CONTROL DEG	HEAT RT	ACCEL G	TIME SEC	AUXILIARY VARIABLES			HEAT	TACC
400000.	36000.	8.09	41.80	484.	0.001	0.	0.2319E 04	0.2568E 05	-0.1475E 11	-0.	-0.
374962.	36022.	7.87	41.99	826.	0.004	5.	0.2321E 04	0.2216E 05	-0.1433E 11	32.	0.
350622.	36042.	7.64	42.60	1390.	0.012	10.	0.2323E 04	0.1876E 05	-0.1391E 11	86.	0.
326989.	36060.	7.40	43.08	2303.	0.033	15.	0.2333E 04	0.1547E 05	-0.1349E 11	177.	0.
304079.	36073.	7.17	43.33	3756.	0.088	20.	0.2368E 04	0.1233E 05	-0.1306E 11	326.	0.
281922.	36072.	6.92	43.45	6020.	0.226	25.	0.2478E 04	0.9339E 04	-0.1263E 11	566.	0.
260590.	36038.	6.64	43.27	9456.	0.557	30.	0.2779E 04	0.6549E 04	-0.1216E 11	947.	1.
240243.	35932.	6.30	42.39	14457.	1.291	35.	0.3501E 04	0.4051E 04	-0.1160E 11	1538.	5.
221227.	35677.	5.83	40.12	21219.	2.713	40.	0.4971E 04	0.2120E 04	-0.1085E 11	2422.	25.
204219.	35177.	5.11	35.72	29220.	4.863	45.	0.7279E 04	0.1499E 04	-0.9779E 10	3681.	97.
190269.	34396.	4.00	29.35	36767.	6.974	50.	0.9664E 04	0.3346E 04	-0.8299E 10	5337.	277.
180462.	33437.	2.57	22.60	41622.	7.881	55.	0.1082E 05	0.7859E 04	-0.6541E 10	7312.	563.
175244.	32469.	1.06	16.99	42591.	7.334	60.	0.1032E 05	0.1342E 05	-0.4781E 10	9434.	861.
174163.	31611.	-0.25	12.78	40218.	5.960	65.	0.8797E 04	0.1825E 05	-0.3245E 10	11515.	1085.
176240.	30915.	-1.22	9.37	35991.	4.396	70.	0.7073E 04	0.2175E 05	-0.2010E 10	13425.	1220.
180421.	30385.	-1.85	6.02	31262.	3.025	75.	0.5583E 04	0.2412E 05	-0.1048E 10	15106.	1289.
185795.	29996.	-2.18	2.13	26822.	2.056	80.	0.4429E 04	0.2582E 05	-0.3013E 09	16556.	1321.
191652.	29704.	-2.28	-2.42	22993.	1.581	85.	0.3569E 04	0.2723E 05	0.2877E 09	17798.	1337.
197493.	29470.	-2.22	-7.21	19827.	1.454	90.	0.2935E 04	0.2859E 05	0.7615E 09	18866.	1348.
203015.	29267.	-2.08	-11.64	17264.	1.413	95.	0.2476E 04	0.2999E 05	0.1152E 10	19791.	1358.
208085.	29081.	-1.90	-15.38	15204.	1.349	100.	0.2153E 04	0.3143E 05	0.1483E 10	20601.	1368.
212671.	28911.	-1.72	-18.40	13548.	1.258	105.	0.1932E 04	0.3290E 05	0.1773E 10	21318.	1376.
216794.	28754.	-1.55	-20.81	12208.	1.155	110.	0.1788E 04	0.3441E 05	0.2035E 10	21961.	1383.
220498.	28611.	-1.40	-22.74	11114.	1.054	115.	0.1700E 04	0.3595E 05	0.2279E 10	22543.	1390.
223829.	28479.	-1.27	-24.28	10211.	0.961	120.	0.1654E 04	0.3754E 05	0.2512E 10	23076.	1395.
226831.	28359.	-1.15	-25.54	9458.	0.878	125.	0.1638E 04	0.3917E 05	0.2739E 10	23567.	1399.
229542.	28248.	-1.04	-26.58	8823.	0.805	130.	0.1646E 04	0.4086E 05	0.2965E 10	24023.	1402.
231997.	28146.	-0.95	-27.45	8283.	0.742	135.	0.1672E 04	0.4261E 05	0.3192E 10	24451.	1405.
234223.	28051.	-0.86	-28.20	7820.	0.687	140.	0.1713E 04	0.4443E 05	0.3423E 10	24853.	1408.
236244.	27963.	-0.79	-28.83	7420.	0.640	145.	0.1766E 04	0.4633E 05	0.3659E 10	25234.	1410.
238080.	27880.	-0.72	-29.38	7072.	0.599	150.	0.1829E 04	0.4832E 05	0.3902E 10	25596.	1412.
239747.	27803.	-0.65	-29.86	6768.	0.563	155.	0.1902E 04	0.5040E 05	0.4154E 10	25942.	1414.
241260.	27729.	-0.59	-30.28	6502.	0.532	160.	0.1983E 04	0.5258E 05	0.4415E 10	26273.	1415.
242629.	27660.	-0.54	-30.66	6268.	0.505	165.	0.2073E 04	0.5486E 05	0.4687E 10	26592.	1416.
243864.	27593.	-0.49	-30.99	6061.	0.482	170.	0.2171E 04	0.5727E 05	0.4971E 10	26900.	1418.
244973.	27530.	-0.44	-31.30	5879.	0.462	175.	0.2276E 04	0.5979E 05	0.5268E 10	27199.	1419.
245963.	27469.	-0.39	-31.57	5718.	0.445	180.	0.2390E 04	0.6245E 05	0.5579E 10	27489.	1420.
246838.	27410.	-0.34	-31.82	5577.	0.430	185.	0.2512E 04	0.6525E 05	0.5905E 10	27771.	1421.
247602.	27353.	-0.30	-32.05	5453.	0.417	190.	0.2643E 04	0.6820E 05	0.6247E 10	28047.	1422.
248258.	27298.	-0.25	-32.25	5344.	0.407	195.	0.2783E 04	0.7131E 05	0.6607E 10	28316.	1423.
248808.	27244.	-0.21	-32.45	5251.	0.398	200.	0.2933E 04	0.7459E 05	0.6985E 10	28581.	1423.
249254.	27191.	-0.17	-32.61	5171.	0.391	205.	0.3094E 04	0.7804E 05	0.7383E 10	28842.	1424.
249596.	27139.	-0.12	-32.78	5104.	0.386	210.	0.3267E 04	0.8169E 05	0.7802E 10	29098.	1425.
249833.	27088.	-0.08	-32.94	5049.	0.382	215.	0.3453E 04	0.8554E 05	0.8244E 10	29352.	1426.
249964.	27037.	-0.03	-33.07	5007.	0.380	220.	0.3654E 04	0.8961E 05	0.8710E 10	29604.	1426.
249992.	27000.	-0.00	-33.17	4984.	0.379	224.	0.3808E 04	0.9269E 05	0.9063E 10	29783.	1427.

TABLE 5.6

OPTIMAL TRAJECTORY FOR K_7 = 250

HEIGHT FT	VELOCITY FT/SEC	FL ANG DEG	CONTROL DEG	HEAT RT	ACCEL G	TIME SEC	AUXILIARY VARIABLES			HEAT	TACC
400000.	36000.	8.09	40.98	484.	0.001	0.	0.2836E 04	0.3303E 05	-0.1801E 11	-0.	-0.
374962.	36022.	7.87	41.93	826.	0.004	5.	0.2840E 04	0.2873E 05	-0.1750E 11	32.	0.
350622.	36042.	7.64	42.53	1390.	0.012	10.	0.2845E 04	0.2456E 05	-0.1699E 11	86.	0.
326989.	36060.	7.40	42.92	2303.	0.033	15.	0.2861E 04	0.2054E 05	-0.1647E 11	177.	0.
304078.	36073.	7.17	43.20	3756.	0.087	20.	0.2910E 04	0.1668E 05	-0.1595E 11	326.	0.
281922.	36072.	6.92	43.29	6020.	0.225	25.	0.3054E 04	0.1301E 05	-0.1541E 11	566.	0.
260590.	36039.	6.64	43.08	9456.	0.555	30.	0.3435E 04	0.9561E 04	-0.1483E 11	947.	1.
240242.	35933.	6.30	42.18	14459.	1.285	35.	0.4334E 04	0.6448E 04	-0.1414E 11	1538.	5.
221225.	35680.	5.83	39.87	21225.	2.697	40.	0.6142E 04	0.4004E 04	-0.1322E 11	2423.	24.
204212.	35185.	5.11	35.38	29243.	4.821	45.	0.8951E 04	0.3132E 04	-0.1190E 11	3682.	95.
190248.	34414.	4.01	28.94	36840.	6.894	50.	0.1182E 05	0.5230E 04	-0.1008E 11	5340.	272.
180400.	33469.	2.59	22.14	41796.	7.775	55.	0.1317E 05	0.1051E 05	-0.7936E 10	7321.	551.
175097.	32514.	1.10	16.52	42901.	7.244	60.	0.1250E 05	0.1702E 05	-0.5798E 10	9455.	841.
173881.	31661.	-0.19	12.34	40653.	5.921	65.	0.1060E 05	0.2269E 05	-0.3939E 10	11555.	1061.
175783.	30961.	-1.15	9.04	36508.	4.421	70.	0.8464E 04	0.2678E 05	-0.2453E 10	13488.	1195.
179773.	30422.	-1.78	5.88	31811.	3.098	75.	0.6625E 04	0.2954E 05	-0.1305E 10	15197.	1266.
184966.	30020.	-2.12	2.33	27365.	2.142	80.	0.5209E 04	0.3149E 05	-0.4219E 09	16674.	1300.
190681.	29717.	-2.23	-1.78	23504.	1.628	85.	0.4160E 04	0.3307E 05	0.2680E 09	17943.	1317.
196430.	29476.	-2.19	-6.18	20293.	1.450	90.	0.3391E 04	0.3458E 05	0.8178E 09	19035.	1328.
201914.	29270.	-2.07	-10.36	17680.	1.389	95.	0.2834E 04	0.3611E 05	0.1267E 10	19982.	1338.
206984.	29085.	-1.90	-14.00	15571.	1.324	100.	0.2438E 04	0.3770E 05	0.1644E 10	20812.	1348.
211595.	28916.	-1.73	-17.01	13870.	1.239	105.	0.2165E 04	0.3932E 05	0.1970E 10	21546.	1356.
215759.	28762.	-1.57	-19.45	12490.	1.142	110.	0.1982E 04	0.4099E 05	0.2261E 10	22204.	1363.
219511.	28620.	-1.42	-21.43	11361.	1.045	115.	0.1867E 04	0.4269E 05	0.2530E 10	22799.	1369.
222892.	28490.	-1.29	-23.03	10428.	0.955	120.	0.1801E 04	0.4444E 05	0.2784E 10	23343.	1374.
225945.	28370.	-1.17	-24.35	9649.	0.874	125.	0.1771E 04	0.4623E 05	0.3030E 10	23845.	1378.
228707.	28261.	-1.06	-25.45	8993.	0.802	130.	0.1770E 04	0.4809E 05	0.3273E 10	24310.	1382.
231210.	28159.	-0.97	-26.37	8435.	0.740	135.	0.1790E 04	0.5001E 05	0.3516E 10	24746.	1384.
233484.	28065.	-0.88	-27.16	7956.	0.685	140.	0.1827E 04	0.5201E 05	0.3762E 10	25155.	1387.
235551.	27978.	-0.81	-27.83	7542.	0.638	145.	0.1879E 04	0.5409E 05	0.4013E 10	25542.	1389.
237431.	27896.	-0.74	-28.42	7182.	0.597	150.	0.1943E 04	0.5625E 05	0.4271E 10	25910.	1391.
239141.	27818.	-0.67	-28.94	6868.	0.561	155.	0.2017E 04	0.5852E 05	0.4538E 10	26261.	1393.
240696.	27746.	-0.61	-29.39	6592.	0.530	160.	0.2100E 04	0.6089E 05	0.4814E 10	26598.	1394.
242106.	27677.	-0.55	-29.80	6350.	0.504	165.	0.2193E 04	0.6337E 05	0.5101E 10	26921.	1396.
243383.	27611.	-0.50	-30.17	6135.	0.480	170.	0.2294E 04	0.6598E 05	0.5401E 10	27233.	1397.
244533.	27548.	-0.45	-30.50	5946.	0.460	175.	0.2404E 04	0.6871E 05	0.5714E 10	27535.	1398.
245563.	27487.	-0.40	-30.80	5779.	0.442	180.	0.2522E 04	0.7159E 05	0.6042E 10	27828.	1399.
246478.	27429.	-0.36	-31.07	5631.	0.427	185.	0.2649E 04	0.7461E 05	0.6385E 10	28113.	1400.
247284.	27373.	-0.31	-31.33	5501.	0.414	190.	0.2785E 04	0.7778E 05	0.6745E 10	28391.	1401.
247982.	27318.	-0.27	-31.56	5388.	0.403	195.	0.2931E 04	0.8113E 05	0.7123E 10	28663.	1402.
248576.	27265.	-0.23	-31.77	5289.	0.394	200.	0.3087E 04	0.8465E 05	0.7520E 10	28930.	1402.
249066.	27212.	-0.18	-31.98	5204.	0.387	205.	0.3254E 04	0.8836E 05	0.7938E 10	29193.	1403.
249454.	27161.	-0.14	-32.16	5132.	0.381	210.	0.3433E 04	0.9227E 05	0.8378E 10	29451.	1404.
249738.	27110.	-0.10	-32.32	5072.	0.377	215.	0.3626E 04	0.9639E 05	0.8841E 10	29706.	1405.
249919.	27060.	-0.05	-32.49	5025.	0.375	220.	0.3833E 04	0.1007E 06	0.9330E 10	29958.	1405.
249995.	27010.	-0.01	-32.64	4989.	0.374	225.	0.4057E 04	0.1053E 06	0.9847E 10	30209.	1406.
249998.	27000.	0.00	-32.67	4983.	0.374	226.	0.4107E 04	0.1063E 06	0.9961E 10	30262.	1406.

HEIGHT	$-x_1$
VELOCITY	$-x_2$
FL ANG	$-x_3$
CONTROL	$-u$
HEAT RT	$-f_4$
ACCEL	$-(f_5)^{\frac{1}{2}}$
TIME	$-t$
AUXILIARY VARIABLES	$-p_1, p_2, p_3$ (from left to right respectively)
HEAT	$-x_4$
TACC	$-x_5$

Some of the more interesting aspects of these results are also shown graphically. Figure 5.1 gives a plot of vehicle altitude $x_1(t)$ for $K_7 = 0$ (the solid line) and for $K_7 = 200$ (the dashed line). The basic result is that when the effects of acceleration are included in the criterion function, the optimal trajectory has a slightly less steep initial dive into the atmosphere, but the dive is extended to a lower minimum altitude. Similarly, the plots of vehicle velocity, flight path angle and angle of attack for these two optimum trajectories are shown in Figures 5.2, 5.3 and 5.4, respectively.

Note that the curve $u(t)$ for the trajectory corresponding to $K_7 = 200$, as shown in Figure 5.4, is smooth, i.e., without discontinuities in its time derivative. This is in contrast to the results given by Denham[45] for an optimum heating trajectory with constraints on the acceleration.

The effectiveness of the criterion function chosen to represent acceleration effects is indicated in Figure 5.5. Here, the values of the acceleration forces along the two optimal trajectories are shown. Both the magnitude and duration of the largest acceleration forces are significantly reduced. Corresponding to this decrease in acceleration forces, the increase in heating rate is as shown in Figure 5.6.

As indicated by the data in Tables 5.1 through 5.5, as K_7 varies from 0 to 200, the optimal trajectory varies smoothly between the two trajectories represented in Figure 5.1 through 5.6.

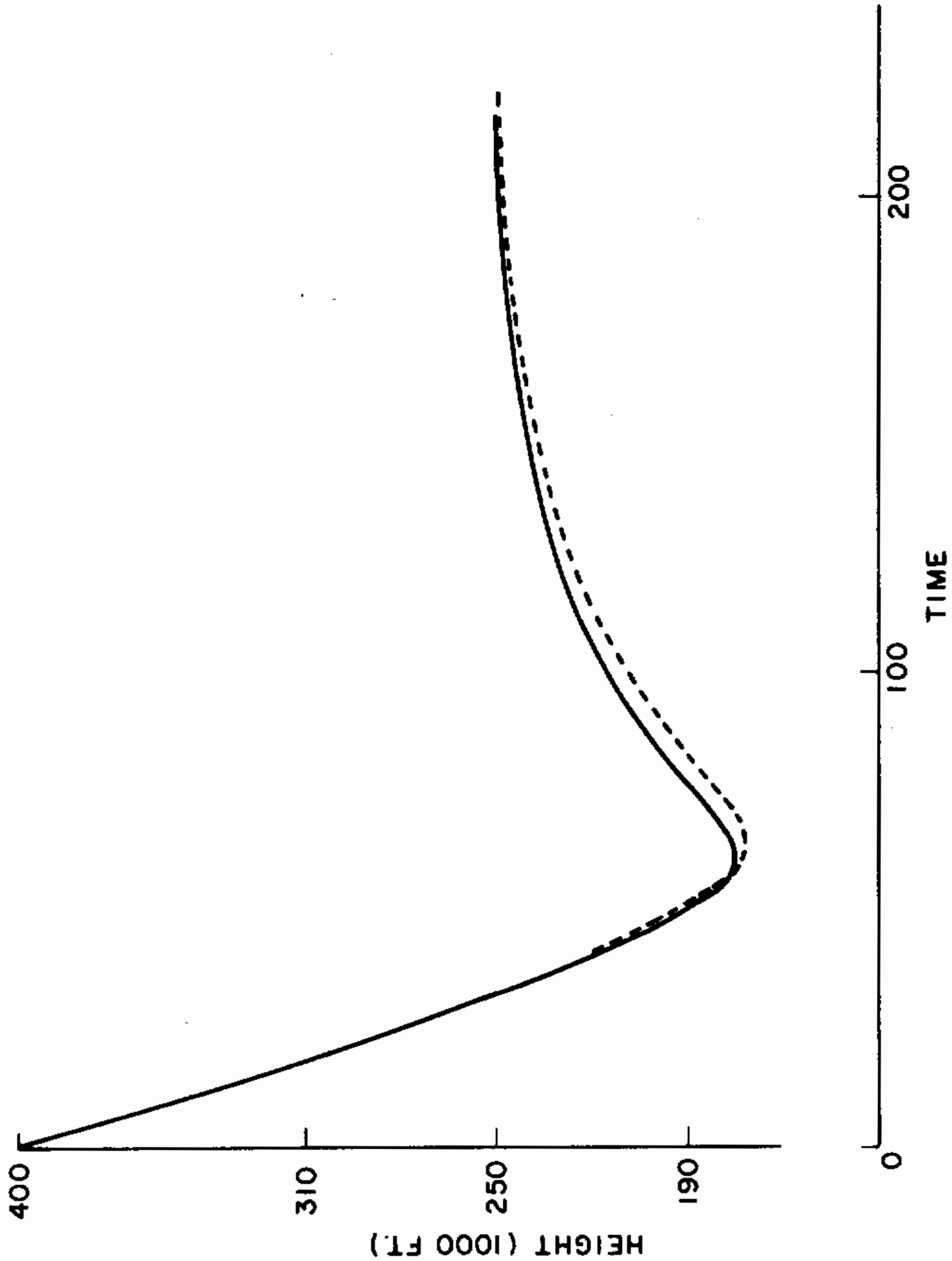

Figure 5.1. Optimal Trajectories, Time-Altitude

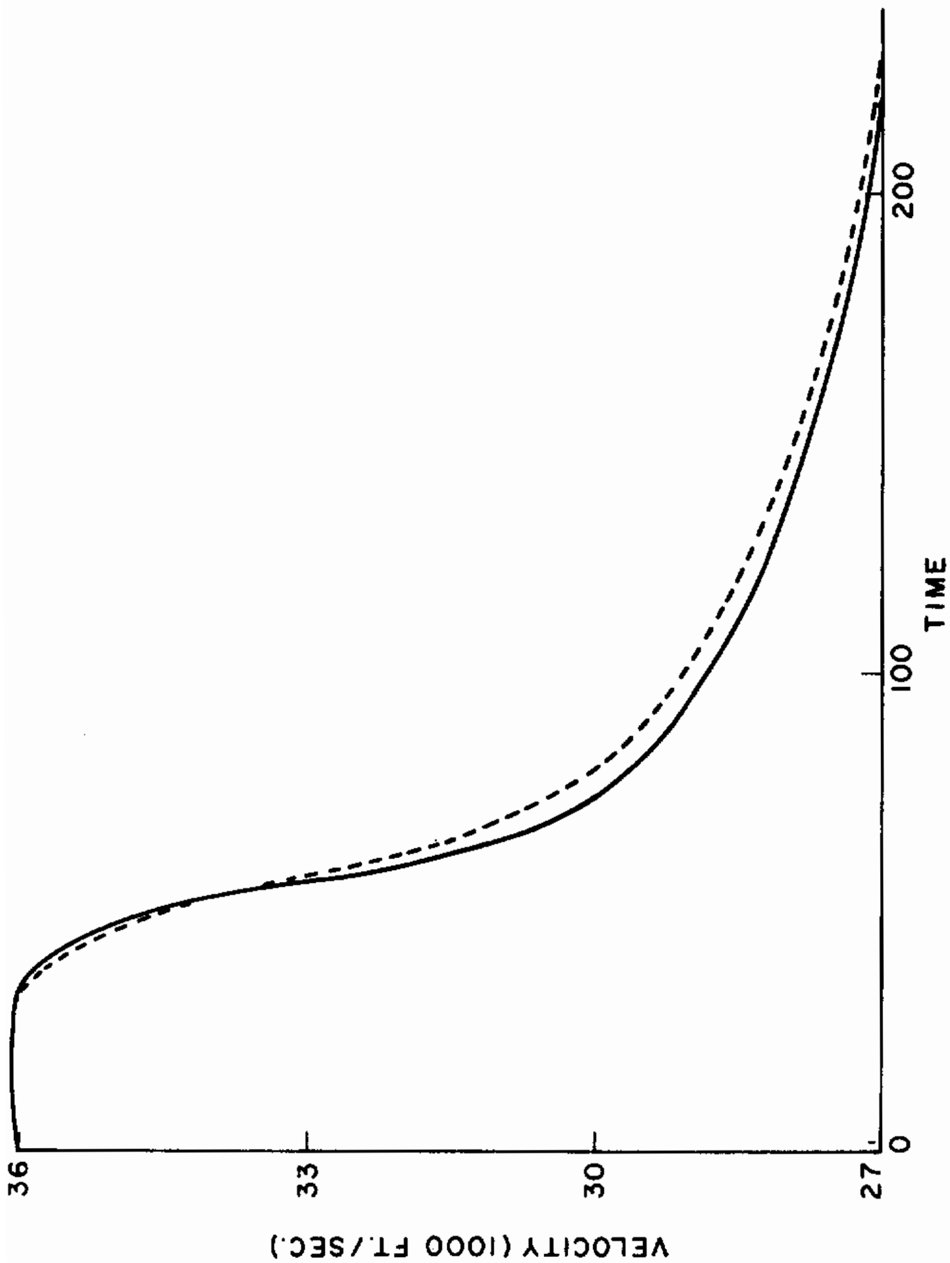

Figure 5.2. Optimal Trajectories, Time-Velocity

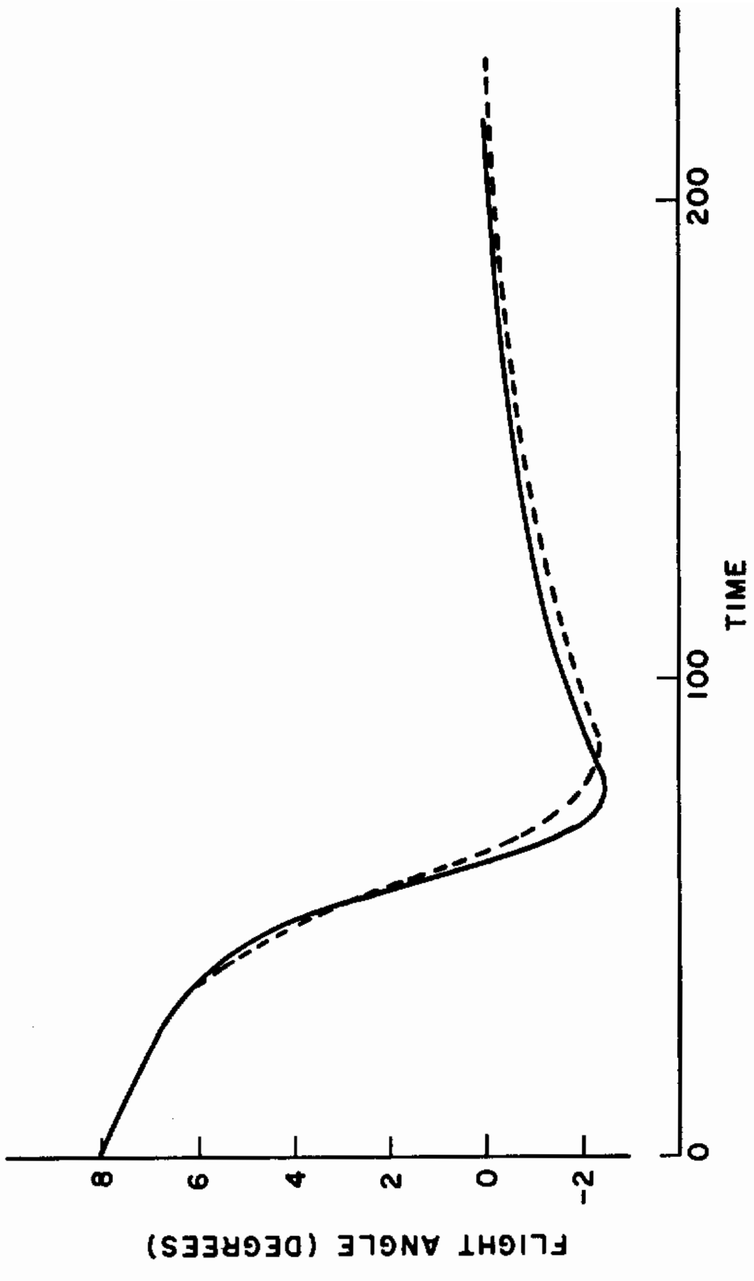

Figure 5.3. Optimal Trajectories, Time-Flight Angle

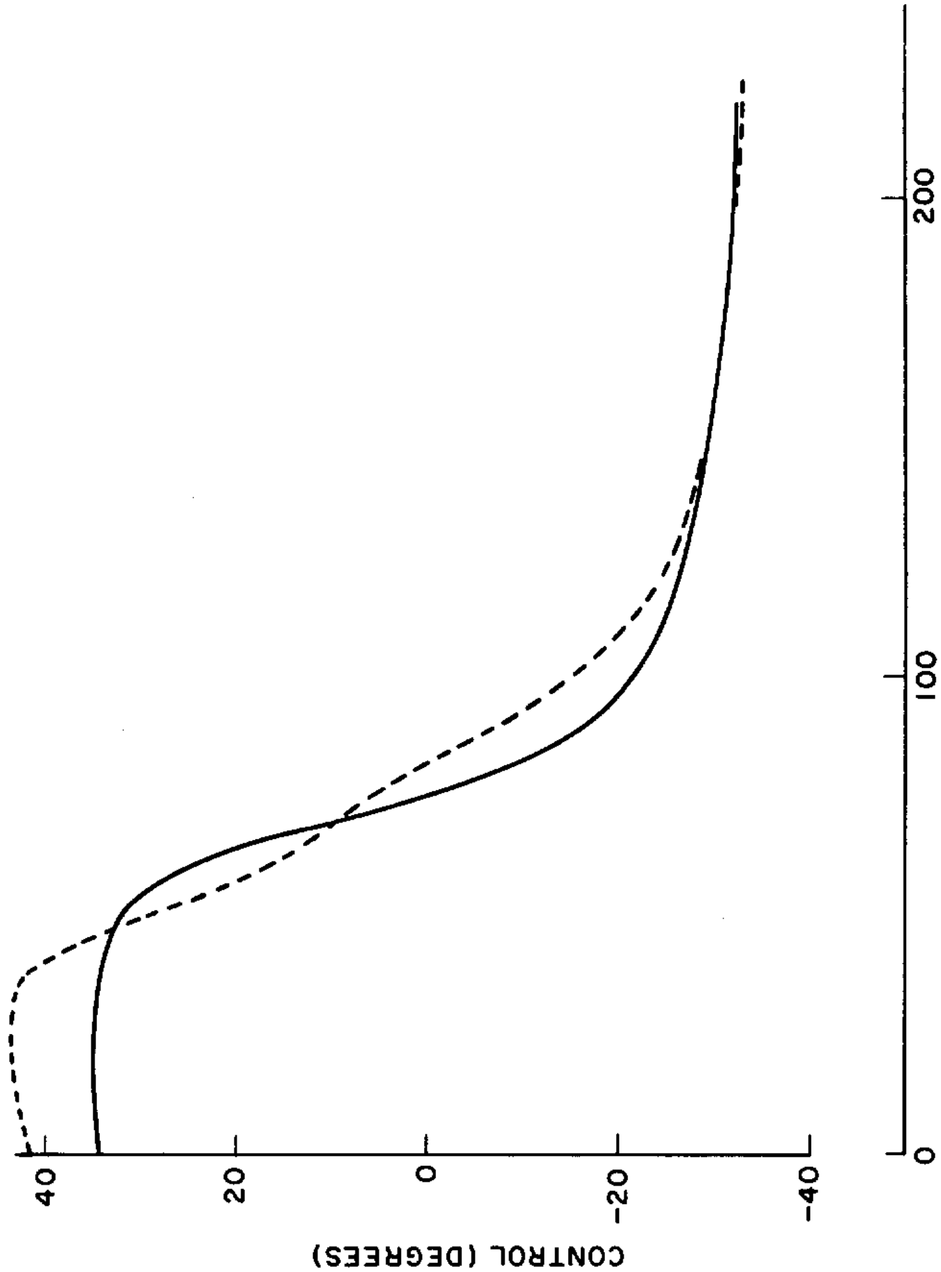

Figure 5.4. Optimal Trajectories, Control Function

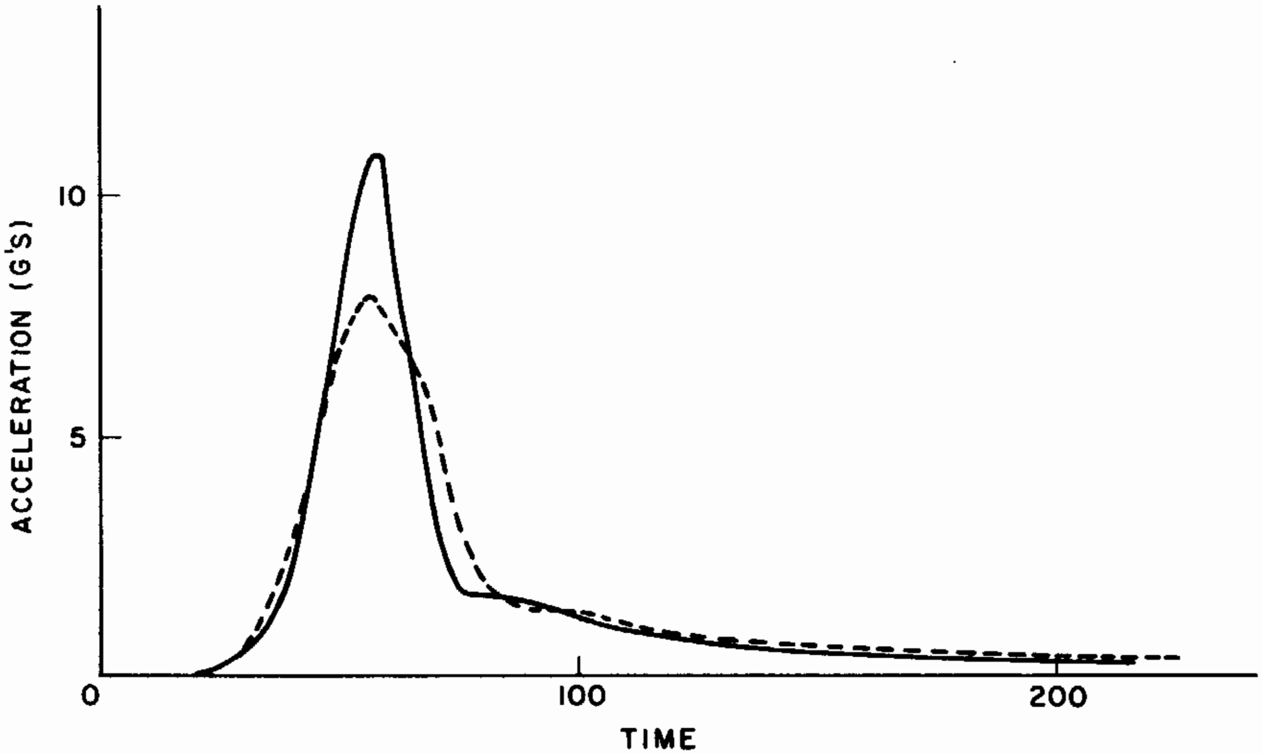

Figure 5.5. Optimal Trajectories, Acceleration Forces

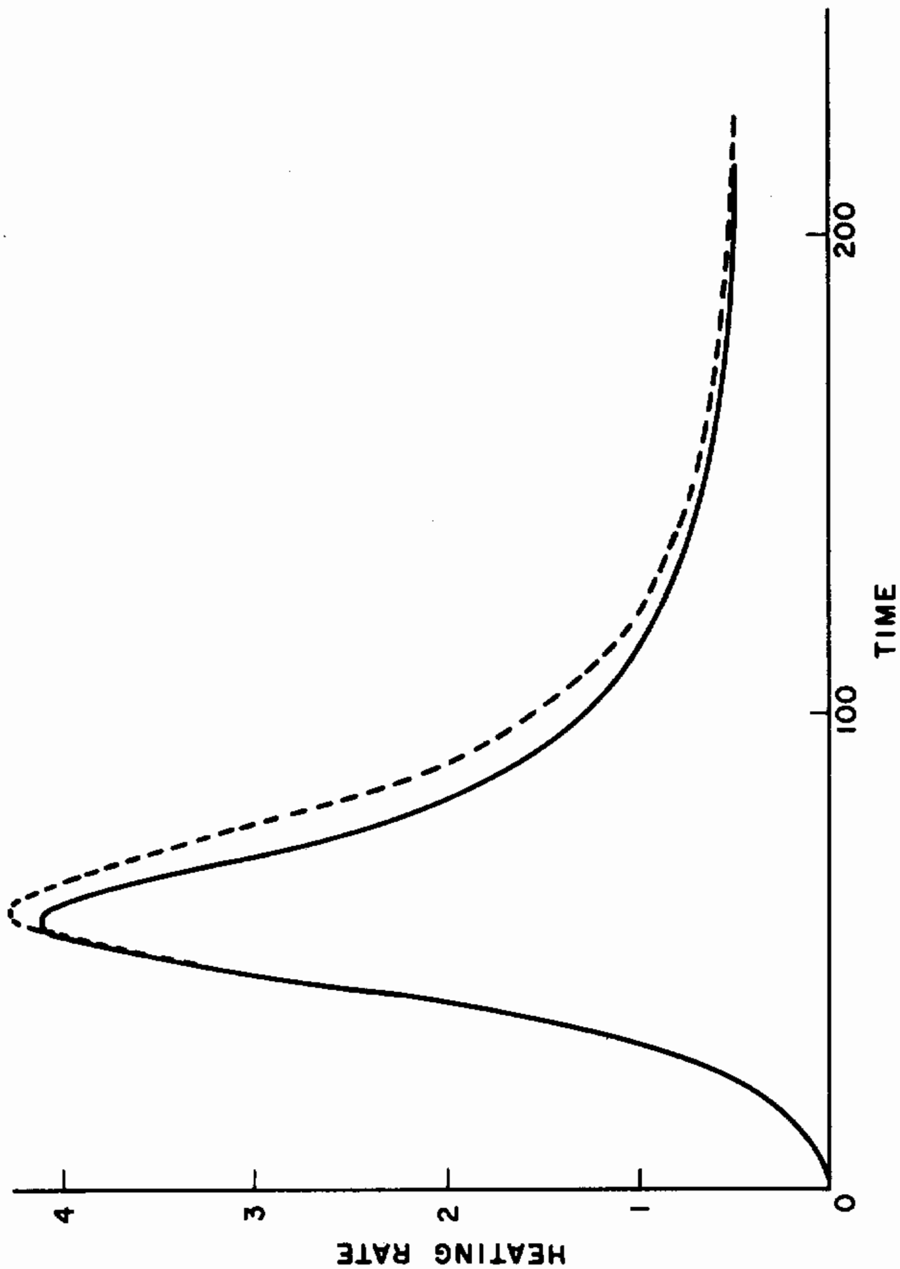

Figure 5.6. Optimal Trajectories, Heating Rate

Figure 5.7 shows the manner in which the total heat and total acceleration effects along the optimal trajectory vary with the parameter K_7. This plot can be used to determine the minimum amount of additional heating which the vehicle must absorb in order to achieve a specified reduction in acceleration effects. For example, to reduce the acceleration as shown in Figure 5.5, it is necessary that the vehicle absorb slightly more than 10% additional heat during the reentry maneuver. Moreover, Figure 5.7 can be used to guide the specification of system performance. In this respect, the most significant feature of the curve of Figure 5.7 is that small decreases in the total acceleration effects, beyond that corresponding approximately to $K_7 = 200$, require extremely large increments in heating effects. This means that to specify a total acceleration effect below, for example, $x_5(t') = 130$, would require an unreasonable increase in the amount of heat the vehicle must absorb.

The behavior of the auxiliary variables along the minimum heat ($K_7 = 0$) trajectory is shown in Figures 5.8, 5.9, and 5.10. It is known[5] that the auxiliary variables are unstable in the forward (in time) direction, i.e., that they increase exponentially. This is verified by Figures 5.8, 5.9, and 5.10. The significance of this effect for the optimization procedure is discussed in the next section.

5.4 Computational Experience with the Optimization Procedure

The most significant characteristics of the computational process were that convergence could only be obtained in a small neighborhood of the correct solution and that convergence was quite rapid in this area. These results are precisely what is indicated by Kantorovich's theorem.

When the initial estimate for γ^o was too far from the correct values, the resulting trajectory either skipped out of the atmosphere or dived into the earth's surface. This gave terminal state values which differed greatly from the desired values. For this reason it did not seem practical to converge to the desired values in a series of intermediate steps. So a trial and error procedure was used to determine sufficiently accurate initial estimates. After the first optimal trajectory was determined, K_7 was changed incrementally throughout its range. In this manner, it was possible to use the previous optimal solutions as initial estimates of γ^o for the succeeding trajectory.

The accuracy of the numerical integration was checked by comparing the results of integrations using various step sizes for the time variable and quantization levels for the control variable. It was found that the accuracy of the optimal trajectory integration was considerably better than that of the perturbation equations. In connection with this, it should be pointed out that the solution of the perturbation equations, corresponding to the initial conditions

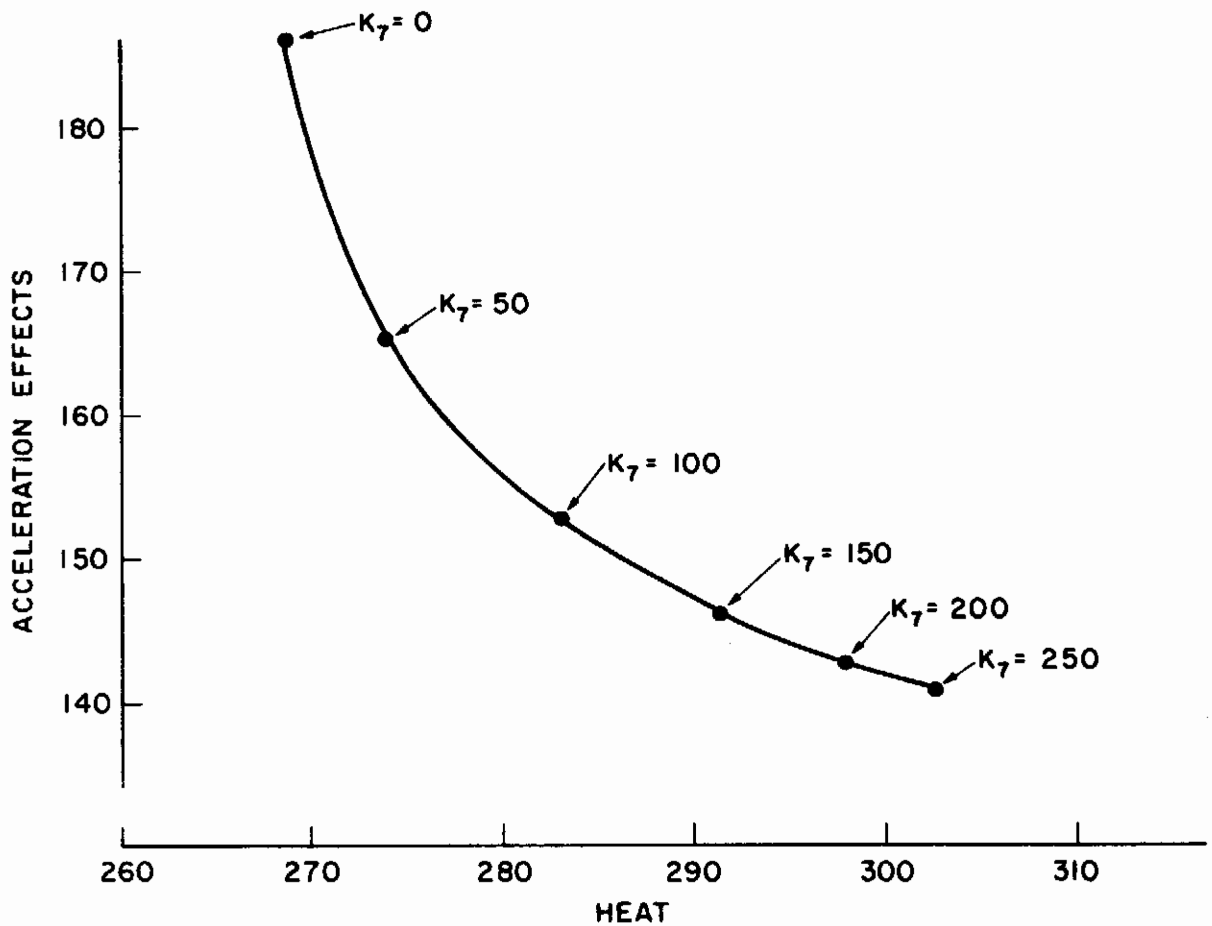

Figure 5.7. Optimal Trade-off, Heat - Acceleration Effects

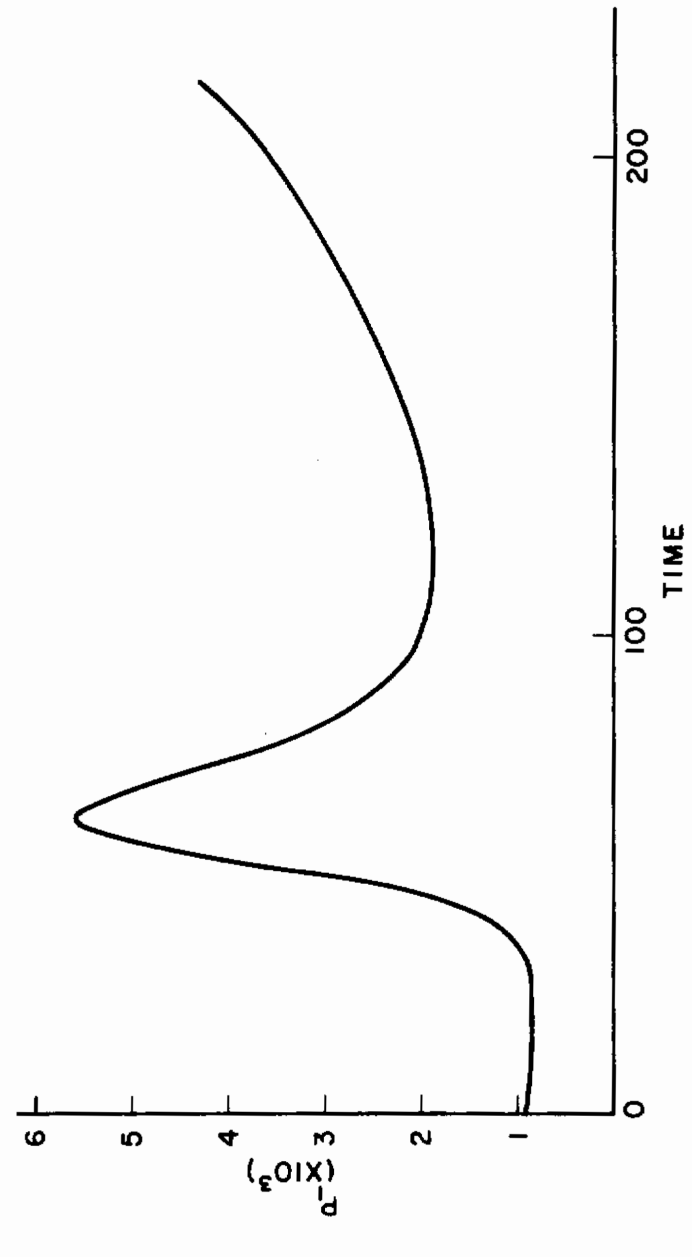

Figure 5.8. Auxiliary Variable p_1

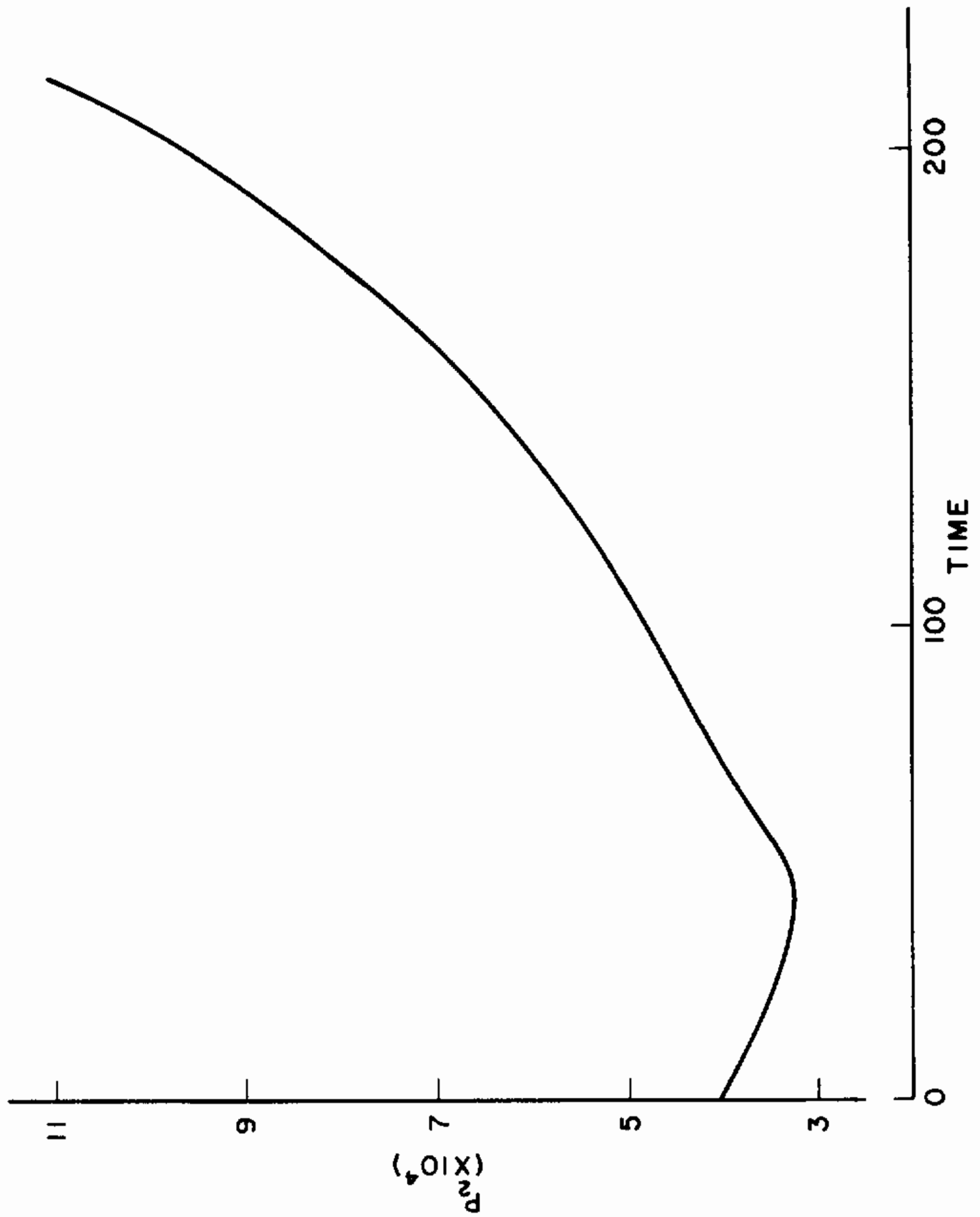

Figure 5.9. Auxiliary Variable p_2

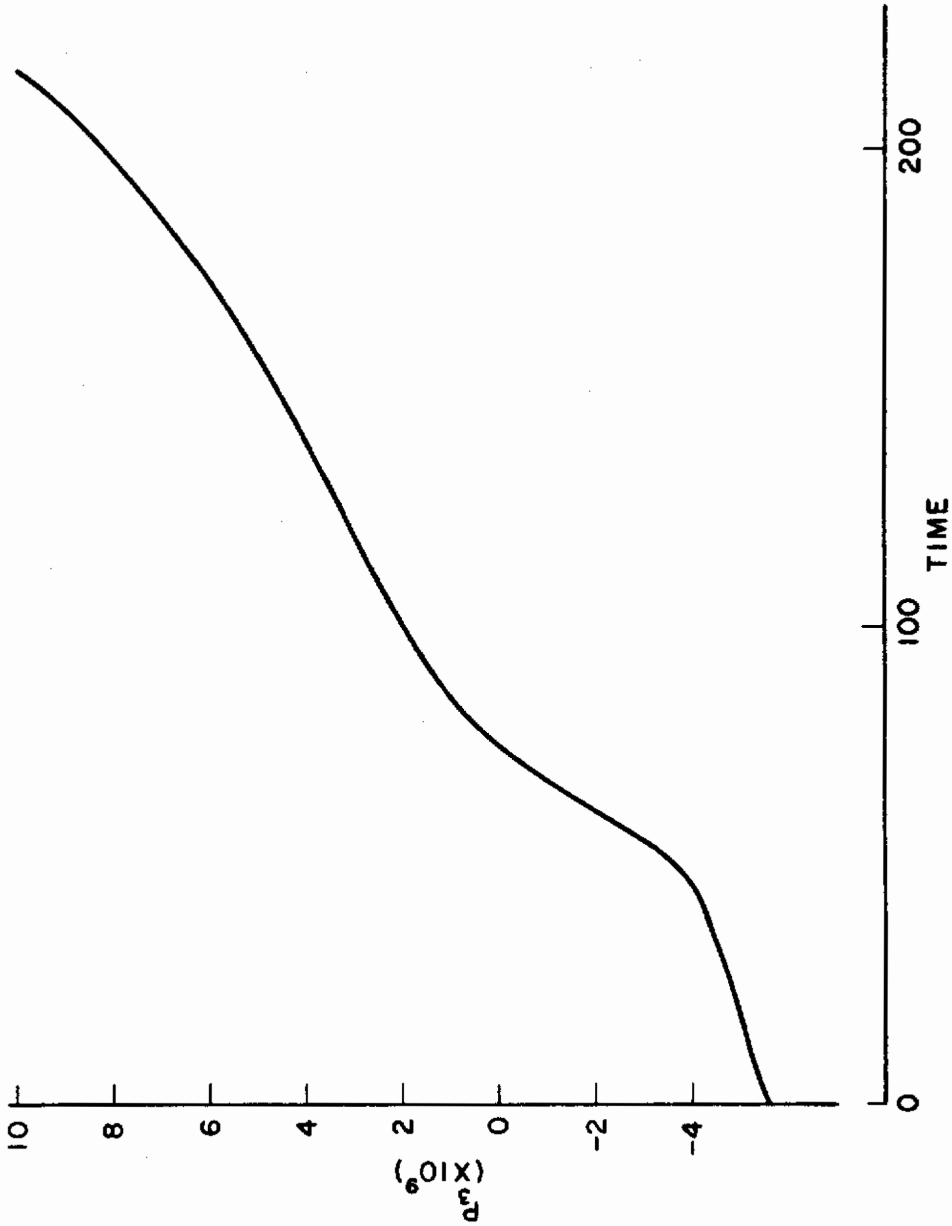

Figure 5.10. Auxiliary Variable p_3

$$\delta p_1(t^o) = 0$$

$$\delta t^o = 1$$

$$\delta p_3(t^o) = 0$$

is simply the derivatives of the state variables, i.e.,

$$\delta x_i(t) = \dot{x}_i(t) = f_i(t) \quad , \quad i = 1, 2, 3$$

and similarly for the auxiliary variables. This means that it is possible to delete this particular integration of the perturbation equations and simply use the state variables derivatives at the terminal time. However, the procedure actually used was to perform this integration and compare the terminal values $\delta x_i(t')$ to $f_i(t')$. This provided a convenient method of checking the integration accuracy for every trajectory.

The optimization procedure was run in both the forward and backward directions. That is, a complete set of initial conditions were determined such that the given terminal conditions were satisfied, by integrating in the forward (time) direction, and also, a complete set of terminal conditions were determined such that the given initial conditions were satisfied, by integrating backward in time. The resulting optimal trajectories, for corresponding values of K_7, differed only slightly. The initial and final values of the state variables were, of course, the same, and the values of the auxiliary variables at the end points differed, generally, in the fourth digit. This difference is due to the fact that the natures of integration errors are not the same for opposite directions of integration. This point is significant in relation to the "adjoint method" for solving two-point boundary value problems.[45] This method is similar to the neighboring optimum method, but uses integration in alternate directions, i.e., first forward integration, then backward, etc. The final accuracy of the adjoint method will, apparently, be limited by the alternating character of the integration errors.

By using various methods of integration it was seen that for a given integration method, the integration error built up more in the forward direction than in the reverse direction. The apparent reason for this is that auxiliary variables (as shown in Figures 5.8, 5.9 and 5.10) are increasing exponentially in the forward direction and this decreases the accuracy of the numerical integration method. The explanation of this effect is given by the manner in which local errors are propagated in the solution.[34]

In most of the computer runs the optimization procedure converged quite rapidly, if it converged at all. When running in the reverse direction, the norm of the terminal error

$$\sum_{i=1}^{3}\left[\left(x_i(t^o) - x_i^o\right) \Big/ |x_i^o|\right]^2$$

was reduced, at each iteration, by two to three orders of magnitude. That is, the norm of the error after each iteration was from 1 to 0.1 percent of the error at the start of the iteration. In these cases, at most four iterations were required to complete the solution. For this situation the Newton-Raphson procedure is fully adequate.

In a few cases, the terminal errors tended to oscillate about the correct values as the solution converged. For these cases, an alternate iterative algorithm (as discussed in Section 3.1), would be helpful.

In every case there was a limit as to the final accuracy that could be obtained. When integrating in the reverse direction it was possible to match the specified initial conditions to six decimal digits and in the forward direction the specified terminal conditions could be matched to five digits. Considering that the program used only eight significant digits for all numbers, this accuracy is very good. The limitation on final accuracy is due primarily to integration errors.

Various experiments in integrating optimal trajectories indicated that when using a fixed number of significant digits, the ultimate accuracy that could be obtained was limited by the accuracy with which the value of the optimal control could be obtained. In standard problems, integration accuracy is limited by round-off errors. But, at least in the problem considered here, the value of the optimal control could not be determined to an accuracy comparable to that of the other variables involved in the integration. Evidently, the reason for this is that the variation of the Hamiltonian with respect to the control is very small in the neighborhood of the optimum control value.

The most convenient way to increase the final accuracy would be to carry out the computation using more significant figures for the variables. The computing machine and compiler (IBM 7094 and FORTRAN IV) used have a double word length capability. It would not require extensive changes to modify the program to use sixteen significant digits for representation of the variables. The time required for double length computation is estimated to be approximately 25% greater than for single length.

The average computing time required for each iteration of the optimization procedure was approximately 45 seconds. This included an integration of the optimal trajectory equations, three integrations of the perturbation equations, a solution of the correction procedure and a second integration of the optimal trajectory to check the results of the correction. All of the integrations used a fixed step-size of 0.5 seconds.

Based on results obtained with other aerodynamic trajectories, it appears that a possible reduction in integration time of up to 50% could be obtained by the use of a variable step-size feature in the integration routine.

CHAPTER 6

AN OPTIMAL LINEAR CONTROL SYSTEM

6.1 Derivation of the Optimum Linear Control Law

The feedback control problem for a reentry vehicle was introduced in Section 4.4, where it was stated that the optimum linear feedback control law is equivalent to the correction procedure of the neighboring optimum optimization method. The procedure for obtaining this control law is discussed in this section. It should be noted that this is not an optimal control law but rather the best approximation which can be obtained by using a linear (time varying) control law.

In Section 2.4 a method was discussed for obtaining n linearly independent solutions of the system of differential equations involving the deviation variables (2.31 and 2.32). If it is assumed that these equations are solved by integrating backward in time from t' to t^o, then the solutions

$$\delta x(t^o) = \left(\delta x_1(t^o),\ \delta x_2(t^o), \ldots, \delta x_n(t^o)\right)$$

$$\delta p(t^o) = \left(\delta p_1(t^o),\ \delta p_2(t^o), \ldots, \delta p_n(t^o)\right)$$

are functions of the terminal conditions $\delta x(t')$, $\delta p(t')$, for a given nominal trajectory. If it is further assumed that these terminal conditions satisfy the boundary conditions given by (2.37), (2.38) and (2.39), then the independent deviation variables at t' may be chosen as

$$\delta\gamma = (\delta p_1,\ \delta p_2, \ldots, \delta p_{q-1},\ \delta t,\ \delta x_{q+1}, \ldots, \delta x_n) \tag{6.1}$$

The n independent solutions of (2.31) and (2.32) may be found by successively using the initial conditions

$$\begin{aligned} \delta\gamma^1 &= (1, 0, 0, \ldots, 0) \\ \delta\gamma^2 &= (0, 1, 0, \ldots, 0) \\ &\ \ \vdots \\ \delta\gamma^n &= (0, 0, 0, \ldots, 1) \end{aligned} \tag{6.2}$$

If we denote by $\theta_1(t)$ the solution $\delta x(t)$, and by $\phi_1(t)$ the solution $\delta p(t)$ corresponding to $\delta\gamma^1$, and similarly define θ_j, and ϕ_j for $j = 2, 3, \ldots, n$ then we may write

$$\begin{bmatrix} \delta x(t) \\ \delta p(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \phi(t) \end{bmatrix} \delta\gamma \tag{6.3}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ and $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$. A necessary condition for the existence of neighboring optimum trajectories is that $\theta(t)$ be nonsingular over the interval $t^o \leqq t < t'$. Note that at the final time t' the solutions θ_i are determined completely by the boundary conditions $\delta\gamma^i$. And since the boundary conditions on δx_i are not independent, $\theta(t')$ will be singular. Points at which $\theta(t)$ is singular are referred to as "conjugate points" in the literature of the Calculus of Variations.[46]

By inverting Equation (6.3) it is possible to obtain $\delta\gamma$ in terms of initial state deviations $\delta x(t^o)$:

$$\delta\gamma = \left[\theta(t^o)\right]^{-1} \delta x(t^o) \tag{6.4}$$

By substituting back into Equation (6.3) we obtain

$$\begin{bmatrix} \delta x(t) \\ \delta p(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \phi(t) \end{bmatrix} \left[\theta(t^o)\right]^{-1} \delta x(t^o) \tag{6.5}$$

Now Equation (3.14) gives the control deviation δu as a function of δx, δp and the nominal optimum trajectory. So by substituting (6.5) into (3.14) we obtain the control deviations in terms of the initial state deviations:

$$\delta u(t) = -\left(\frac{\partial^2 H}{\partial u^2}\right)^{-1} \left[\left(\frac{\partial^2 H}{\partial u \partial x}\right)\theta(t) + \left(\frac{\partial^2 H}{\partial u \partial p}\right)\phi(t)\right] \left[\theta(t^o)\right]^{-1} \delta x(t^o) \tag{6.6}$$

This equation gives a control correction which will cause the system to follow a neighboring optimum trajectory $x(t)$, $t^o \leqq t < t'$ to a nominal trajectory $\bar{x}(t)$, $t^o \leqq t \leqq t'$ based on a measured initial derivation $\delta x(t^o) = x(t^o) - \bar{x}(t^o)$. That is, the control input $u(t) = \bar{u}(t) + \delta u(t)$, where $\bar{u}(t)$ is the control corresponding to the nominal trajectory and $\delta u(t)$ is determined by (6.6), applied to the system will cause the system to follow an optimal trajectory from the initial point $x(t^o)$ to the terminal point $\bar{x}(t')$ (assuming the initial deviation $\delta x(t^o)$ is sufficiently small).

The use of time t^o in Equation (6.4) was arbitrary, and any time t, $t^o \leqq t < t'$ could have been used. Thus Equation (6.6) can be rewritten to give the control deviation in terms of the state deviations measured at any time in the interval $t^o \leqq t < t'$

$$\delta u(t) = -\left(\frac{\partial^2 H}{\partial u^2}\right)^{-1} \left[\left(\frac{\partial^2 H}{\partial u \partial p}\right) + \left(\frac{\partial^2 H}{\partial u \partial p}\right) \phi(t) \, [\theta(t)]^{-1}\right] \delta x(t) \qquad (6.7)$$

Equation (6.7) gives the control law for a continuous feedback system of the type shown in Figure 4.4. The feedback compensation K is identified immediately by rewriting (6.7) as:

$$\delta u(t) = K(t) \; \delta x(t) \qquad (6.8)$$

From (6.7) and (6.8) it is clear that the feedback gains matrix K(t) is a function of the nominal trajectory and of the n linearly independent solutions of the perturbation differential equations.

The neighboring optimum feedback control law for the more general case, which includes deviations from the specified terminal conditions as well as deviations from the nominal state trajectory, is given by Breakwell, Speyer and Bryson.[16]

6.2 The Time Distortion Problem in Neighboring Optimum Control

The neighboring optimum control scheme discussed in the previous section has been considered from various viewpoints by several authors.[16,17,47,48] In all of these studies it was assumed that the deviations from the nominal state trajectory would be measured by comparing the actual state x(t), $t^o \leqq t < t'$ to the value of the nominal state $\bar{x}$ at the corresponding time t. This assumption neglected the fact that the nominal trajectory and the neighboring optimum trajectory have, in general, different time durations. Specifically, the nominal trajectory time $(\bar{t})$ is related to the neighboring optimum trajectory time by

$$t' - t = \bar{t}' - \bar{t} + \delta t \qquad (6.9)$$

where δt may be determined as a function of $\delta x(t)$ by Equation (6.4). Since the neighboring optimum control method is only a first order approximation to optimum control, it is not immediately obvious that neglecting this time difference δt would introduce additional errors. However, Kelley[17] has demonstrated the significant effects that occur in a simple example when this time difference is included when measuring state deviations. This point can best be understood by considering the example.

Zermelo's Problem: This example considered the motion of a ship driven at a constant velocity V relative to the water through a current which has constant velocity components v and w in the x_1 and x_2 directions, as shown in Figure 6.1. The problem to be solved is that of finding the path of minimum time from the initial point (x_1^o, x_2^o) to the terminal point $(x_1^!, x_2^!)$. This path is a function of the ship steering direction u, which (as in Figure 6.1) is considered to be the angle between the axis x_2 and the direction in which the ship's power is applied. The equations of motion are

$$\dot{x}_1 = V \sin u + v \tag{6.10}$$

$$\dot{x}_2 = V \cos u + w$$

The criterion function for the minimum time problem is

$$J = \int_{t^o}^{t'} (\dot{x}_o)\, dt = \int_{t^o}^{t'} dt = (t' - t^o) \tag{6.11}$$

Using the procedure of the Maximum Principle, we obtain

$$\dot{p}_1 = 0$$

$$\dot{p}_2 = 0 \tag{6.12}$$

$$H = -1 + p_1(V \sin u + v) + p_2(V \cos u + w) = 0 \tag{6.13}$$

$$\frac{\partial H}{\partial u} = p_1 V \cos u - p_2 V \sin u = 0 \tag{6.14}$$

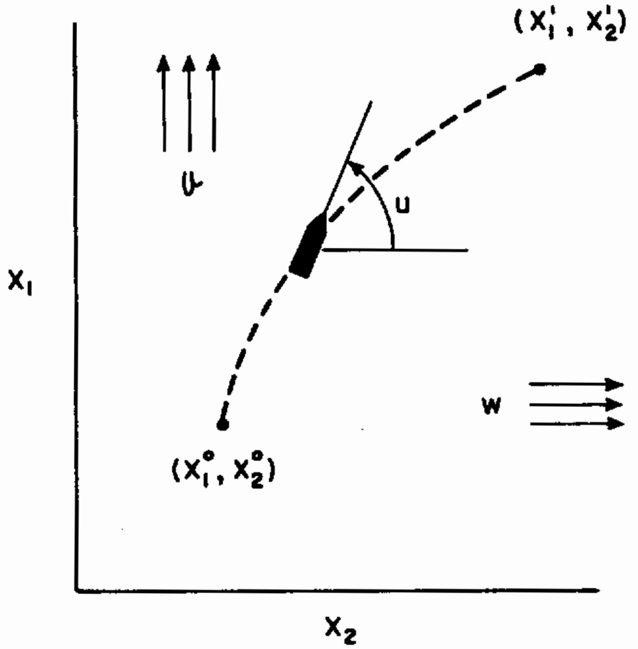

Figure 6.1. Diagram for Zermelo's Problem

Equation (6.14) implies that the optimal control satisfies the equation

$$u = \arc\tan \frac{p_1}{p_2} \tag{6.15}$$

Also, by (6.12), the auxiliary variables p_1 and p_2 are constants. So the optimal control is a constant, and the optimal paths (trajectories) in the x_1- x_2 plane are straight lines from x^o to x'.

For this example, given values for the constants V, v and w, and specified boundary points $\left(x_1^o = x_1(t^o),\ x_2^o = x_2(t^o),\ x_1' = x_1(t'),\ x_2'(t')\right)$, the equations

$$x_1' = (V \sin u + v)\,(t' - t^o) + x_1^o$$
$$x_2' = (V \cos u + w)\,(t' - t^o) + x_2^o \tag{6.16}$$

can be solved for the optimal (constant) control u and the minimum time $(t' - t^o)$.

If we assume the values V = 1, v = 0.5, w = 0 and the nominal boundary points $\bar{x}^o = (0,0)$, $\bar{x}^1 = (1,2)$, the nominal optimal control is $\bar{u} = 0$ and the minimum time is $t' - \bar{t}^o = 2$. This corresponds to the nominal optimal trajectory

$$\left.\begin{array}{l}\bar{x}_1(t) = 0.5\,(t - \bar{t}^o)\\[2ex] \bar{x}_2(t) = (t - \bar{t}^o)\end{array}\right\} ,\quad \bar{t}^o \leqq t \leqq \bar{t}' \tag{6.17}$$

For this simple problem, the neighboring optimum control law can be obtained as an analytic expression. The differential equations for the deviation variables about a nominal optimal trajectory are:

$$\delta\dot{x}_1 = V \cos \bar{u}\ \delta u$$
$$\delta\dot{x}_2 = -V \sin \bar{u}\ \delta u$$
$$\delta\dot{p}_1 = 0$$
$$\delta\dot{p}_2 = 0 \tag{6.18}$$

And the perturbation equation of the optimal control condition is

$$\delta p_1 \cos \bar{u} - \delta p_2 \sin \bar{u} - [\bar{p}_1 \sin \bar{u} + \bar{p}_2 \cos \bar{u}]\ \delta u = 0 \tag{6.19}$$

If we assume that the nominal and neighboring optimum paths have the same starting time (i.e., $t^o = \bar{t}^o$), then the deviation variables satisfy the boundary conditions

$$\delta x_1(t^o) = x_1(t^o) - \bar{x}_1(t^o)$$

$$\delta x_2(t^o) = x_2(t^o) - \bar{x}_2(t^o) \qquad (6.20)$$

$$\delta x_1(\bar{t}') + (V \sin \bar{u} + v)\, \delta t' = 0$$

$$\delta x_2(\bar{t}') + (V \cos \bar{u} + w)\, \delta t' = 0 \qquad (6.21)$$

Finally, the perturbation equation for the transversality condition (6.13) is

$$\delta p_1(\bar{t}') [V \sin \bar{u} + v] + \delta p_2(\bar{t}') [V \cos \bar{u} + w] = 0 \qquad (6.22)$$

Using the nominal optimal control value $\bar{u} = 0$, we find that

$$\delta \dot{x}_1 = V\, \delta u$$

$$\delta \dot{x}_2 = 0$$

$$\delta u = (\delta p_1 / p_2) = V\, \delta p_1 \qquad (6.23)$$

The independent deviation variables for this problem are

$$\delta \gamma = [\delta p_1(t'),\ \delta t'] \qquad (6.24)$$

The solutions of (6.18) corresponding to $\delta\gamma = [1, 0]$ are

$$\delta x_1(t) = -V^2(t' - t)$$

$$\delta x_2(t) = 0$$

$$\delta p_1(t) = 1$$

$$\delta p_2(t) = -2$$

and corresponding to $\delta\gamma = [0, 1]$, they are

$$\delta x_1(t) = -v$$

$$\delta x_2(t) = -V$$

$$\delta p_1(t) = 0$$

$$\delta p_2(t) = 0$$

Thus the matrices θ and ϕ are:

$$\theta = \begin{bmatrix} -V^2(t'-t) & -v \\ 0 & -V \end{bmatrix}$$

$$\phi = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

And the equation for the control deviations (6.7) is

$$\delta u(t) = -(-1)\left[[0,0] + [V,0]\left(\frac{1}{V^3(t'-t)}\right)\begin{bmatrix} -V & v \\ 0 & -V^2(t'-t) \end{bmatrix}\right]\delta x(t)$$

$$\delta u(t) = -\frac{\delta x_1(t)}{V(t'-t)} + \frac{v\ \delta x_2(t)}{V^2(t'-t)} \qquad (6.25)$$

Also, by use of Equation (6.4) the time difference is found to be

$$\delta t = -\delta x_2/V \qquad (6.26)$$

Now that the neighboring optimum control law (6.25) is available, we shall consider a particular numerical example that demonstrates the action of this type of control system. Assume that in the above example the starting point of the ship is $x^o = (0, 1)$ and the destination point is $x' = (1, 2)$. This problem can be considered to be the same problem as solved above for the nominal starting point $\bar{x}^o = (0, 0)$ but with an initial error of

$$\delta x_1(t^o) = x_1^o - \bar{x}_1^o = 0$$

$$\delta x_2(t^o) = x_2^o - \bar{x}_2^o = 1 \qquad (6.27)$$

Using this value of $\delta x(t^o)$ and the values $V = 1$, $v = 0.5$ and $w = 0$, Equation (6.25) gives the control correction

$$\delta u(t^o) = +.25 \text{ (radians)} \qquad (6.28)$$

and the change in time is, from (6.26),

$$\delta t' = -1 \qquad (6.29)$$

The values given in (6.28) and (6.29) represent the approximate solution given by the neighboring optimum method. The exact solution, found by solving (6.16), corresponds to the corrections

$$\delta u(t^{o}) = +.425$$
$$\delta t' = -.9 \tag{6.30}$$

The corrections (6.28) and (6.29) correspond to making a single error measurement at time t^{o}. Next, assume that the approximate correction (6.28) is used and the system operates over the interval $0 \leqq t \leqq 0.5$, then the values for the state variables would be

$$x_1(0.5) = .3735$$
$$x_2(0.5) = 1.4845 \tag{6.31}$$

If the error $\delta x(1/2)$ is now measured, the corresponding neighboring optimum corrections will be

$$\delta u(.5) = +.2456$$
$$\delta t' = -.9845 \tag{6.32}$$

whereas the exact solution is

$$\delta u(.5) = +.56$$
$$\delta t = -.89 \tag{6.33}$$

The trajectory of the neighboring optimum control system for this case is shown in Figure 6.2, where the nominal optimal trajectory is also shown.

We now solve the above numerical example making use of the time change δt in measuring the state errors as suggested by Kelley. The procedure is to first measure the error (6.27) and calculate the corresponding time change (6.29). Then the next step is to remeasure the state error but define time along the neighboring optimum to be $t = \bar{t} - \delta t$, thus

$$\delta x(t) = x(t) - \bar{x}(\bar{t} - \delta t) \tag{6.34}$$

And

$$\delta x_1(0) = x_1(0) - \bar{x}_1(0+1) = -0.5$$
$$\delta x_2(0) = x_2(0) - \bar{x}_2(1) \quad = 0 \tag{6.35}$$

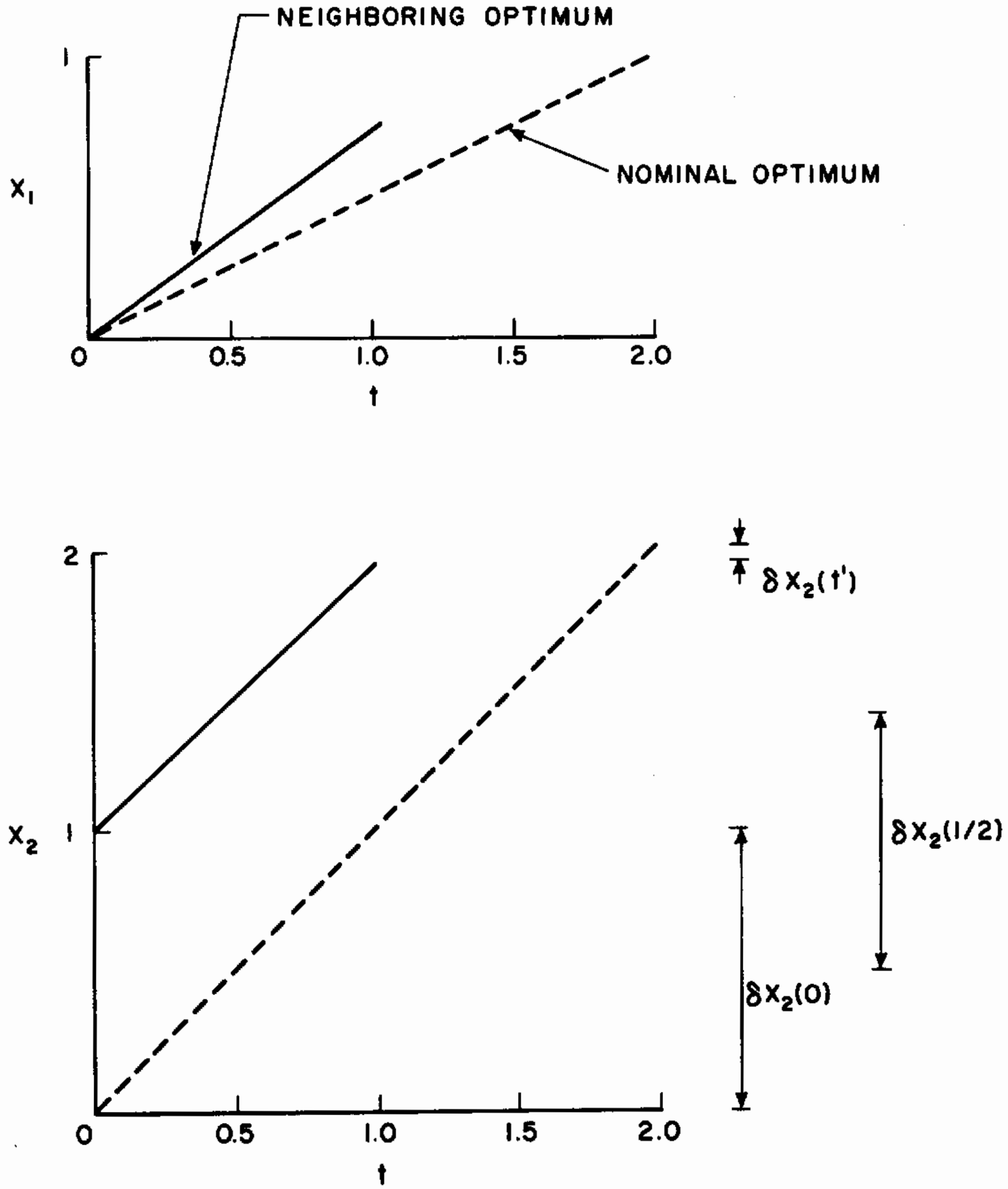

Figure 6.2. Control Results - Standard Time Comparison

Using the error (6.35) the neighboring optimum control correction is

$$\delta u(0) = + .5 \tag{6.36}$$

The exact solution is, of course, (6.30). Again we assume that this control correction is used over the period $0 \leqq t \leqq 0.5$, giving the following values for the state variables

$$x_1(0.5) = 0.4895$$
$$x_2(0.5) = 1.439 \tag{6.37}$$

The corresponding change in time is

$$\delta t = -\delta x_2 = -(1.439 - .5) = - .939 \tag{6.38}$$

Using (6.38) with (6.34) gives the state errors as

$$\delta x_1(.5) = x_1(.5) - \bar{x}_1(1.439) = - .23$$
$$\delta x_2(.5) = x_1(.5) - \bar{x}_1(1.439) = 0 \tag{6.39}$$

This gives the neighboring optimum control correction

$$\delta u(.5) = + .41 \tag{6.40}$$

The exact solution corresponding to (6.37) is

$$\delta u = + .36$$
$$\delta t = - .898 \tag{6.41}$$

The neighboring optimum control system trajectory for this case is shown in Figure 6.3, where the time coordinate corresponds to time along the nominal optimal trajectory.

It is clear from the above numerical results, that the terminal error of the neighboring optimum control scheme is greatly reduced by using the time change δt in measuring the state errors. And it is interesting to note that in the first case, making the second correction (at $t = 0.5$) did not improve the control value, while, in the second case, there was significant improvement. An intuitive explanation for this result is clear from a consideration of Figures 6.2 and 6.3. From these figures it is seen that the errors in the first case are larger than in the second case and they do not decrease with time as in the second case. Now since the neighboring optimum control method uses only the first order terms in

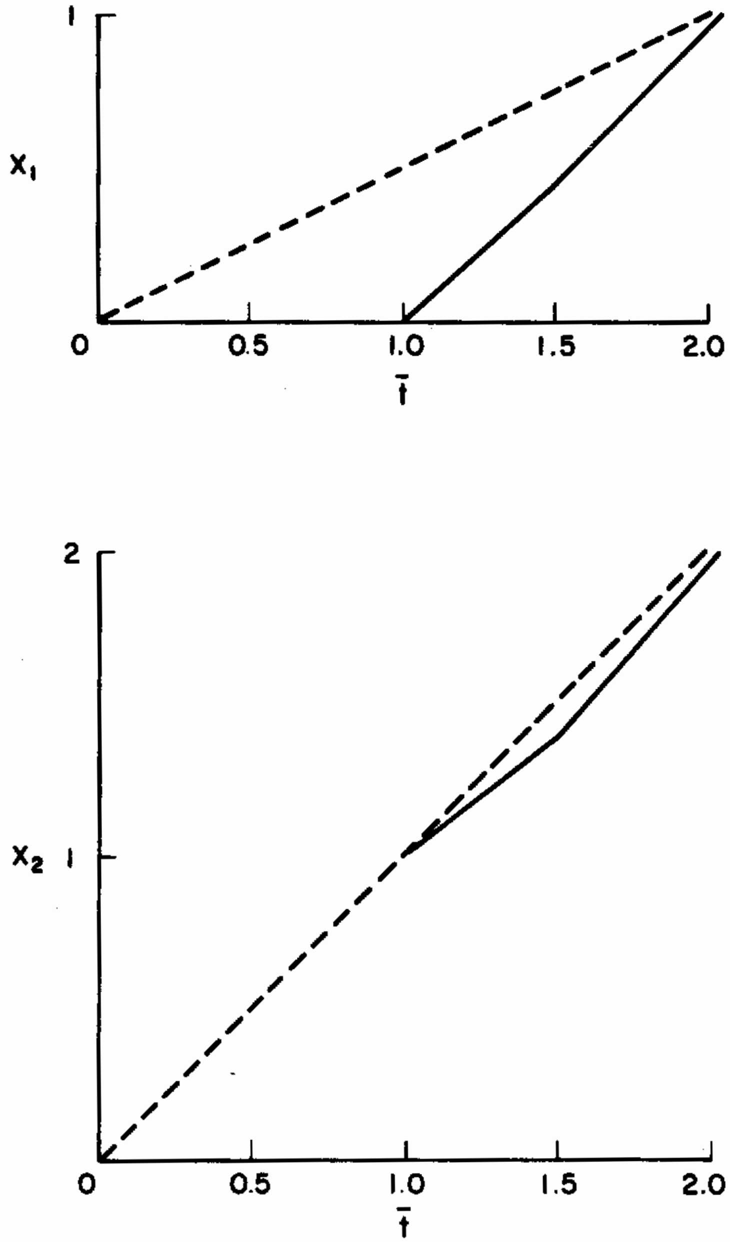

Figure 6. 3. Control Results - Transversal Time Comparison

approximating the optimal control, the approximation becomes less accurate as the size of the state variable error increases. Thus in the second case, where the state variable error δx is smaller, the control correction δu is more accurate. Furthermore, as δx decreases, the accuracy of δu increases, which means that, in the second case, continuous error measurement and continuous computation of δu will completely cancel the terminal state error.

It should be noted that, as in the above example, a smaller value of state variable error does not necessarily mean a smaller value for the control correction.

In the above example the criterion function was simply time, and including the time change δt corresponded to making the comparison between the nominal trajectory and the actual trajectory at the same "time-to-go". Kelley refers to this as an example of transversal comparison, where in general the comparison would be on the basis of "distance-to-go" for the criterion function. That is, the actual trajectory at time t would be compared to the nominal trajectory at time $\bar{t}$ which satisfies the relation

$$J(x,t) - J(x',t') = \bar{J}(\bar{x}, \bar{t}) - \bar{J}(x', \bar{t}') \tag{6.42}$$

In order to find $\bar{t}$ it is necessary to estimate the change in the criterion function at the end point (t') of the neighboring optimal trajectory and the value of $\bar{J}(x',\bar{t}')$. Such estimates are given by Kelley;[17] he also makes the statement that such a comparison "...is more natural, and, in an error propagation sense, optimal." However, this statement is not justified in that paper.

It appears that on the basis of the theoretical derivation of the neighboring optimum control method, the natural method of comparing the nominal and neighboring trajectories is by use of the time change δt computed by use of Equation (6.4). Also such an approach is much easier to implement than is the transversal comparison. However, it intuitively seems that the optimum method of comparison would be to use that time along the nominal trajectory $\bar{t}$ which gives a minimum value (in the sense of some norm) to the error

$$\delta x(t) = x(t) - \bar{x}(\bar{t}) \tag{6.43}$$

Because of the difficulty in implementing this approach, or the transversal comparison, and because the three methods will probably not give significantly different results for the terminal state, the method of using the computed value of δt will be used in the simulation studies of the reentry vehicle control system. The justification for assuming that the three methods will not differ greatly in their terminal state error is that the three methods should give the same time $\bar{t}$ over the final portion of the state trajectory.

Assuming that δt is the q'th component of $\delta\gamma$, then by Equation (6.4) we may write

$$\delta t(t) = \psi(t)\ \delta x(t) \tag{6.44}$$

where ψ is the q'th row of $[\theta(t)]^{-1}$. As pointed out previously, t^o in Equation (6.4) may be replaced by t.

The continuous control system for this case is represented in Figure 6.4. As indicated, the total time change $\delta t = \bar{t} - t$ is not continuously recomputed, but is only corrected by computing $(\delta t)'$ which makes use of the state error measured by comparison with $\bar{x}(\bar{t})$. Note that the nominal control $\bar{u}$, the feedback gain matrix K and the time change vector ψ are all evaluated at $\bar{t}$.

The time change δt, at any instant, represents a time shift between the actual time t and the nominal trajectory time $\bar{t}$. However, δt is not a constant throughout the trajectory time interval $t^o \leqq t \leqq t'$, so this method of comparison introduces a time distortion of the nominal trajectory rather than simply a time shift.

6.3 The Nominal Reentry Trajectory and Control Law

The reentry trajectory corresponding to the weighting function value, $K_7 = 100$, was chosen as the nominal trajectory for the simulation studies. This trajectory is shown in Table 6.1.

The time variable in Table 6.1 goes in the negative direction from t'(0 sec.) to t^o (-218 sec.), corresponding to the manner in which the integration was performed. Comparing the trajectory of Table 6.1 to that of Table 5.3, the only significant difference is in the value of the control variable in the neighborhood of t^o. This reflects the difference in accuracy for the forward and backward integration procedures.

Values for the components of the control gains vector K(t) of Equation (6.8) are listed in Table 6.1. At each step of the integration (i.e., at each 0.5 second interval) the values of the components K_i, i = 1, 2, 3, were computed and stored. The computation of K was accomplished by performing the operations indicated by Equation (6.7). It is interesting to note that the values of the functions $K_i(t)$ do not have constant signs. This means that the corrections δu resulting from an error δx do not always tend to cancel the error. Hence the control system is not a path following type as shown in Figure 4.5, but instead it will generate a trajectory having the characteristic of the neighboring optimum path shown in Figure 4.6.

The true values for the control gains at the terminal time t' are infinite. This corresponds to a singular point for the matrix θ. The

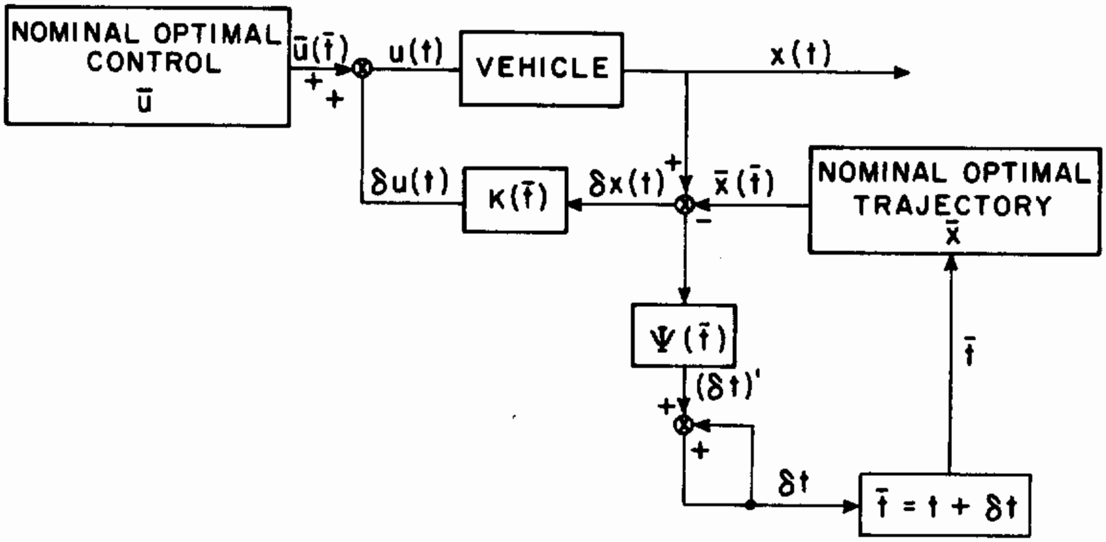

Figure 6.4. Block Diagram of Neighboring Optimum Control with Time Change Correction

Contrails

TABLE 6.1

THE NOMINAL TRAJECTORY

HEIGHT FT	VELOCITY FT/SEC	FL ANG DEG	CONTROL DEG	TIME SEC	CONTROL GAINS K 1	K 2	K 3	TIME FUNCTION PSI 1	PSI 2	PSI 3
250000.	27000.	-0.	-33.96	0.	0.	0.	0.	0.	0.	0.
249945.	27052.	-0.05	-33.85	-5.	-0.4157E-01	0.5058E-01	0.8983E 04	0.6331E-02	0.1554E-00	0.1547E 05
249781.	27105.	-0.09	-33.74	-10.	-0.1009E-01	0.2458E-01	0.4431E 04	0.5828E-02	0.1487E-00	0.1468E 05
249509.	27158.	-0.14	-33.63	-15.	-0.4345E-02	0.1575E-01	0.2889E 04	0.5443E-02	0.1421E-00	0.1389E 05
249131.	27212.	-0.18	-33.46	-20.	-0.2411E-02	0.1163E-01	0.2152E 04	0.5077E-02	0.1359E-00	0.1314E 05
248646.	27267.	-0.23	-33.35	-25.	-0.1498E-02	0.8786E-02	0.1659E 04	0.4741E-02	0.1299E-00	0.1241E 05
248054.	27323.	-0.27	-33.18	-30.	-0.1028E-02	0.7074E-02	0.1352E 04	0.4412E-02	0.1241E-00	0.1172E 05
247353.	27381.	-0.32	-32.96	-35.	-0.7557E-03	0.5964E-02	0.1145E 04	0.4100E-02	0.1187E-00	0.1105E 05
246540.	27441.	-0.36	-32.79	-40.	-0.5724E-03	0.4938E-02	0.9636E 03	0.3802E-02	0.1134E-00	0.1041E 05
245612.	27502.	-0.41	-32.57	-45.	-0.4529E-03	0.4221E-02	0.8305E 03	0.3518E-02	0.1084E-00	0.9795E 04
244565.	27567.	-0.46	-32.29	-50.	-0.3711E-03	0.3704E-02	0.7294E 03	0.3247E-02	0.1036E-00	0.9201E 04
243391.	27634.	-0.51	-32.01	-55.	-0.3097E-03	0.3237E-02	0.6396E 03	0.2989E-02	0.9897E-01	0.8627E 04
242085.	27705.	-0.57	-31.73	-60.	-0.2621E-03	0.2808E-02	0.5583E 03	0.2745E-02	0.9453E-01	0.8069E 04
240636.	27779.	-0.63	-31.34	-65.	-0.2279E-03	0.2529E-02	0.4980E 03	0.2515E-02	0.9027E-01	0.7528E 04
239034.	27858.	-0.69	-30.95	-70.	-0.1993E-03	0.2242E-02	0.4393E 03	0.2300E-02	0.8614E-01	0.6998E 04
237265.	27942.	-0.76	-30.50	-75.	-0.1759E-03	0.1997E-02	0.3873E 03	0.2102E-02	0.8216E-01	0.6480E 04
235313.	28032.	-0.84	-29.94	-80.	-0.1571E-03	0.1812E-02	0.3439E 03	0.1920E-02	0.7829E-01	0.5970E 04
233158.	28128.	-0.92	-29.33	-85.	-0.1403E-03	0.1630E-02	0.3022E 03	0.1758E-02	0.7452E-01	0.5467E 04
230776.	28232.	-1.01	-28.60	-90.	-0.1254E-03	0.1467E-02	0.2639E 03	0.1620E-02	0.7083E-01	0.4964E 04
228138.	28346.	-1.12	-27.71	-95.	-0.1120E-03	0.1333E-02	0.2296E 03	0.1507E-02	0.6719E-01	0.4462E 04
225209.	28470.	-1.24	-26.59	-100.	-0.9931E-04	0.1218E-02	0.1983E 03	0.1426E-02	0.6357E-01	0.3955E 04
221947.	28606.	-1.38	-25.25	-105.	-0.8659E-04	0.1100E-02	0.1679E 03	0.1381E-02	0.5997E-01	0.3443E 04
218303.	28755.	-1.53	-23.52	-110.	-0.7380E-04	0.9969E-03	0.1397E 03	0.1382E-02	0.5636E-01	0.2924E 04
214218.	28920.	-1.71	-21.28	-115.	-0.6044E-04	0.8969E-03	0.1131E 03	0.1436E-02	0.5272E-01	0.2398E 04
209633.	29100.	-1.91	-18.38	-120.	-0.4628E-04	0.7904E-03	0.8787E 02	0.1551E-02	0.4909E-01	0.1870E 04
204499.	29296.	-2.12	-14.52	-125.	-0.3137E-04	0.6803E-03	0.6450E 02	0.1737E-02	0.4551E-01	0.1346E 04
198811.	29507.	-2.30	-9.55	-130.	-0.1642E-04	0.5541E-03	0.4331E 02	0.2002E-02	0.4220E-01	0.8372E 03
192685.	29739.	-2.41	-3.52	-135.	-0.3311E-05	0.4078E-03	0.2547E 02	0.2350E-02	0.3950E-01	0.3575E 03
186459.	30022.	-2.32	2.68	-140.	0.3994E-05	0.2375E-03	0.1270E 02	0.2793E-02	0.3794E-01	-0.9664E 02
180782.	30415.	-1.92	8.04	-145.	0.4159E-05	0.8447E-04	0.6406E 01	0.3333E-02	0.3763E-01	-0.5571E 03
176591.	30982.	-1.14	12.40	-150.	0.8690E-07	-0.2376E-04	0.5093E 01	0.3840E-02	0.3751E-01	-0.1054E 04
175034.	31760.	0.06	16.59	-155.	-0.2823E-05	-0.7400E-04	0.5902E 01	0.3979E-02	0.3595E-01	-0.1545E 04
177301.	32730.	1.58	21.51	-160.	-0.2176E-06	-0.6099E-04	0.5645E 01	0.3550E-02	0.3355E-01	-0.1985E 04
184241.	33787.	3.17	27.37	-165.	0.6834E-05	0.1250E-04	0.1222E 01	0.2744E-02	0.3271E-01	-0.2441E 04
195863.	34744.	4.52	33.57	-170.	0.1254E-04	0.1361E-03	-0.9252E 01	0.1743E-02	0.3185E-01	-0.2774E 04
211266.	35429.	5.46	38.43	-175.	0.1179E-04	0.2333E-03	-0.1947E 02	0.9246E-03	0.2874E-01	-0.2737E 04
229252.	35815.	6.06	41.28	-180.	0.8944E-05	0.2741E-03	-0.2476E 02	0.5126E-03	0.2593E-01	-0.2569E 04
248906.	35992.	6.46	42.56	-185.	0.7314E-05	0.2873E-03	-0.2696E 02	0.3464E-03	0.2469E-01	-0.2470E 04
269708.	36059.	6.77	42.96	-190.	0.5978E-05	0.2906E-03	-0.2784E 02	0.2817E-03	0.2451E-01	-0.2443E 04
291409.	36075.	7.03	42.96	-195.	0.5449E-05	0.2910E-03	-0.2823E 02	0.2562E-03	0.2485E-01	-0.2454E 04
313896.	36069.	7.27	42.79	-200.	0.5146E-05	0.2902E-03	-0.2840E 02	0.2459E-03	0.2543E-01	-0.2482E 04
337121.	36053.	7.51	42.51	-205.	0.4984E-05	0.2878E-03	-0.2833E 02	0.2414E-03	0.2611E-01	-0.2518E 04
361061.	36034.	7.74	42.23	-210.	0.4690E-05	0.2857E-03	-0.2825E 02	0.2391E-03	0.2683E-01	-0.2556E 04
385703.	36013.	7.96	41.95	-215.	0.4524E-05	0.2839E-03	-0.2816E 02	0.2376E-03	0.2757E-01	-0.2595E 04
400000.	36000.	8.09	41.73	-218.	0.4418E-05	0.2832E-03	-0.2813E 02	0.2369E-03	0.2800E-01	-0.2617E 04

computational procedure fails at this point and therefore generates the zero values shown in Table 6.1.

Table 6.1 also gives values for the components of the time change function $\psi(t)$. The values are the second row of the inverse of the matrix θ and were obtained during the computations of the control gain function K.

The values for the control gains are theoretically independent of the values selected for the initial conditions on the perturbation equations. However, because of round-off in the numerical operations, the control gains values do change with changes in these initial conditions. This effect was investigated and the result was that changes of several orders of magnitude in the values of the initial conditions did not give any changes in the first four significant digits of the values for the control gains. Thus this effect is not considered to be significant. Nevertheless, the final choice of values for the perturbation equation's initial values were such that the values for the three independent solutions were of approximately the same order of magnitude. These initial conditions were:

$$\delta\gamma = \{\delta p_1(t'),\ \delta t',\ \delta p_3(t')\}$$

$$\delta\gamma^1 = \{10^1,\ 0,\ 10^7\}$$

$$\delta\gamma^2 = \{0,\ 1,\ 0\}$$

$$\delta\gamma^3 = \{10^1,\ 0,\ -10^7\}$$

Because of the fact that $f_1(t')$ was equal to zero, it was necessary to have either $\delta t'$ or $\delta p_3(t')$ nonzero in every case. This was, of course, also true when solving the optimization problem in the negative time direction.

6.4 Techniques for Simulation of Neighboring Optimum Control

The purpose of the simulation studies was to verify the results previously obtained for the control law and to determine the range of disturbances for which the resulting control system is adequate.

Three methods of control were investigated. The first of these methods corresponded to making state error measurements at the initial point and using Equation (6.6) to determine corrections for the nominal control function. The second method consisted of simulating the continuous feedback system as shown in Figure 4.4, using Equation (6.7) to determine the feedback gain K. The third method was similar to the second method but also used the time shift feature as indicated in Figure 6.4.

A brief study of these methods leads to the following conclusions: (1) the first method is less accurate in meeting the terminal conditions and is valid for a smaller range of initial disturbances than the second method, (2) the second method is very accurate in satisfying the terminal conditions and is valid over a large range of initial conditions provided that the simulated trajectory is stopped when one of the terminal conditions is met and that the time duration of the simulated trajectory is smaller than that of the nominal trajectory, and (3) the third method is unstable for fairly small initial disturbances, but for those cases where it is stable, errors in all directions are uniformly reduced until the final few steps of the integration.

In view of these conclusions it became clear that the proper method of control was a combination of the three methods investigated. Specifically, the procedure chosen was to use the second method during the first part of the trajectory until the errors were sufficiently small that the third method would not be unstable. Then the third method was used until the final three seconds of the simulated trajectory, at which time the system switched to the first method.

The reason for using the first method during the last few seconds, is that at the final step the nominal control correction is incorrect due to the fact that the matrix θ is singular. For a few steps prior to the final step, θ is so close to being singular that the results obtained by inverting θ are not very accurate. Since the first method does not use the inverse of θ except at its initial point, it is preferable to the other two methods over the terminal portion of the trajectory.

The use of the third method made it possible to match the final times for the simulated and nominal trajectories. This avoided the difficulty encountered with the second method when the simulation terminal time exceeded the nominal time. Also, it resulted in smaller terminal errors for all cases. Moreover, using the time shift feature gave, in all cases, smaller values for the criterion function than did the second method.

In performing numerical integration for the simulation procedure it was necessary to predict future values for the control variable, just as it was in the optimization procedure. This could be done by computing the error correction δu, based on errors δx_i of the predicted state values x_i generated in the integration routine. It could also be done by extrapolation of past values of the control variable, as discussed in Section 3.4, or by extrapolation of past values of δu. However, the reduction in computation which was obtained when integrating optimal trajectories, does not occur in this case. The reason for this is that δu is available explicitly in the simulation equations. Both of these methods were used in various integrations. A comparison of results did not indicate which method was more accurate.

The neighboring optimum control method does not include any limitation on the size of the control correction δu. Therefore, the control values resulting from the addition of δu to the nominal control value may exceed the specified bounds. The procedure in this case was to use the maximum admissible value in the direction of the computed control value.

6.5 Simulation Results for the Reentry Problem

The results obtained from the simulation were that for a region about the nominal optimal trajectory, the neighboring optimum control method was extremely accurate in meeting terminal conditions and in obtaining the optimum value for the criterion function. This region is sufficiently large to include significant physical disturbances from the nominal trajectory. Hence, these results indicate that this control method may have practical utility in reentry problems.

The terminal errors resulting from a range of initial errors in each state variable are shown graphically in Figures 6.5 through 6.13. The results indicate a controllable region about the initial state with approximate bounds of +22,000 and -10,000 feet in altitude, +280 and -200 feet per second in velocity and +.0015 and -.0028 radians for the flight path angle.

The results obtained for the criterion function values are indicated by the data of Table 6.2. This table gives the true optimum values and the values obtained over the neighboring optimum for the criterion function and its components, x_4 and x_5. Three trajectories are represented, the nominal and two which correspond to initial errors in x_2. In all cases the value for the criterion function over the simulated trajectory is larger than the true optimum and this difference increases slightly with increased initial error.

As indicated in Figures 6.5 through 6.13, the terminal errors increase rapidly beyond rather sharply defined limits on the initial errors. Generally, this limit corresponds to the point at which the computed control values, $\bar{u}(t) + \delta u(t)$, exceed the specified bounds on $u(t)$ at some time along the simulated trajectory. But in some cases, the build-up in terminal errors is due to instability in the time shift loop.

Bounds on the control variable are fundamental limitations on the neighboring optimum control method and cannot be avoided. However, modifying the time shift feature can result in changes of the controllable region.

In the simulation, the time shift feature was initiated at the point in time 100 seconds prior to the terminal time of the nominal trajectory. The choice of this point was based on observation of the PSI function in Table 6.1. Over the last 100 seconds of the trajectory, these functions are reasonably smooth and it was assumed that in this region the time shift

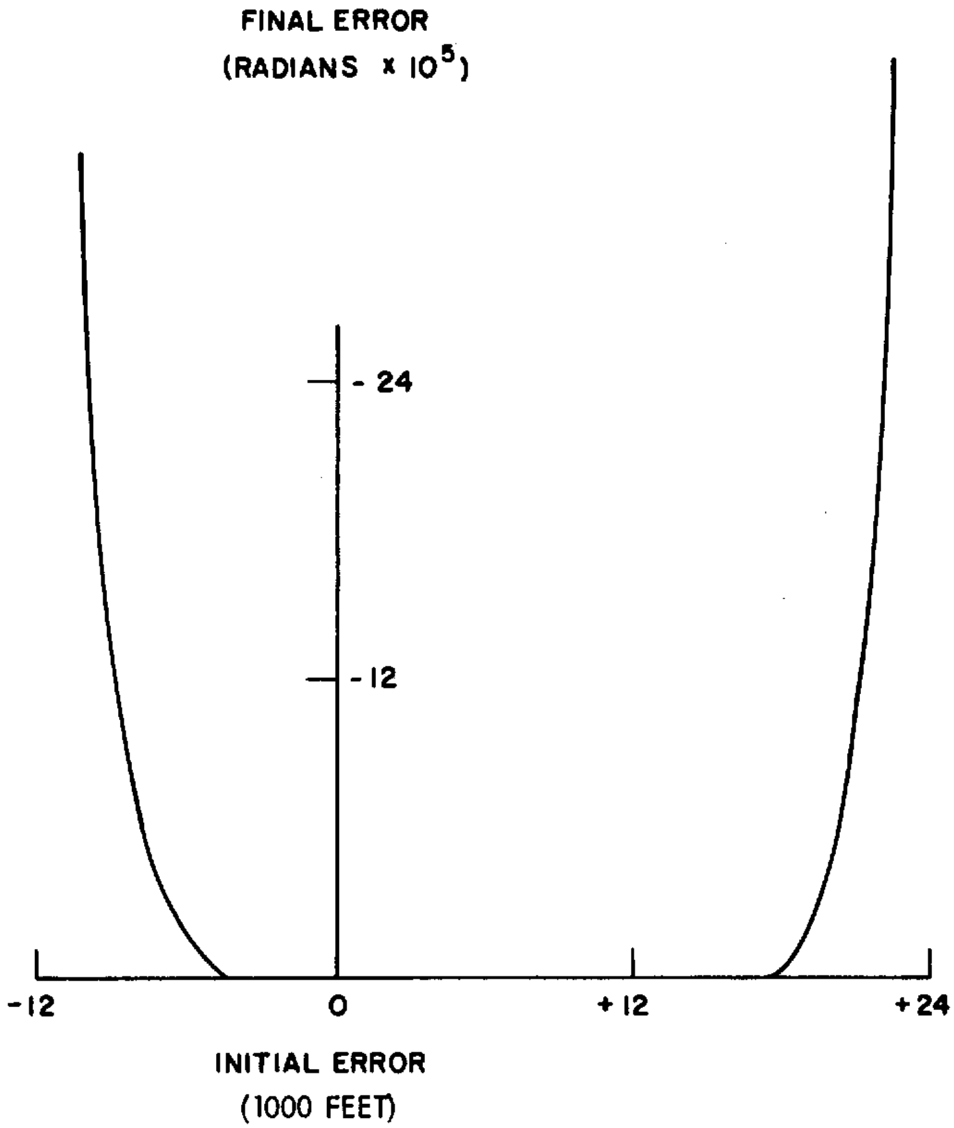

Figure 6.5. Final Error in x_1 vs. Initial Error in x_1

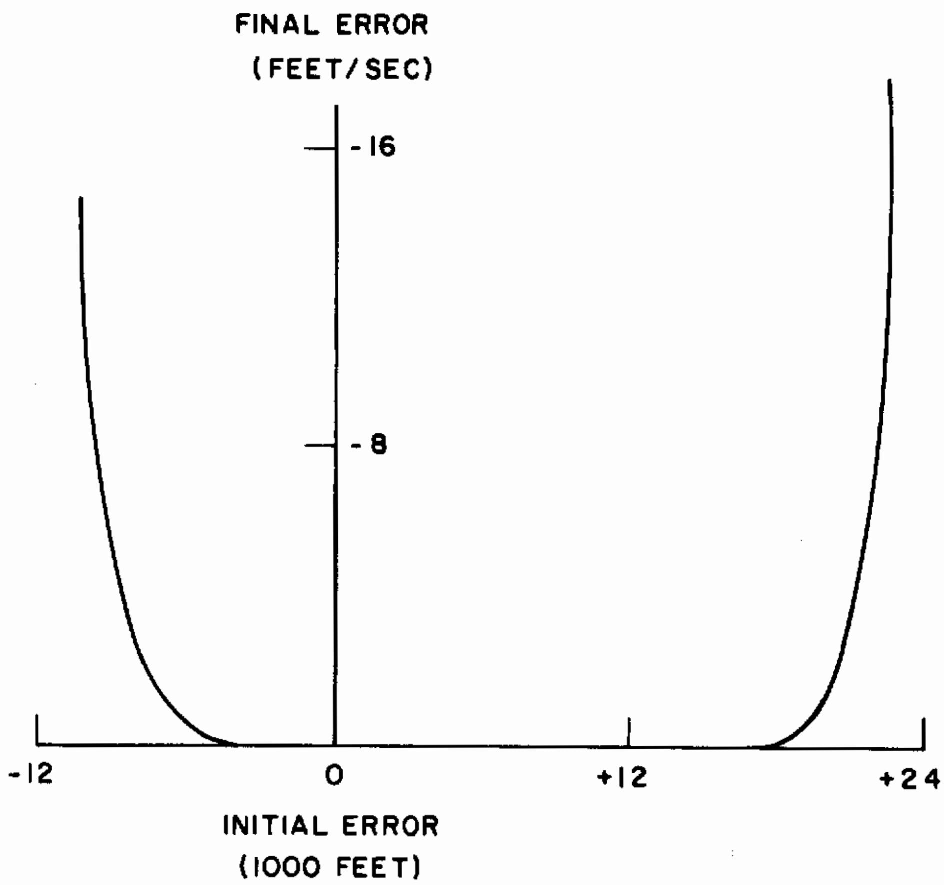

Figure 6.6. Final Error in x_2 vs. Initial Error in x_1

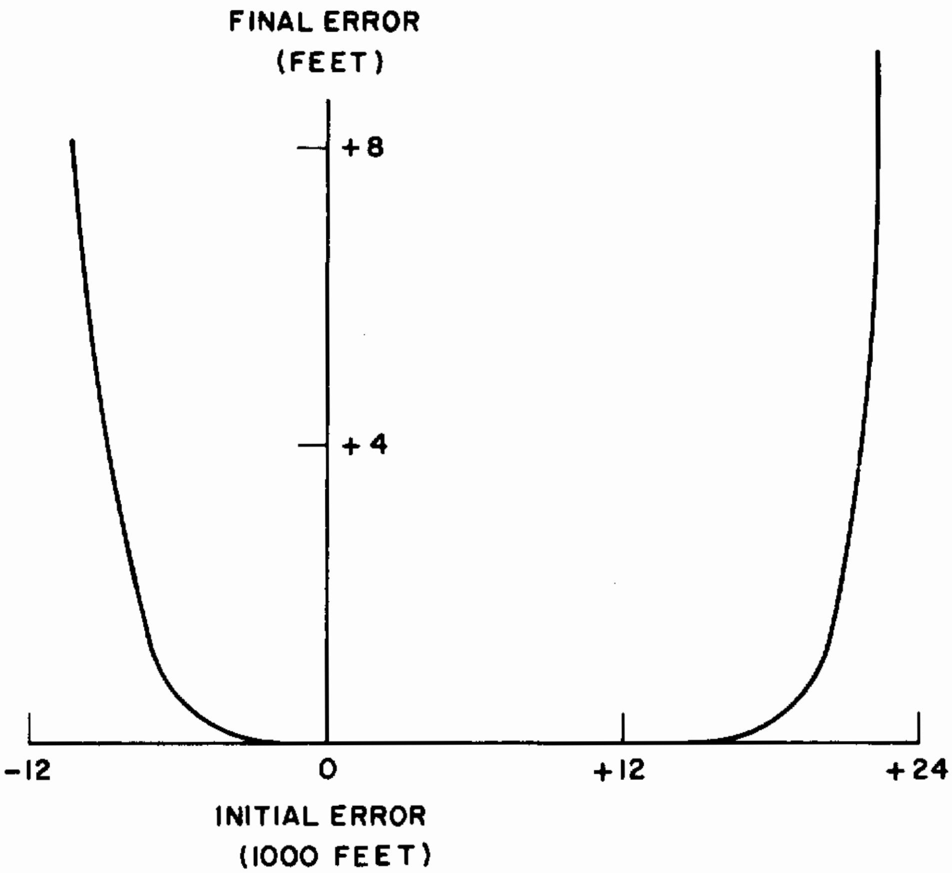

Figure 6.7. Final Error in x_3 vs. Initial Error in x_1

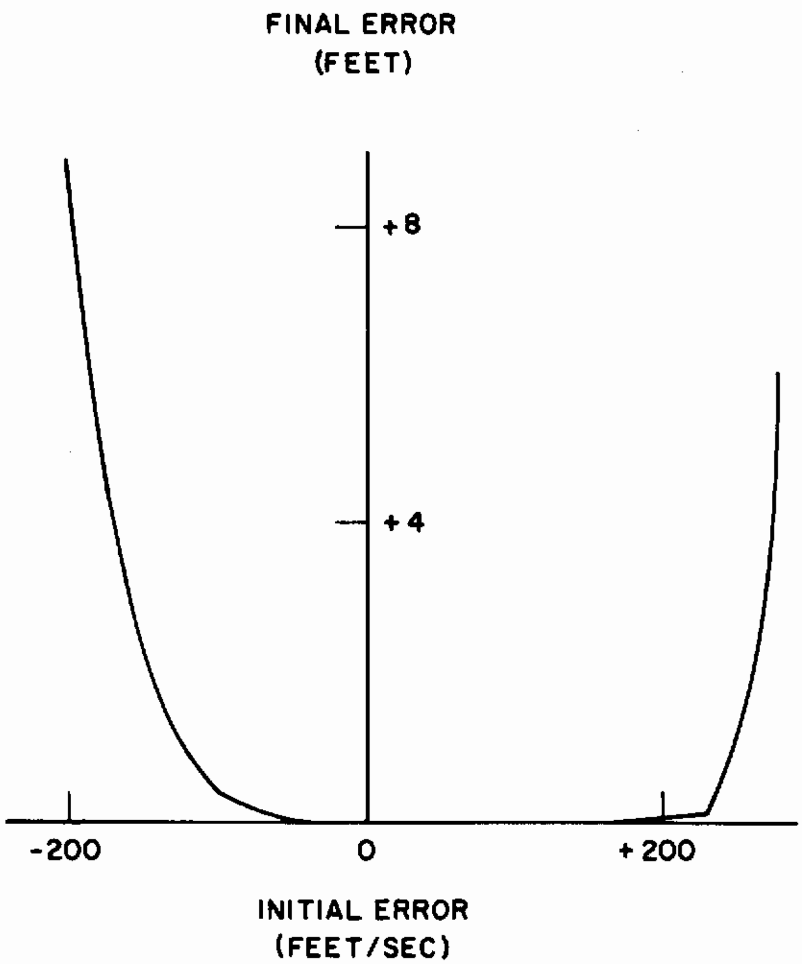

Figure 6.8. Final Error in x_1 vs. Initial Error in x_2

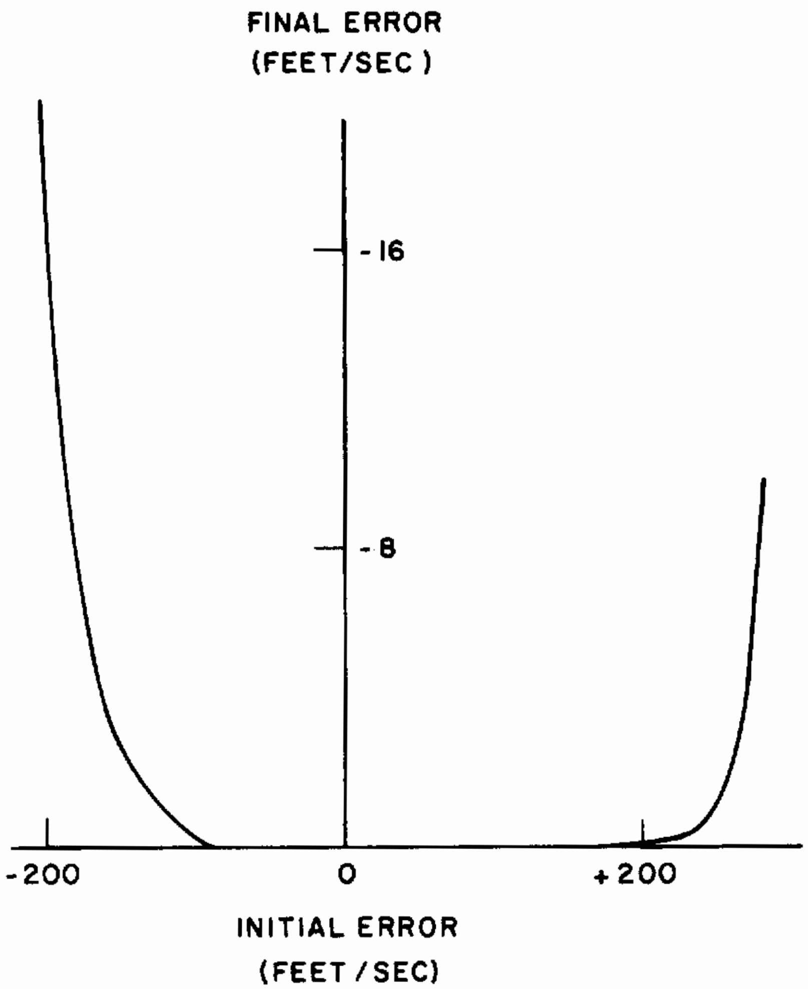

Figure 6.9. Final Error in x_2 vs. Initial Error in x_2

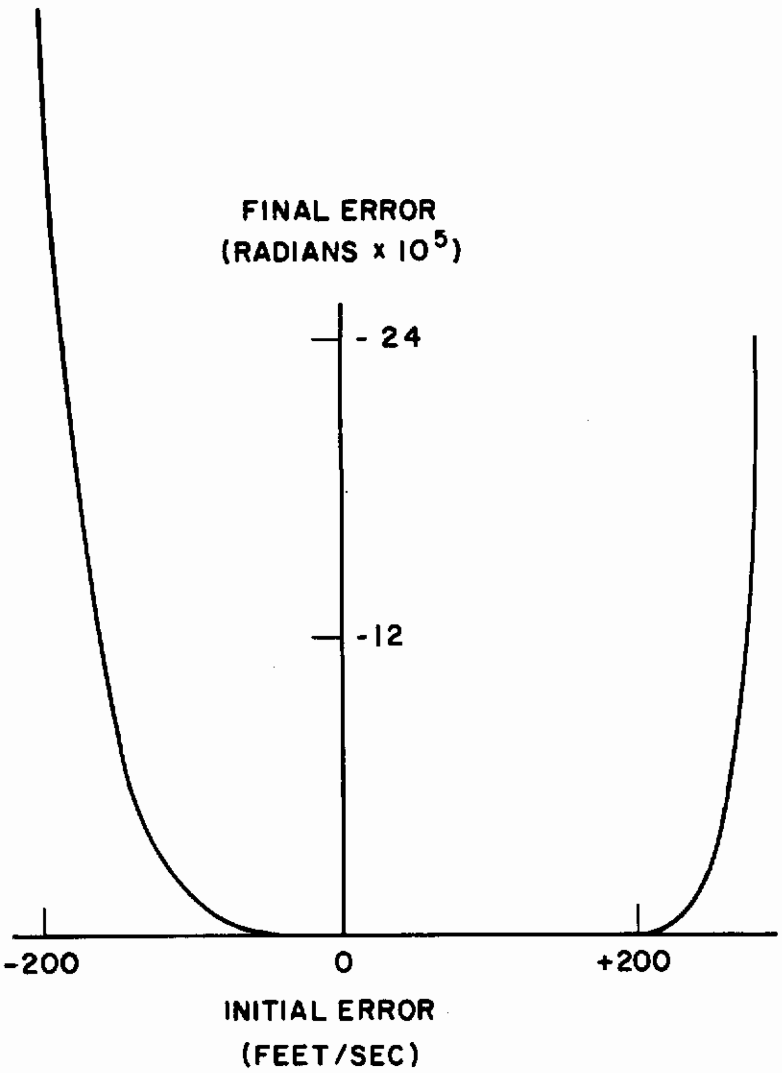

Figure 6.10. Final Error in x_3 vs. Initial Error in x_2

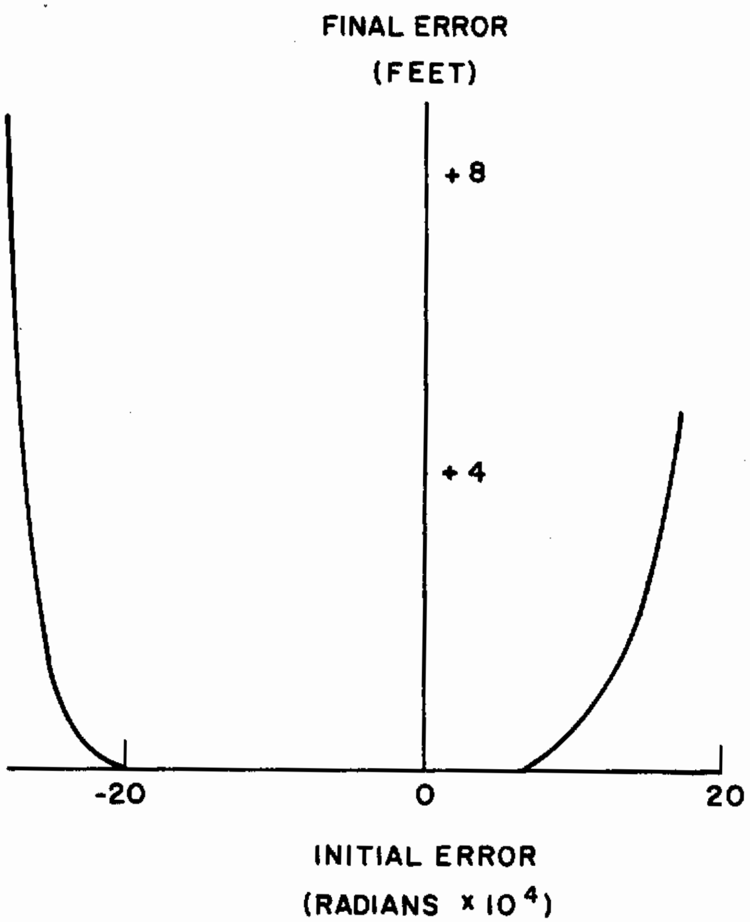

Figure 6.11. Final Error in x_1 vs. Initial Error in x_3

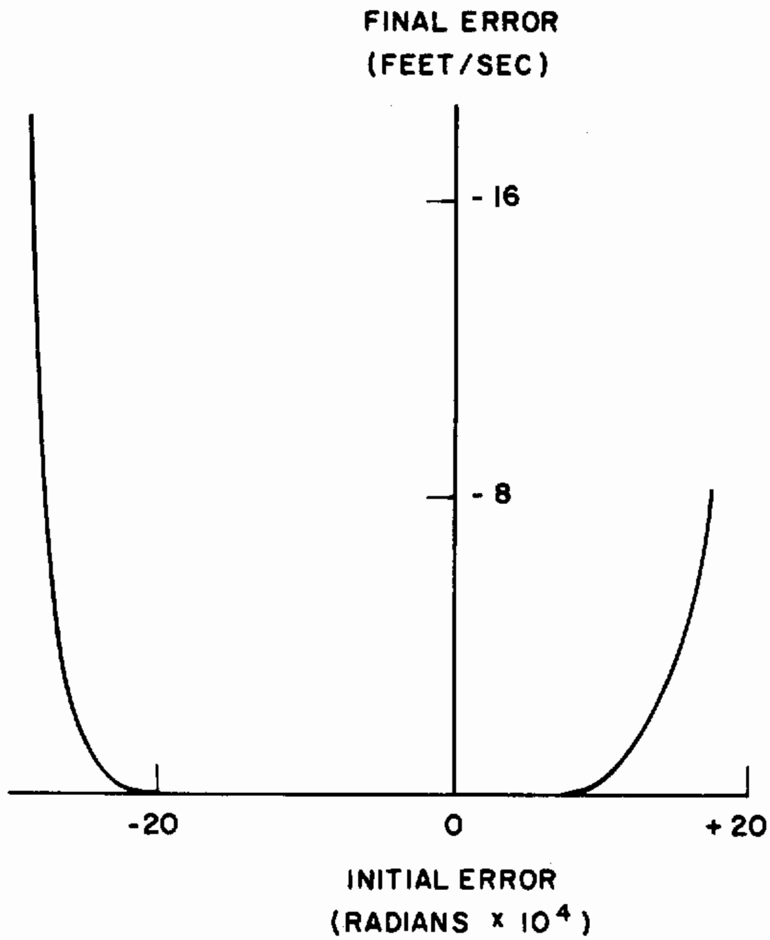

Figure 6.12. Final Error in x_2 vs. Initial Error in x_3

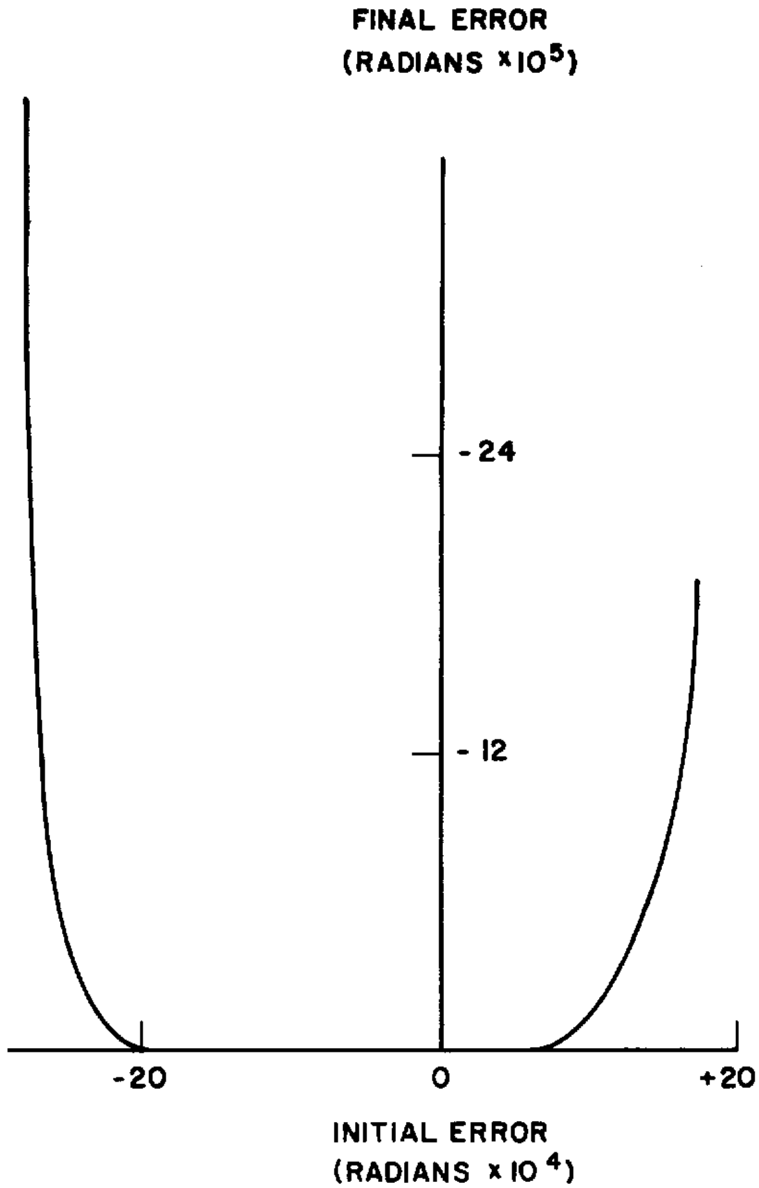

Figure 6.13. Final Error in x_3 vs. Initial Error in x_3

TABLE 6.2

REPRESENTATIVE VALUES FOR THE CRITERION FUNCTION

Initial error in (feet/sec.)	Variable	Optimum values	Simulated values
0	x_4	$.28307685 \times 10^9$	$.28310035 \times 10^9$
	x_5	$.15845571 \times 10^7$	$.15845466 \times 10^7$
	J	$.44153256 \times 10^9$	$.44155501 \times 10^9$
+100	x_4	$.28131823 \times 10^9$	$.28155203 \times 10^9$
	x_5	$.15843487 \times 10^7$	$.15873962 \times 10^7$
	J	$.43975310 \times 10^9$	$.44029165 \times 10^9$
-100	x_4	$.28546295 \times 10^9$	$.28559856 \times 10^9$
	x_5	$.15861780 \times 10^7$	$.15851684 \times 10^7$
	J	$.44408075 \times 10^9$	$.44411540 \times 10^9$

loop was less likely to be unstable. If the time shift feature had been used over a shorter period, the controllable region in some directions would be larger. However, this would result in a slight decrease in terminal accuracy and an increase in the value of the criterion function.

The bounds on the control variable, in this simulation, were $\pm\pi/4$ radians.

CHAPTER 7

SUMMARY AND CONCLUSIONS

7.1 Summary of Results with the Optimization Procedure

The results presented in this dissertation demonstrate that optimal control theory can successfully be employed to study the mathematical model of a reentry problem.

The mathematical model used was adequate to represent the large scale dynamic behavior of a variable lift aerodynamic vehicle during a reentry maneuver into the earth's atmosphere. No special properties of this model were used in order to modify the method of computation. Hence, more detailed models will not require any significant change in the computational procedure.

The optimization of the model was performed with respect to a variable criterion function. The specific criterion function used was the sum of two terms which measured the heating and acceleration effects experienced by the vehicle. The weighting between these two terms was the variable element in the criterion function. The equations and conditions which specify the system trajectory which minimizes this criterion function were derived by means of optimal control theory.

A family of solutions was computed, with the criterion function weighting factor as the parameter. The characteristics of these optimal trajectories are presented in Tables 5.1 through 5.6. As a result of obtaining these trajectories, the optimal trade-off between the competitive effects of heating and acceleration can be determined. This function is shown in Figure 5.7.

The use of a variable criterion function is a natural approach in applying system optimization techniques to the analysis of complex control systems. For this reason, computational methods for applying this approach are important for engineering applications.

In problems of this type the most difficult aspect is the formulation and development of effective computational procedures. There are several different approaches to computing solutions of optimal control problems. Each of these approaches has advantages and disadvantages in specific applications. For the problem considered here, the primary interest was in the manner in which the optimal trajectory changed for small changes in the criterion function. For this reason, the computational procedure was required to have good convergence characteristics in the neighborhood of the optimum. This led to the use of a second variation or "neighboring optimum" optimization method. In this procedure, most of the computational effort was involved in performing numerical integration. The special features of the optimal control problem that affect the choice of a numerical integration procedure were considered. Also, a special technique was presented which is useful in reducing the amount of computation required to numerically integrate optimal trajectories.

Experience with this optimization procedure leads to the conclusion that for many dynamic control systems (specifically, those which can be described by a reasonably small number of first order, nonlinear, ordinary differential equations), the amount of computation required to obtain good approximations to optimal trajectories is not excessive.

7.2 Additional Requirements for the Optimization Procedure

The computational procedure used to determine the optimal reentry trajectories is sufficiently general to be useful with many control problems. But there are several additional features which, if added, would greatly increase the area of application for the procedure.

The most logical feature to add to the optimization program is the capability to handle state variable inequality constraints. A theoretical method to accomplish this was discussed in Section 2.6. However, there are several difficulties to instrumenting this approach. The primary problem is in determining what numerical accuracy is required at the junction point between two subarcs.

If the optimization program is to be used for a wide range of problems, it is desirable to include the capability of switching to an alternate iterative algorithm such as those mentioned in Section 3.1. To accomplish this, it will be necessary to devise some method for selecting the best algorithm in a sequential manner, i.e., switching algorithms on the basis of the previous convergence behavior.

Finally, a method should be provided for increasing the final accuracy of the convergent solution. An obvious means of doing this is to use double precision in the computer operations. However, this alone may not be enough to sufficiently reduce the final error. If this is the case, it will be necessary to reduce the quantization level for the control variables and increase the accuracy of the numerical integration.

7.3 Comments on the Neighboring Optimum Control Method

The neighboring optimum control method results in the best linear approximation to the optimum feedback control law. It is best in the sense that it yields the minimum value of the criterion function of any linear control system. Because of this property it is obviously of great potential value for comparing and evaluating other control schemes.

In earlier results it was not clear that the neighboring optimum method would provide adequate terminal accuracy. The results presented in Chapter 6 indicate that with a time shift introduced into the nominal trajectory, the neighboring optimum method becomes essentially a terminal guidance scheme.

Only one method of performing this time shift was investigated, but this appears to be the most natural and easily instrumented method. Based on the fact that the final value of the criterion function varied inversely with the length of time over which the time shift was used, it is concluded that the time shift should be included throughout the control period in order to obtain true neighboring optimum control. However, introducing the time shift did reduce the range of disturbances which could be compensated for by the control system. This led to theuse of a combination of control methods. Experience with the simulation indicates that the neighboring optimum method can be quite flexible and is capable of achieving a wide variety of control system specifications.

The main drawback to using this method for either system evaluation or control is that the optimization problem must be solved first. For complex, nonlinear systems, optimal control theory appears to be the proper method of analysis. At present, the computation of solutions to optimal control problems cannot be considered a routine task. But when optimization is applied to a system's mathematical model, then the neighboring optimum method is the logical means to derive a feedback control law.

7.4 An Extension to the Control Method

The primary limitation on the neighboring optimum control method is that it is only valid in a neighborhood of the nominal trajectory where the linear terms of the Taylor series expansion of the optimal control function are dominant. In order to extend the region of state space over which the control system is adequate, it will be necessary to utilize some type of nonlinear control scheme.

One approach to accomplish this is to solve for the quadratic terms of the optimal control law and introduce these into the control system.[17, 18] These quadratic terms are functions of the second order variations of the state and auxiliary variables, $\delta^2 x$ and $\delta^2 p$. These terms satisfy differential equations of the form

$$\frac{d}{dt}(\delta^2 x) = A_1(t)\,\delta^2 x + A_2(t)\,\delta^2 p + \delta x\, B_1(t)\,\delta x + \delta x\, B_2(t)\,\delta p + \delta p\, B_3(t)\,\delta p \tag{7.1}$$

$$\frac{d}{dt}(\delta^2 p) = A_3(t)\,\delta^2 x + A_4(t)\,\delta^2 p + \delta x\, B_4(t)\,\delta x + \delta x\, B_5(t)\,\delta p + \delta p\, B_6(t)\,\delta p$$

where the A_i are n vectors and the B_i are nxn matrices which are functions of the nominal trajectory. Because Equations (7.1) are nonlinear, it is not possible to obtain a general solution by solving the equations for one set of initial conditions. Solutions must be computed over the range of disturbances for which the control system is to operate. This is an interesting extension of the theory of neighboring optimum control but for many problems the computational requirements for deriving such a control law may cause this approach to be impractical.

An alternate approach to approximating the quadratic control terms can be based on combining neighboring nominal trajectories. Consider two optimal trajectories $x^1(t)$ and $x^2(t)$, with initial states differing by small amounts and having the same terminal state. Assume that the optimal linear control laws are,

$$\delta u^1 = K^1(t)\ \delta x^1$$

$$\delta u^2 = K^2(t)\ \delta x^2$$

Then if the deviations δx^1 from the $x^1(t)$ trajectory lie between the $x^1(t)$ and $x^2(t)$ states, it can be shown that the optimal control may be approximated by

$$\delta u^1 = \left\{ K^1(t) - K^2(t) + \left(\frac{u^1(t) - u^2(t)}{x^1(t) - x^2(t)} \right) \right\} \delta x^1 + \delta x^1 \left\{ \frac{K^1(t) - K^2(t)}{x^1(t) - x^2(t)} \right\} \delta x^1 \tag{7.2}$$

Similarly, additional neighboring trajectories can be combined. The elements appearing in Equation (7.2) can be obtained directly from the neighboring optimum control solutions. This is a convenient approach to extending the controllable region and should be investigated.

REFERENCES

1. J.H. Westcott, An Exposition of Adaptive Control, New York, The Macmillan Company: Pergamon Press, 1962. p. 1-3.

2. I.M. Gelfand and S.V. Fromin, Calculus of Variations, Englewood Cliffs, N.J., Prentice-Hall, 1963.

3. M.R. Hestenes, "A General Problem in the Calculus of Variations with Applications to Paths of Least Time", Rand Corporation, RM-100, 1950.

4. L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze and E.F. Mishchenko, The Mathematical Theory of Optimal Processes, New York, John Wiley: Interscience Publishers, 1962.

5. W. Kipiniak, Dynamic Optimization and Control, New York, John Wiley: The M.I.T. Press, 1961.

6. C.W. Merriam III, Optimization Theory and the Design of Feedback Control Systems, New York, McGraw-Hill, 1964.

7. J.L. Speyer, "Optimization and Control Using Perturbation Theory to Find Neighboring Optimum Paths", Raytheon Company, BR-2142, December 1962.

8. D.K. Scharmack, G.D. Swanlund, R.G. Johnson, M.D. Ward and E.R. Rang, "Feasibility Study of New Techniques for Control of Reentry Vehicles, Volume 1, Calculation of Optimal Trajectories and Synthesis of Control Function", Flight Dynamics Laboratory, Wright-Patterson Air Force Base, FDL-TDR-64-13, March 1964.

9. T.L. Saaty and J. Bram, Nonlinear Mathematics, New York, McGraw-Hill, 1964.

10. R. Bellman, Dynamic Programming, Princeton, N.J., Princeton University Press, 1957.

11. R. Bellman, Adaptive Control Processes: A Guided Tour, Princeton, N.J., Princeton University Press, 1961.

12. Y.C. Ho and P.B. Brentani, "On Computing Optimal Control with Inequality Constraints", J.S.I.A.M. on Control, Series A, 1:319-348, 1963.

13. W.L. Nelson, "On the Use of Optimization Theory for Practical Control System Design", Joint Automatic Control Conference, Preprints of papers, June 24-26, 1964, Stanford University, Stanford, California, p. 422-428.

14. Z.V. Rekasius, "Sub-optimal Design of Intentionally Nonlinear Controllers", Joint Automatic Control Conference, Preprints of papers, June 24-26, 1964, Stanford University, Stanford, California, p. 436-440.

15. H.J. Kelley, "Guidance Theory and Extremal Fields", IRE Transactions on Automatic Control, AC-7: 75-82, October 1962.

16. J.V. Breakwell, J.L. Speyer and A.E. Bryson, "Optimization and Control of Nonlinear Systems Using the Second Variation", J.S.I.A.M. on Control, Series A, 1:193-223, 1963.

17. H.J. Kelley, "An Optimal Guidance Approximation Theory", IEEE Transactions on Automatic Control, AC-9: 375-380, October 1964.

18. A.H. Jazwinski, "Quadratic and Higher Order Feedback Gains for Control of Nonlinear Systems", Paper Presented at the AIAA/ION Astrodynamics Guidance and Control Conference, August 24-26, 1964, Los Angeles, California.

19. A.E. Taylor, Introduction to Functional Analysis, New York, John Wiley, 1958.

20. L.A. Zadeh, "Optimality and Non-scalar-valued Performance Criteria", IEEE Transactions on Automatic Control, AC-8: 59-60, January 1963.

21. E.B. Lee, "A Sufficient Condition in the Theory of Optimal Control", J.S.I.A.M. on Control, Series A, 1:241-245, 1963.

22. R.A. Gambill, "Generalized Curves and the Existence of Optimal Controls", J.S.I.A.M. on Control, Series A, 1:246-260, 1963.

23. L.V. Kantorovich, "Functional Analysis and Applied Mathematics", Uspekhi Mat. Nauk., 3: 89-185, 1948. Translated by C.D. Benster and G.E. Forsythe as Nat. Bur. Standards, No. 1590.

24. P. Henrici, Discrete Variable Methods in Ordinary Differential Equations, New York, John Wiley, 1962.

25. A.E. Bryson, W.F. Denham, F.J. Carroll and K. Mikami, "Determination of Lift or Drag Programs to Minimize Reentry Heating", Journal of the Aerospace Sciences, Volume No. 29: 420-430, April 1962.

26. H.J. Kelley, "Gradient Theory of Optimal Flight Paths", A.R.S. Journal, 30: 947-953, October 1960.

27. R. Kalaba, "On Nonlinear Differential Equations, the Maximum Operation and Monotone Convergence", Journal of Mathematics and Mechanics, 8: 519-574, 1959.

28. A.E. Bryson and W.F. Denham, "A Steepest-ascent Method for Solving Optimum Programming Problems", Journal of Applied Mechanics, 29: 247-257, June 1962.

29. R. McGill and P. Kenneth, "A Convergence Theorem on the Iterative Solution of Nonlinear Two-point Boundary Value Systems", paper presented at the XIVth I.A.F. Congress, September 1963, Paris, France.

30. W.F. Denham, "Steepest-ascent Solution of Optimal Programming Problems", Raytheon Company, BR-2393, April 1963.

31. S.E. Dreyfus, "Variational Problems with State Variable Inequality Constraints", Rand Corporation, P-2605-1, July 1962. Revised, August 1963.

32. F.A. Valentine, The Problem of Lagrange with Differential Inequalities as Side Conditions, Ph.D. Dissertation, Department of Mathematics, University of Chicago, Chicago, Illinois, 1937.

33. D.K. Scharmack, "A Modified Newton-Raphson Method for the Control Optimization Problem", paper presented at the Joint AIAA-IMS-SIAM-ONR Symposium on Control and System Optimization, January 27-29, 1964, U.S. Naval Postgraduate School, Monterey, California.

34. P. Henrici, Error Propagation for Difference Methods, New York, John Wiley, 1963.

35. J.F. Traub, Iterative Methods for the Solution of Equations, Englewood Cliffs, N.J., Prentice-Hall, 1964.

36. W. Kizner, "A Numerical Method for Finding Solutions of Nonlinear Equations", J.S.I.A.M., 12: 424-428, June 1964.

37. R.W. Hamming, Numerical Methods for Scientists and Engineers, New York, McGraw-Hill, 1962.

38. R. Ralston and H.S. Wilf (Editors), Mathematical Methods for Digital Computers, New York, John Wiley, 1960. (Matrix inversion and related topics by direct methods, Chapter 2.)

39. J.R. Rice, The Approximation of Functions, Volume 1: Linear Theory, Reading, Massachusetts, Addison-Wesley, 1964.

40. R.H. Schorsch, The Re-entry Trajectory and a Re-entry Optimization Problem Analysis, M.S. in Engineering, University of California, Los Angeles, 1964.

41. G.E. Solomon, "Nature of Re-entry", Astronautics, 4:20-21, March 1959.

42. J.H. Allen, "Aerodynamic Problems of the Future", Presentation to the Fluid Mechanics Panel of Advisory Group for Aeronautical Research and Development, April 1962, Brussels, Belgium.

43. E.S. Levinsky, "Application of Inequality Constraints to Variational Problems of Lifting Re-entry", Journal of the Aerospace Sciences, 29:400-409, April 1962.

44. W.F. Denham and A.E. Bryson, "Optimal Programming Problems with Inequality Constraints II", A.I.A.A. Journal, 2:25-34, 1964.

45. S.A. Jurovics and J.E. McIntyre, "The Adjoint Method and its Application to Trajectory Optimization", A.R.S. Journal, 32:1354-1358, September 1962.

46. G.A. Bliss, Lectures on the Calculus of Variations, Chicago, The University of Chicago Press, 1946.

47. S.E. Dreyfus, "An Optimal Linear Feedback Guidance Scheme", Rand Corporation, RM-3604-PR, May 1963.

48. A.H. Jazwinski, "Optimal Trajectories and Linear Control of Nonlinear Systems", paper presented at the Joint AIAA-IMS-SIAM-ONR Symposium on Control and System Optimization, January 27-29, 1964, U.S. Naval Postgraduate School, Monterey, California.

APPENDIX A

The first partial derivatives which enter into the perturbation Equations (5.1) and (5.2) are as follows:

$$\frac{\partial f_1}{\partial x_1} = 0$$

$$\frac{\partial f_1}{\partial x_2} = -\sin x_3$$

$$\frac{\partial f_1}{\partial x_3} = -x_2 \cos x_3$$

$$\frac{\partial f_1}{\partial u} = 0$$

$$\frac{\partial f_2}{\partial x_1} = -(K_5^2 K_6 K_{10}/K_3)\exp[K_6 x_1]\, x_2^2\, (K_{12} + K_{13}\sin^2 u)$$

$$\frac{\partial f_2}{\partial x_2} = -(2\, K_5^2 K_{10}/K_3)\, \exp[K_6 x_1] x_2 (K_{12} + K_{13} \sin^2 u)$$

$$\frac{\partial f_2}{\partial x_3} = K_2 \cos x_3$$

$$\frac{\partial f_2}{\partial u} = -(K_5^2 K_{10} K_{13}/K_3)\exp[K_6 x_1] x_2^2\, (2 \sin u \cos u)$$

$$\frac{\partial f_3}{\partial x_1} = \left[x_2 \cos x_3/(K_1 + x_1)^2\right] - (K_5^2 K_6 K_{10}/K_3)\left[\exp[K_6 x_1]\, x_2 \sin u \cos u\right]$$

$$\frac{\partial f_3}{\partial x_2} = -\left[K_2 \cos x_3/x_2^2\right] - \left[\cos x_3/(K_1 + x_1)\right] - (K_5^2\, K_{10} K_{11}/K_3)\, \exp[K_6 x_1] \sin u \cos u$$

$$\frac{\partial f_3}{\partial x_3} = -[K_2 \sin x_3/x_2] + \left[x_2 \sin x_3/(K_1 + x_1)\right]$$

$$\frac{\partial f_3}{\partial u} = -(K_5^2 K_{10} K_{11}/K_3)\exp[K_6 x_1] x_2 (\cos^2 u - \sin^2 u)$$

$$\frac{\partial f_4}{\partial x_1} = (K_4K_5K_6/2) \exp [K_6x_1/2] x_2^3$$

$$\frac{\partial f_4}{\partial x_2} = (3 K_4K_5) \exp [K_6x_1/2] x_2^2$$

$$\frac{\partial f_4}{\partial x_3} = 0$$

$$\frac{\partial f_4}{\partial u} = 0$$

$$\frac{\partial f_5}{\partial x_1} = (2 K_5^4K_6K_7K_{10}^2/K_3^2) \exp [2 K_6x_1]x_2^4 \Big| K_{11}^2 \sin^2 u \cos^2 u + K_{12}^2 + 2K_{12}K_{13}\sin^2 u + K_{13}^2 \sin^4 u \Big|$$

$$\frac{\partial f_5}{\partial x_2} = (4 K_5^4K_7K_{10}^2/K_3^2)\exp [2 K_6x_1]x_2^3 \Big| K_{11}^2\sin^2 u \cos^2 u + K_{12}^2 + 2 K_{12}K_{13}\sin^2 u + K_{13}^2\sin^4 u \Big|$$

$$\frac{\partial f_5}{\partial x_3} = 0$$

$$\frac{\partial f_5}{\partial u} = (K_5^4K_7K_{10}^2/K_3^2) \exp [2 K_6x_1]x_2^4 \Big| K_{11}^2(2 \sin u \cos^3 u - 2 \sin^3 u \cos u) + 2 K_{12}K_{13}(2 \sin u \cos u) + K_{13}^2(4 \sin^3 u \cos u) \Big|$$

It is clear from the manner in which the variables x_4 and x_5 were introduced into the problem that all partial derivatives with respect to x_4 and x_5 are equal to zero.

Expressions for all of the second partial derivatives, i.e.,

$$\frac{\partial f_i}{\partial x_j \partial x_k} \quad ; \quad \Big((i = 1, 2, 3),\ j = 1, 2, \ldots, 5\Big),\ k = 1, 2, \ldots, 5$$

$$\frac{\partial f_i}{\partial x_j \partial u} \quad ; \quad (i = 1, 2, 3),\ j = 1, 2, \ldots, 5$$

$$\frac{\partial f_i}{\partial u^2} \quad ; \quad i = 1, 2, 3$$

are also required in order to write the perturbation equations and the Equation (5.3) satisfied by the control deviation variable. These expressions are easily obtained by simple differentiation of the above first partials.