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A brief review of the unsteady aerodynamics in lifting surface theory is reported. For utility in design application, simplified but accurate analytical expressions were developed for generalized unsteady aerodynamics of wings of finite aspect ratio and sweep. This includes a means of calculating the unsteady lift load on the wing and tail surface of arbitrary shape and angle of attack variation. In addition, a formulation to include the indicial aerodynamics in the longitudinal equations of motion of the aircraft

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20. (Cont'd.) was performed. Sample numerical solutions to the longitudinal equations for a step input in elevator deflection are presented for the Navion aircraft. From these calculations, a noted difference in angle of attack and pitch rate response can be seen for the cases of unsteady and quasi-steady aerodynamics. The apparent increase in damping effects in the aircraft response due to the unsteady aerodynamics indicate that various design constraints might be relaxed somewhat if the unsteady aerodynamics is modeled properly.

FOREWORD

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LIST OF SYMBOLS

A	Aspect ratio of wing or horizontal tail
a_1, a_2	Constants
b	Wing span
b_1, b_2	Constants
c	Chord
c_ℓ	Section lift coefficient, $\frac{\text{Lift}}{1/2\ \rho U^2 c}$
Δc_ℓ	Indicial section lift function
$c_{\ell_{\dot{\alpha}}}$	$\frac{\partial c_\ell}{\partial \frac{\dot{\alpha} c}{2U}}$
C_L	Lift coefficient, $\frac{\text{Lift}}{1/2\ \rho U^2 S}$
ΔC_L	Indicial lift function
C_x	Perturbed axial force coefficient, $\frac{\text{Axial force}}{1/2\ \rho U^2 S}$
C_z	Perturbed normal force coefficient, $\frac{\text{Normal force}}{1/2\ \rho U^2 S}$
C_m	Perturbed pitching moment coefficient, $\frac{\text{Pitching moment}}{1/2\ \rho U^2 Sc}$
C_{x_α}	$\frac{\partial C_x}{\partial \alpha}$
C_{L_α}	$\frac{\partial C_L}{\partial \alpha}$
C_{m_q}	$\frac{\partial C_m}{\partial \frac{qc}{2U}}$
$C_{z_{\delta_e}}$	$\frac{\partial C_z}{\partial \delta_e}$
$C_{m_{\delta_e}}$	$\frac{\partial C_m}{\partial \theta_e}$

$C_{L_{\dot{\alpha}}}$	$\frac{\partial C_L}{\partial \frac{\dot{\alpha}c}{2U}}$
$C_{m_{\dot{\alpha}}}$	$\frac{\partial C_m}{\partial \frac{\dot{\alpha}c}{2U}}$
E	$\frac{A + 1}{A}$
f_1, g_1, h_1	Constants
g	Acceleration due to gravity
I_y	Moment of inertia about pitch axis
k', K	Constants
k_1	Wagner function
L	Lift force
ℓ_w	Wake length
ℓ_t	Tail length
ℓ, m	Direction cosines
m	Mass
p	Pressure
q	Pitch rate
q'	Total velocity
q'_u, q'_ℓ	Velocity on upper and lower surface respectively
r	Radial polar coordinate
r_1, r_2	Distances measured from the shed vortex and its image
s	Laplace transform variable
S	Wing surface
t, τ	Time

LIST OF SYMBOLS (Cont'd.)

U	Forward velocity
u_r, u_β	Radial and tangential velocities respectively
w	Downwash
Δw	Indicial downwash function
x, y	Body axes
X, Y	Body forces
x_o	Initial location of shed vortex
z	$x_o + \frac{Ut}{2}$
α	Angle of attack
β	Angular polar coordinate
γ	Vorticity distribution
$\Gamma, \Gamma', \Gamma_1, \Gamma_0$	Circulation functions
$\Gamma_{ij}^{(k)}$	Circulation of bound vortex ring
δ_e	Elevator deflection
$\delta_i^{(j)}$	Strength of free vortex in wake
ε	Downwash angle
$\Delta\varepsilon$	Indicial downwash angle
ξ	Distance measured in z-plane
η	Distance measured in z-plane
θ	Pitch angle
Λ	Sweep angle
ν	Euler constant = 0.5772156
ρ	Air density
ϕ	Velocity potential function
ψ	Stream function
ω	Frequency

Subscripts:

w	Wing
t	Horizontal tail
∞	Steady state

Superscripts:

$-$	Laplace transformed quantity

SECTION I

INTRODUCTION

The work reported in this study concerns indicial unsteady aerodynamic theory and its role in conventional aircraft dynamics. The basic objective of this effort was to investigate the extent to which improved knowledge of the unsteady derivatives $C_{L_{\dot{\alpha}}}$, $C_{m_{\dot{\alpha}}}$, etc. would allow for extension of the boundaries which constrain the full potential operation of existing high performance aircraft. To accomplish that task, it was found necessary to develop a simplified but accurate theory for the indicial unsteady aerodynamics. The simplified concepts developed in this report have been extended to a wider range of wing geometries and considered for application to the parameter estimation problem as reported in references [1] - [2]. The results of the analysis are presented following a brief overview of classical and contemporary unsteady aerodynamic theory.

SECTION II

REVIEW OF UNSTEADY WING THEORY

1. Classical Research

A systematic study of unsteady aerodynamics began shortly after Prandtl's theory of bound vortices was published in 1918. This was followed by an analysis of Birnbaum [3] in which he calculated the aerodynamic force on a harmonically oscillating two dimensional flat plate. One of the more referenced and fundamental studies was made by Wagner [4] in which he calculated the unsteady lift on an airfoil impulsively started from rest or equivalently an abrupt change in angle of attack of a plate in a uniform stream. Glauert [5] extended Wagner's method to the problem of computing the forces and moments on an oscillating airfoil. In 1935, Theodorsen [6] published his exact solution for the lift on a harmonically oscillating airfoil with a flap. This work introduced the Theodorsen F and G functions to unsteady aerodynamics theory. Küssner [7], more or less closed this creative era with his solution for the lift on an airfoil as it penetrates a sharp edged gust. The mathematical functions associated with the works of Wagner, Theodorsen, and Küssner were shown to be fundamentally related by Garrick [8].

During the next two decades these fundamental solutions were extended and sharpened by another generation of researchers [9-16]. Von Kármán and Sears [9] developed a uniform circulation theory which treats the previous studies as special cases. The Kármán-Sears theory is outlined in more detail in the Appendix. Jones [10] used the basic results of Wagner to solve for the unsteady lift on a wing of finite aspect ratio. Another significant contribution in Jones' work was the development of an expression for the start-up lift on the wing. Further work was performed on the finite wing by Reissner [12] in which he formulated, in terms of an integral equation, and solved for the lift distribution on oscillating wings. Extension of these results to include compressibility effects were performed by Lomax, et al. [15] for two and three dimensional wings at high speeds.

2. Contemporary Research

The direction of unsteady lifting surface theory during the past decade has been toward digital computer application to the classical formulations [17-25]. Ashley, et al. [17] started this movement in a research article in which he proposed the use of modern digital computers to obtain an exact solution to the oscillating wing problem. In particular, he outlined the method now referred to as the "doublet lattice method" for numerically solving the singular integral equation for the normal velocity at the surface. Albano and Rodden [19] developed the method to an operational form adaptable to complex wing geometries. The method is outlined in detail in Reference 14. The basic idea is to divide the wing surface into trapezoidal panels. The 1/4 chord line of each trapezoid contains a distribution of acceleration potential doublets of uniform but unknown strength. The strength of the doublets is then determined by use of the flow tangency condition at the control points of each trapezoid of the wing.

A vortex-lattice method of determining the transient lift buildup on a finite wing has been presented by Belotserkovskii [21]. This method has been further developed by Atta [22] to account for the wake geometry as well as the wing-tip vortex system. Due to the likely importance of this particular numerical method in computing indicial aerodynamic effects, a brief description of the method will be given using Figure 1 which indicates the manner in which the lattice is formed in the wake and on the wing surface. At time t=1, the initial

free vortex with strength $\delta_i^{(1)}$ is shed in the wake and is placed at an appropriate location forcing satisfaction of the Kutta condition at the trailing edge. At this time, the strengths of the wing bound vortex $\Gamma_{ij}^{(1)}$ are solved for while satisfying the flow tangency conditions at the control points. At time t = 2, the free vortex of strength $\delta_i^{(1)}$ is carried downstream while its strength remains constant. Another free vortex with strength $\delta_i^{(2)}$ forms where $\delta_i^{(1)}$ was. Again, the strengths of the entire vortex field $\Gamma_{ij}^{(2)}$ and $\delta_i^{(2)}$ are solved for from the flow tangency conditions. This process is repeated until no further change occurs in $\Gamma_{ij}^{(k)}$ at which time steady state conditions are reached.

Morino [23] has developed a numerical procedure to solve for the unsteady lift on a wing based on a Green's function formulation. The method is applicable to either oscillatory or transient flow phenomena and applies to both subsonic and supersonic flow. Basically, Morino develops an integral equation for the unsteady velocity potential which involves values of the potential and its normal derivative on the surface of the body and wake. He then applies a finite element method for solving for the potential function.

The most recent contribution to the field of unsteady aerodynamics is due to Edwards [25]. He addresses the problem of active control of supercritical flutter modes. His approach is to solve the basic linear governing equation for the velocity potential by use of Laplace transform techniques.

SECTION III
APPROXIMATE AERODYNAMICS ANALYSES

For use in preliminary design and stability and control analyses, simplified analytical solutions are preferred to more exact numerical ones. Accordingly, an approximate solution to the basic Wagner problem was sought which preserves much of the accuracy of the exact solution and physical principles involved. The simplified aerodynamic model obtained is based on the physical principle of Prandtl's lifting line theory and trailing vortex concept. The wake is assumed to be compressed to a single shed vortex element of appropriate strength moving downstream at a speed sufficient to approximate the Wagner function.

1. Two-Dimensional Wings

In the simplified analysis, the wing is represented by a bound vortex located at the wing quarter-chord line (Figure 2). The boundary condition of no flow through the wing is satisfied at the 3/4 chord line. A step change in wing angle of attack increases the strength of the bound vortex by an amount Γ. An equal amount of vorticity of opposite sign is shed behind the wing to form a continuous vortex sheet which increases in length with time. For the approximation, the shed vortex sheet was replaced by a single shed vortex having the same strength as the bound one and moving with a velocity KU as shown in Figure 2. The distance from the 3/4 chord point at which the vortex is shed, x_o, is determined by forcing the start-up lift to match the results of Jones [10]. For two dimensional wings, $x_o = c/2$. The downwash at the 3/4 chord point is

$$w = \frac{\Gamma}{2\pi}\left[\frac{1}{c/2} + \frac{1}{c/2 + KUt}\right] \tag{1}$$

For a unit step change in angle of attack, w = U resulting in

$$\Gamma = \frac{\pi Uc}{1 + \dfrac{1}{1+\dfrac{2KUt}{c}}} \tag{2}$$

Use of this value of Γ in the Kutta-Joukowski equation results in the lift coefficient

$$\Delta c_\ell = 2\pi\left[1 - \frac{1}{2+\dfrac{2KUt}{c}}\right] \tag{3}$$

It was found by comparison with exact results that a value of K = 1/2 resulted in excellent agreement. It is interesting to note that this value for Δc_ℓ agrees with an empirical value obtained by a curve fit to Wagner's solution by Garrick [8].

To get the change in lift coeffecient for an arbitrary change in angle of attack $\alpha(t)$, Duhamel's integral formula is used; that is,

$$c_\ell(t) = 2\pi \int_o^t \left[1 - \frac{1}{2 + \frac{U}{c}(t-\tau)}\right] \dot{\alpha}(\tau)d\tau \tag{4}$$

To illustrate the use of equation (4) to compute $c_{\ell_{\dot{\alpha}}}$ consider the case where $\alpha = k't$. From equation (4), we get

$$c_\ell(t) = 2\pi k't - \frac{2\pi k'c}{U} \ln\left(1 + \frac{Ut}{2c}\right) \tag{5}$$

To get $c_{\ell_{\dot{\alpha}}}$ differentiate with respect to $\frac{k'c}{2U}$

$$c_{\ell_{\dot{\alpha}}} = \frac{4U}{c} \pi t - 4\pi \ln\left(1 + \frac{Ut}{2c}\right) \tag{6}$$

The hysteresis effect due to the unsteadiness as computed from equation (5) is shown in Figure 3 for U = 100 fps and c = 5 ft. The curve labeled $\dot{\alpha} > 0$ corresponds to α increasing linearly up to 10° ($k' = \pi/180$). The angle of attack is then held constant until steady state is reached. The curve labeled $\dot{\alpha} < 0$ corresponds to α decreasing linearly from 10° to 0° ($k' = -\pi/180$). The curve labeled "quasi-steady" corresponds to the instantaneous lift buildup due to a finite change in angle-of-attack.

2. Three-Dimensional Untapered Wings

The Wagner function for 3-dimensional wings can be approximated utilizing the same concepts as for 2-dimensional wings as shown in Figure 4 for a straight wing. At the control point P, a unit change in α results in a downwash of value

$$\Delta w = \frac{\Gamma}{\pi b}\left[\sqrt{A^2+1} + \sqrt{1 + \left(\frac{b}{2z}\right)^2}\right] \tag{7}$$

where b is the wing span, A the aspect ratio and

$$z = x_o + \frac{Ut}{2} \tag{8}$$

The corresponding values of circulation and lift coefficient are computed as

$$\Delta\Gamma = \frac{\pi b U}{\sqrt{A^2+1} + \sqrt{1+(b/2z)^2}} \tag{9}$$

$$\Delta C_L = \frac{2\pi A}{\sqrt{A^2+1} + \sqrt{1+(b/2z)^2}} \tag{10}$$

For an arbitrary change in angle of attack, the unsteady lift coefficient is computed by Duhamel's integral formula as

$$C_L = 2\pi A \int_0^t \frac{\dot{\alpha}(\tau)\, d\tau}{\sqrt{A^2+1} + \sqrt{1+\left(\frac{b}{2x_o+U(t-\tau)}\right)^2}} \qquad (11)$$

To find the location of the starting point x_o, we use the initial value due to Jones [10] for an elliptical planform wing, i.e.,

$$\Delta C_L(o) = \pi/E \qquad (12)$$

where $E = \frac{\text{Semi-perimeter}}{\text{span}} = \frac{A+1}{A}$

This results in

$$x_o/c = \frac{A/2}{[(2(A+1) - \sqrt{A^2+1})^2 - 1]^{1/2}} \qquad (13)$$

A plot of x_o/c versus aspect ratio is shown in Figure 5.

The lift as computed from equation (11) is well suited for computation of the unsteady stability derivatives $C_{L_{\dot{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$. For instance, $C_{L_{\dot{\alpha}}}$ due to a linear change in α is computed from equation (11) as (letting $x_o = c/2$ which is the limiting value for large aspect ratio wings)

$$C_{L_{\dot{\alpha}}} = \frac{4\pi U}{c}\,\frac{\sqrt{A^2+1}}{A}\left\{t - \frac{c}{U\sqrt{A^2+1}}\left[\sqrt{A^2+2\left(1+\frac{Ut}{c}\right)} - \sqrt{A^2+1}\right]\right.$$

$$\left. - \frac{c}{U}\,\ell n\left[\left(1+\frac{Ut}{2c}\right)\left(\frac{2\sqrt{A^2+1}}{\sqrt{A^2+1} + \sqrt{A^2+2\left(1+\frac{Ut}{c}\right)}}\right)\right]\right\} \qquad (14)$$

The Wagner function for the three-dimensional swept wing can be calculated by consideration of the geometry shown in Figure 6. The calculation is carried out for a swept wing with zero taper undergoing a unit step change in angle of attack.

The change in downwash at point P is computed as

$$w = \frac{\Gamma}{\pi b}\left[\sqrt{A^2\sec^2\Lambda - 2A\tan\Lambda + 1} + \frac{1}{2zb}\sqrt{b^2+4z^2}\right. \tag{15}$$

$$\left. + \frac{A}{2}\tan\Lambda\left(1-\frac{c}{2z}\right) + \frac{b\tan\Lambda}{\sqrt{b^2+4z^2}}\right]$$

The corresponding change in lift is

$$\Delta C_L = 2\pi A\left[\sqrt{A^2\sec^2\Lambda - 2A\tan\Lambda + 1} + \frac{1}{2zb}\sqrt{b^2+4z^2}\right. \tag{16}$$

$$\left. + \frac{A}{2}\tan\Lambda\left(1-\frac{c}{2z}\right) + \frac{b\tan\Lambda}{\sqrt{b^2+4z^2}}\right]^{1/2}$$

3. Horizontal Tail Lift

In order to calculate the unsteady tail lift load, an expression must first be obtained for the unsteady downwash at the tail surface to the wing. For this calculation, the shed vortex is allowed to move downstream at the free stream velocity. This assumption causes the initial disturbance to reach the tail surface at the proper time. Numerical tests indicate that the time variation in downwash at this station is not substantially different for either assumption of shed vortex velocity equal to U or U/2. The geometry for the calculation of downwash at the quarter chord point of the horizontal tail surface (point P) is presented in Figure 7. The downwash angle at point P due to the effect of the wing wake is (for unit step in wing angle of attack)

$$\Delta\varepsilon = \frac{\Gamma}{\pi Ub}\left[\frac{1}{\ell_t}\sqrt{b^2+4\ell_t^2} + \frac{\sqrt{b^2+4(c/2+x_o-\ell_t+Ut)^2}}{c/2+x_o-\ell_t+Ut}\right] \tag{17}$$

where

$$\Gamma = \frac{\pi Ub}{\sqrt{1+A^2} + \sqrt{1+\left[\frac{b}{2(x_o+Ut)}\right]^2}} \tag{18}$$

The effective angle of attack at the tail surface (for small rotation) is equal to the wing angle of attack less the downwash angle $\alpha-\varepsilon$. This provides an estimate of the tail lift for a unit step change in effective angle of attack as

$$\Delta C_{L_t}(t) = \frac{2\pi A_t}{\sqrt{A_t^2 + 1} + \sqrt{1 + \left(\frac{b_t}{2x_{o,t}+Ut}\right)^2}} \tag{19}$$

where the subscript "t" refers to quantities based on horizontal tail surface geometry. For arbitrary changes in angle attack, the unsteady tail surface lift coefficient is

$$C_{L_t}(t) = \int_o^t \Delta C_{L_t}(t-\tau)[\dot{\alpha}(\tau)-\dot{\varepsilon}(\tau)]\, d\tau \tag{20}$$

where

$$\varepsilon(t) = \int_o^t \Delta\varepsilon(t-\tau)\, \dot{\alpha}(\tau)\, d\tau$$

SECTION IV

AIRCRAFT DYNAMICS: LONGITUDINAL MODES

1. Equations of Motion

For a dynamic model of the longitudinal motion, we assume the following perturbation equations of motion

$$\dot{u} = g\theta + \frac{\rho U^2 S}{2m} C_x \tag{21}$$

$$\dot{\alpha} = q + \frac{\rho U S}{2m} C_z \tag{22}$$

$$\dot{q} = \frac{\rho U^2 Sc}{2I_y} C_m \tag{23}$$

$$\dot{\theta} = q \tag{24}$$

The axial force coefficient for the airplane is assumed to be a function of angle of attack only according to

$$C_x = C_{x_\alpha}\,\alpha \tag{25}$$

The normal force coefficient for the airplane is taken as the sum of the unsteady wing lift, the unsteady tail lift and elevator contribution according to

$$C_z = -\int_o^t \Delta C_{L_w}(t-\tau)\dot{\alpha}(\tau)d\tau - \frac{S_t}{S}\int_o^t \Delta C_{L_t}(t-\tau)[\dot{\alpha}(\tau)-\dot{\epsilon}(\tau)]d\tau + C_{z_{\delta_e}}\delta_e \tag{26}$$

The moment coefficient for the airplane is taken as the sum of the effects of the tail lift, the contribution due to pitch, and that due to elevator according to

$$C_m = -\frac{\ell_t S_t}{cS}\int_o^t \Delta C_{L_t}(t-\tau)[\dot{\alpha}(\tau) - \dot{\epsilon}(\tau)]d\tau \tag{27}$$

$$+ C_{m_q}\frac{qc}{2U} + C_{m_{\delta_e}}\delta_e$$

Reference to equations (21)-(24) indicates that u and θ can be computed once q and α are determined. Integro-differential equations for α and q are obtained upon substitution from equations (26) and (27) into equations (22) and (23) resulting in

$$\dot{\alpha} = q - \frac{\rho US}{2m}\left\{\int_o^t \Delta C_{L_w}(t-\tau)\dot{\alpha}(\tau)d\tau\right. \tag{28}$$

$$\left. + \frac{S_t}{S}\int_o^t \Delta C_{L_t}(t-\tau)[\dot{\alpha}(\tau) - \dot{\epsilon}(\tau)]d\tau - C_{z_{\delta_e}}\delta e\right\}$$

$$\dot{q} = -\frac{\rho U^2 Sc}{2I_y}\left\{\frac{\ell_t S_t}{cS}\int_o^t \Delta C_{L_t}(t-\tau)[\dot{\alpha}(\tau) - \dot{\epsilon}(\tau)]d\tau\right. \tag{29}$$

$$\left. -C_{m_q}\frac{qc}{2U} - C_{m_{\delta_e}}\delta_e\right\}$$

Equations (28) and (29) can be solved for α and q numerically or by Laplace transform techniques provided certain approximations for ΔC_L and $\Delta\varepsilon$ are used as shown in the following section.

2. Approximate Solution in Frequency Domain

The time domain equations for α and q as given from equations (26) and (27) are difficult to solve even with modern high-speed digital computers due to the presence of the convolution integrals. However, they are considerably more tractable if they are transformed to the frequency domain after making the following approximations:

$$\Delta C_{L_w} = (C_{L_{\alpha_w}})_\infty \left[1 - a_1 \exp\left(-\frac{b_1 Ut}{c_{w/2}}\right)\right] \tag{30}$$

$$\Delta C_{L_t} \simeq (C_{L_{\alpha_t}})_\infty \left[1 - a_2 \exp\left(-\frac{b_2 Ut}{c_{t/2}}\right)\right] \tag{31}$$

$$\Delta\varepsilon = \left(\frac{\partial\varepsilon}{\partial\alpha}\right)_\infty \left[1 - \frac{f_1}{\frac{\ell_t}{c} - 1 - \frac{Ut}{c}} - g_1 \exp\left(-\frac{h_1 Ut}{c/2}\right)\right] \tag{32}$$

where $\left(C_{L_{\alpha_w}}\right)_\infty$, $\left(C_{L_{\alpha_t}}\right)_\infty$ and $\left(\frac{\partial\varepsilon}{\partial\alpha}\right)_\infty$ are final steady state values of ΔC_{L_w}, ΔC_{L_t} and $\Delta\varepsilon$ respectively. The constants a_1, b_1, a_2, b_2, f_1, g_1, and h_1 are determined by curve fits to the results represented by Equations (16)-(19). These approximations result in the following expressions for $\bar{C}_z$, $\bar{C}_m$, and $\bar{\varepsilon}$:

$$\bar{C}_z(s) = -s\bar{\alpha}(s)\,\Delta\bar{C}_{L_w}(s) - \frac{S_t}{S}\,s\,(\bar{\alpha}(s) - \bar{\varepsilon}(s))\,\Delta\bar{C}_{L_t}(s) + C_{z_{\delta_e}}\bar{\delta}_e(s) \tag{33}$$

$$\bar{C}_m(s) = -\frac{\ell_t S_t}{cS}\,s\,(\bar{\alpha}(s) - \bar{\varepsilon}(s))\,\Delta\bar{C}_{L_t}(s) + C_{m_q}\frac{c}{2U}\bar{q}(s) + C_{m_{\delta_e}}\bar{\delta}_e(s) \tag{34}$$

$$\bar{\varepsilon}(s) = s\bar{\alpha}(s)\Delta\bar{\varepsilon}(s) \tag{35}$$

where

$$\Delta\bar{C}_{L_w}(s) = \left(C_{L_{\alpha_w}}\right)_\infty \left[1/s - \frac{a_1}{s + b_1\left(\frac{U}{c_{w/2}}\right)}\right] \tag{36}$$

$$\Delta\bar{C}_{L_t}(s) = \left(C_{L_{\alpha_t}}\right)_\infty \left[1/s - \frac{a_2}{s + b_2\left(\frac{U}{c_{w/2}}\right)}\right] \tag{37}$$

and

$$\Delta\bar{\varepsilon}(s) = \left(\frac{\partial\varepsilon}{\partial\alpha}\right)_\infty \left\{\frac{1}{s} - \frac{g_1}{s-h_1 Ut/c/2} + \frac{c}{U} f_1 \exp\left[\frac{sc}{U}\left(\frac{\ell_t}{c} - 1\right)\right] \right. \quad (38)$$
$$\left. \times \text{Ei}\left[\frac{sc}{U}\left(1 - \frac{\ell_t}{c}\right)\right]\right\}$$

Ei is the error integral whose series form is

$$\text{Ei}(s) = \nu + \ell n\, s + \sum_{n=1}^{\infty} \frac{s^n}{n \cdot n!} \quad (39)$$

ν is the Euler constant with numerical value 0.5772156.

We note that $\bar{C}_z(s)$ and $\bar{C}_m(s)$ can be rewritten in the convenient form

$$\bar{C}_z(s) = f(s)\,\bar{\alpha}(s) + C_{z_{\delta_e}}\,\bar{\delta}_e(s) \quad (40)$$

$$\bar{C}_m(s) = g(s)\,\bar{\alpha}(s) + C_{m_q}\,\frac{c_w}{2U}\,\bar{q}(s) + C_{m_{\delta_e}}\,\bar{\delta}_e(s) \quad (41)$$

Upon taking the Laplace transforms of equations (21)-(22) and substitution from equations (40)-(41), the following matrix equation for $\bar{\alpha}(a)$ and $\bar{q}(s)$ evolves

$$\begin{bmatrix} s - \frac{\rho U S_w}{2m} f(s) & -1 \\ -\frac{\rho U^2 S_w c_w}{2I_y} g(s) & s - \frac{\rho U S_w c_w^2}{4I_y} C_{m_q} \end{bmatrix} \begin{bmatrix} \bar{\alpha}(s) \\ \bar{q}(s) \end{bmatrix} = \begin{bmatrix} \frac{\rho U S_w}{2m} C_{z_{\delta_e}} \\ \frac{\rho U^2 S_w c_w}{2I_y} C_{m_{\delta_e}} \end{bmatrix} \bar{\delta}_e(s) \quad (42)$$

from which expressions for the transfer functions

$\frac{\bar{\alpha}(s)}{\bar{\delta}_e(s)}$ and $\frac{\bar{q}(s)}{\bar{\delta}_e(s)}$ can be found.

SECTION V

APPLICATIONS

In order to assess the effect of the unsteady terms on the aircraft's response to control inputs, calculations have been made using the Navion (Figure 8) aircraft as subject. This aircraft was chosen since its characteristics have been studied extensively from flight research data and well documented by recent NASA technical reports [26]. The geometric and mass characteristics of the Navion aircraft are listed in Table I and the quasi-steady stability and control derivatives are listed in Table II. For the simulation, a trimmed flight condition at 5,000 feet altitude and speed of 240 ft/sec was used. At this time a step input of -30° elevator deflection was applied and held for 4 seconds. The subsequent responses of the aircraft using both quasi-steady and unsteady theory is shown in Figures 9-10. In Figure 9, the aircraft's angle of attack and pitch rate exhibit a more sluggish response to the elevator input when unsteady aerodynamics are included. This is to be expected due to the hysteresis previously noted in Figure 5. The reduction in oscillatory behavior in pitch rate is particularly noticeable. From Figure 10, it can be seen that the unsteady aerodynamics has considerably less effect on the aircraft's forward speed and attitude response. These effects are summarized in Table III.

Frequency response curves for the two transfer functions defined by Equation (42) are shown in Figure 11 for the Navion aircraft. For comparison purposes the quasi-steady form of the transfer function is also presented. The unsteadiness is seen to have a pronounced effect on the pitch rate response of the aircraft.

SECTION VI

CONCLUSIONS AND RECOMMENDATIONS

A simplified aerodynamic model based on the principle of Prandtl's lifting line theory and trailing vortex concept has been developed for use in preliminary design and stability and control analyses. The model proved to be applicable to both two and three dimensional wing theory in calculating unsteady lift load for arbitrary angle of attack variation. Sample numerical solutions to the longitudinal equations incorporating the unsteady model are presented for the Navion aircraft. From these calculations, an apparent increase in damping effects in the aircraft response is noted.

For future analyses, it is recommended that the control input rates be included in the aircraft equations of motion to more realistically simulate and assess the role of the unsteadiness in the indicial response. Further, a study of the aircraft response should be performed with aeroelastic effects and unsteady aerodynamics included.

TABLE 1

GEOMETRIC AND MASS CHARACTERISTICS OF NAVION

Aircraft Mass, slugs	92.17
Aircraft Moment of Inertia (I_y), slug-ft^2	2772.86
Wing:	
Area, ft^2	184.0
Aspect Ratio	6.04
Span, ft	33.38
Mean chord, ft	5.70
Horizontal Tail:	
Area, ft^2	43.0
Aspect ratio	4.0
Span, ft	12.70
Mean chord, ft	3.30
Tail length (center of gravity to tail aerodynamic center), ft	15.12

TABLE 2

STABILITY AND CONTROL DERIVATIVES OF NAVION

C_{x_α}	0.262
C_{z_α}	-4.33
C_{z_q}	-15.90
$C_{z_{\delta_e}}$	-0.511
C_{m_α}	-0.63
C_{m_q}	-18.10
$C_{m_{\delta_e}}$	-1.42

TABLE 3

STABILITY CHARACTERISTICS

Specification	α		q	
	Quasi-steady	Unsteady	Quasi-steady	Unsteady
Percent overshoot	4.5%	1%	85%	50%
Time to peak	0.6 sec	0.9 sec	0.3 sec	0.2 sec
Rise time (90% of steady state value)	0.35 sec	0.45 sec	0.1 sec	0.1 sec
Settling time	1.6 sec	1.3 sec	1.3 sec	0.8 sec

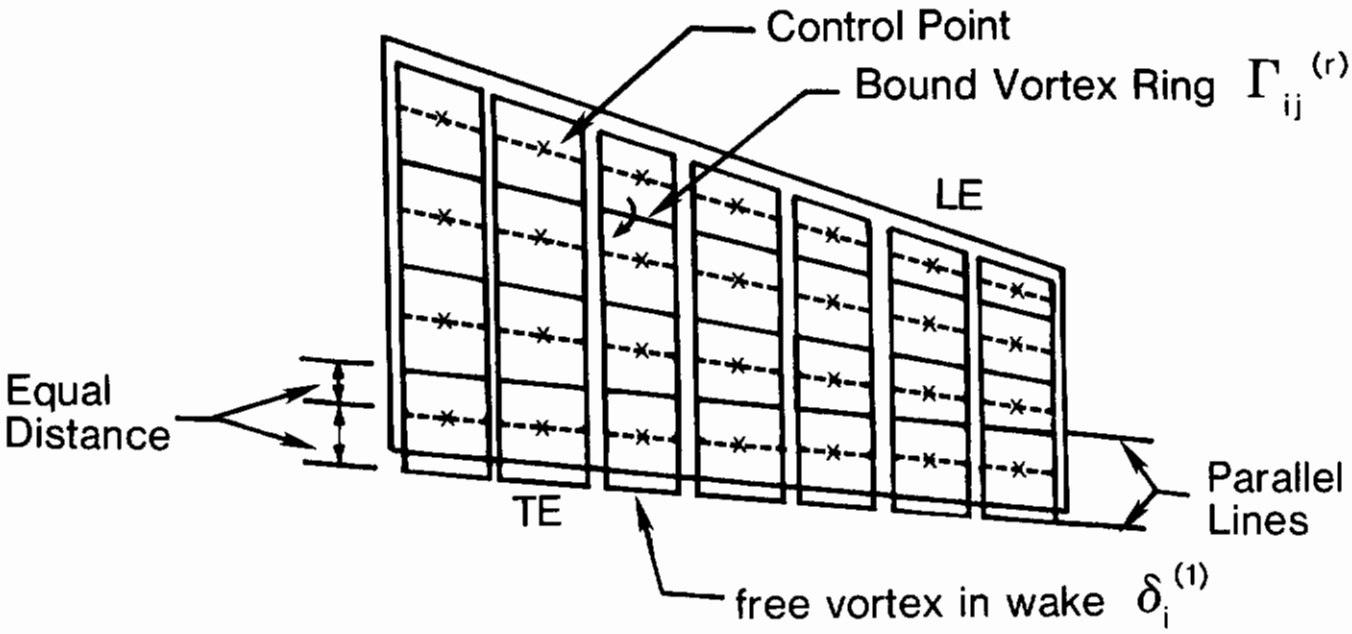


Figure 1. Vortex-lattice Method of Belotserkovskii

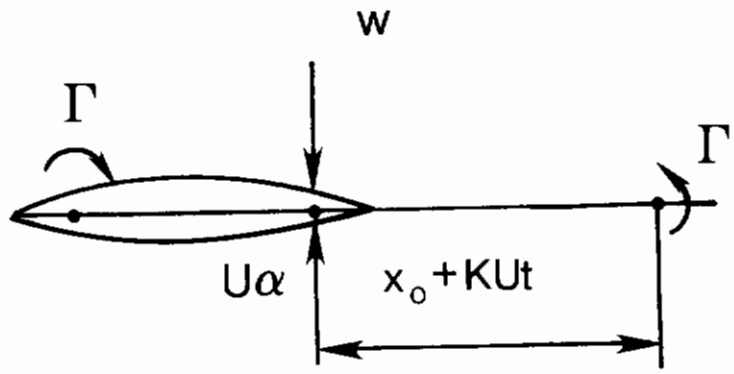


Figure 2. Simplified Vortex System

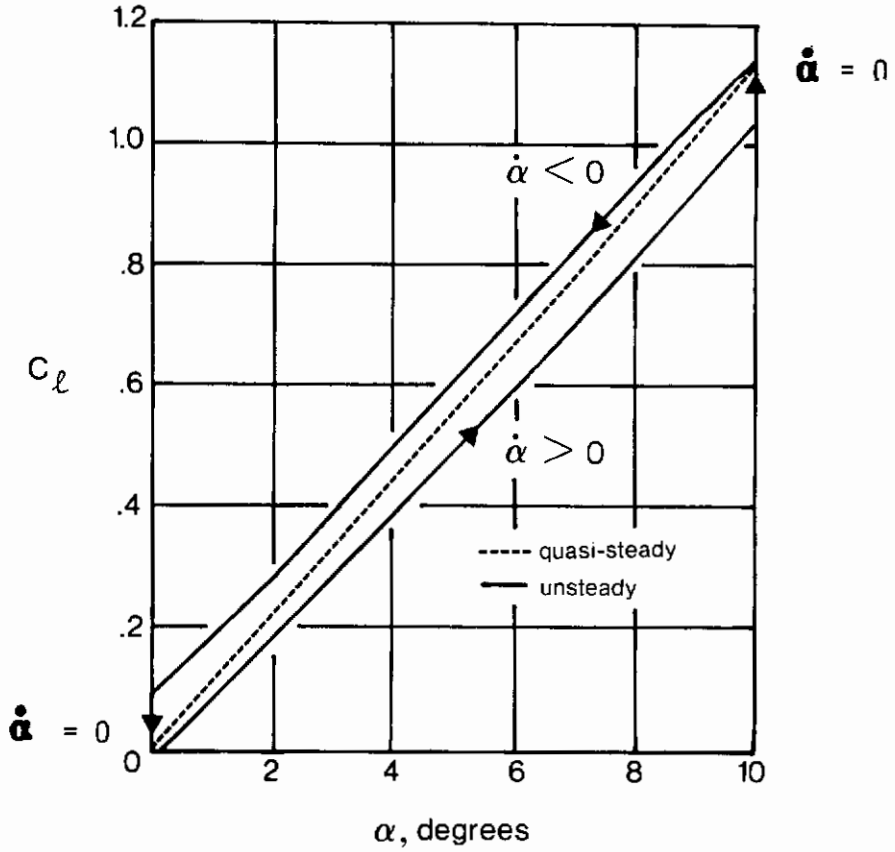


Figure 3. Hysteresis Effect Due to Unsteadiness

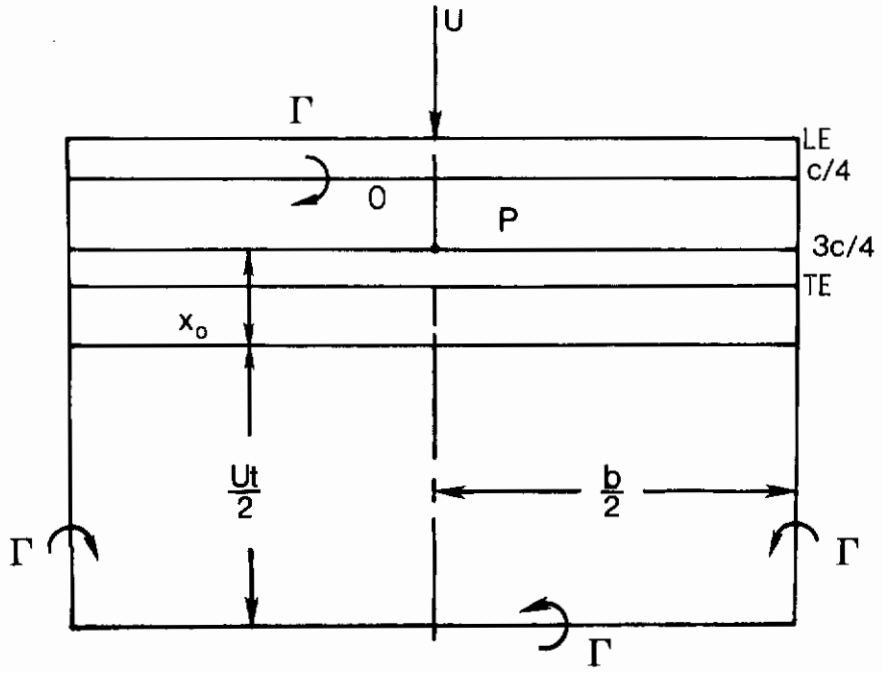


Figure 4. Simplified Vortex System for Straight Wings

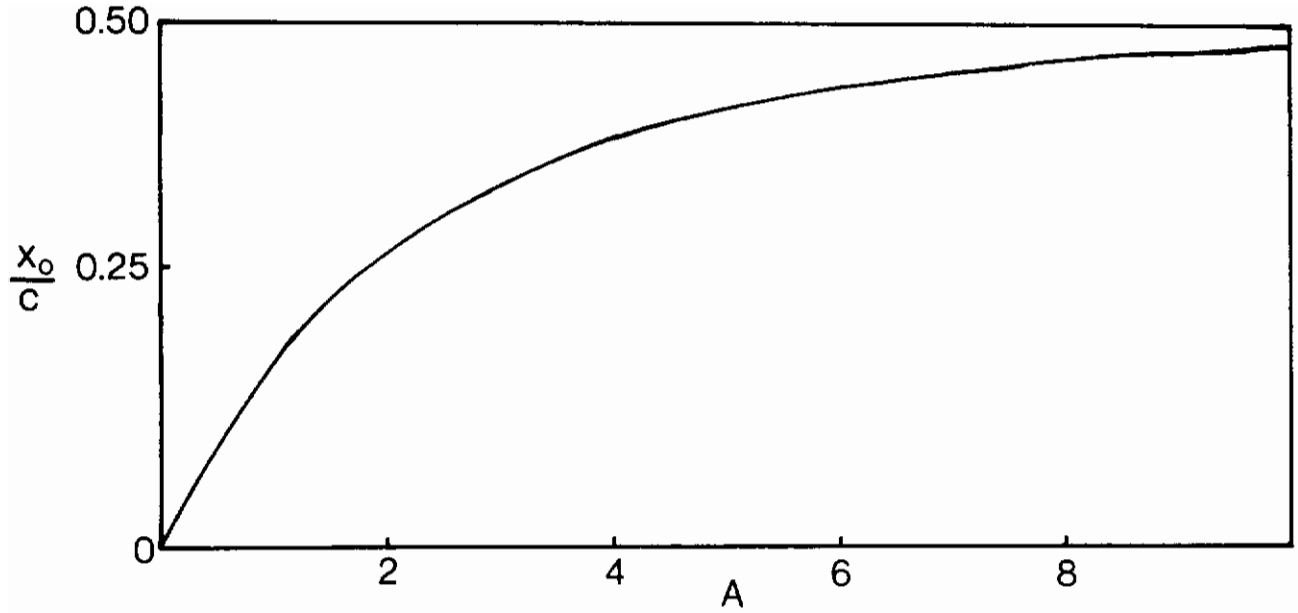


Figure 5. Variation of x_o/c With Aspect Ratio

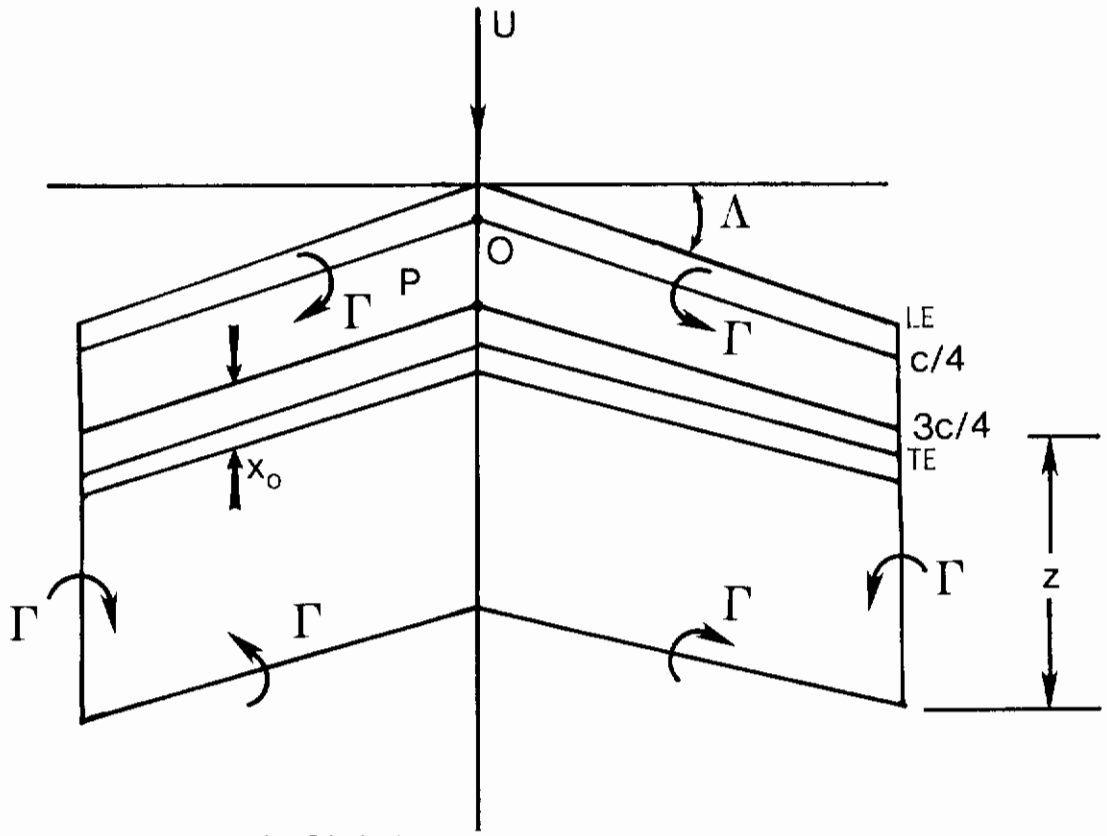


Figure 6. Simplified Vortex System for Swept Wings

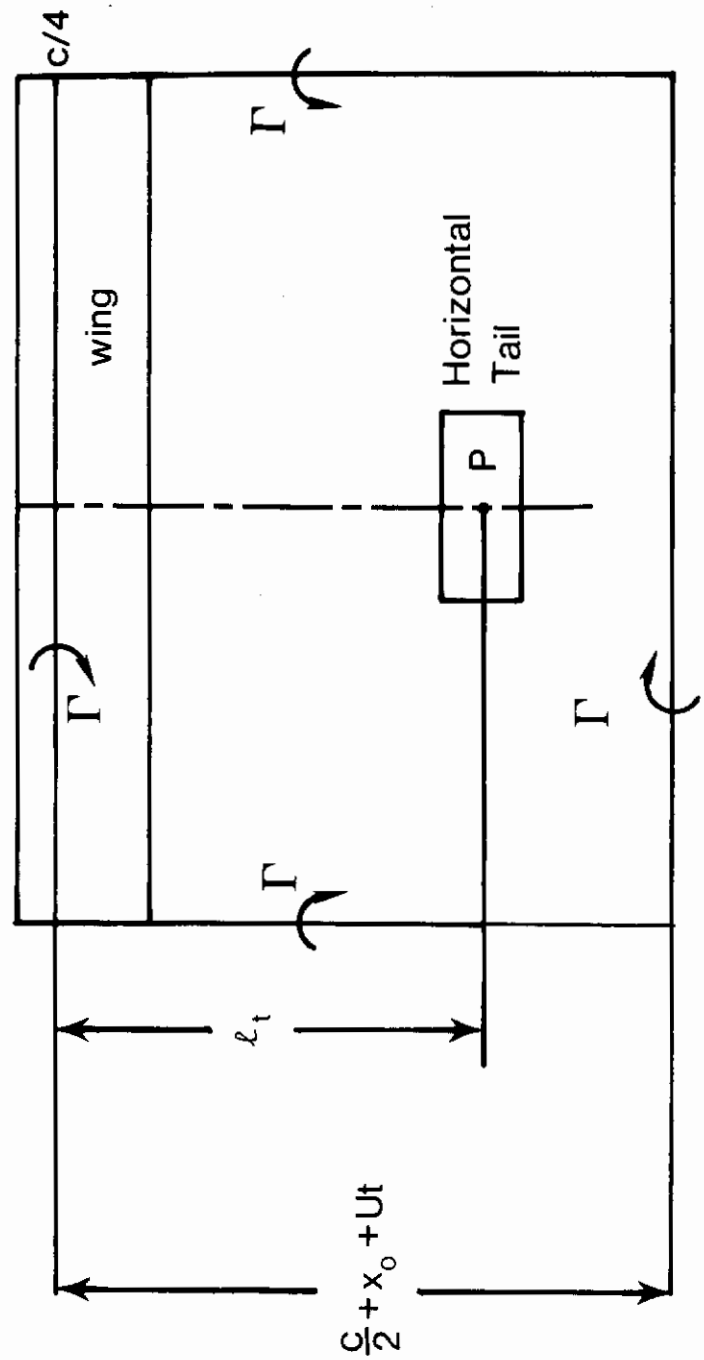
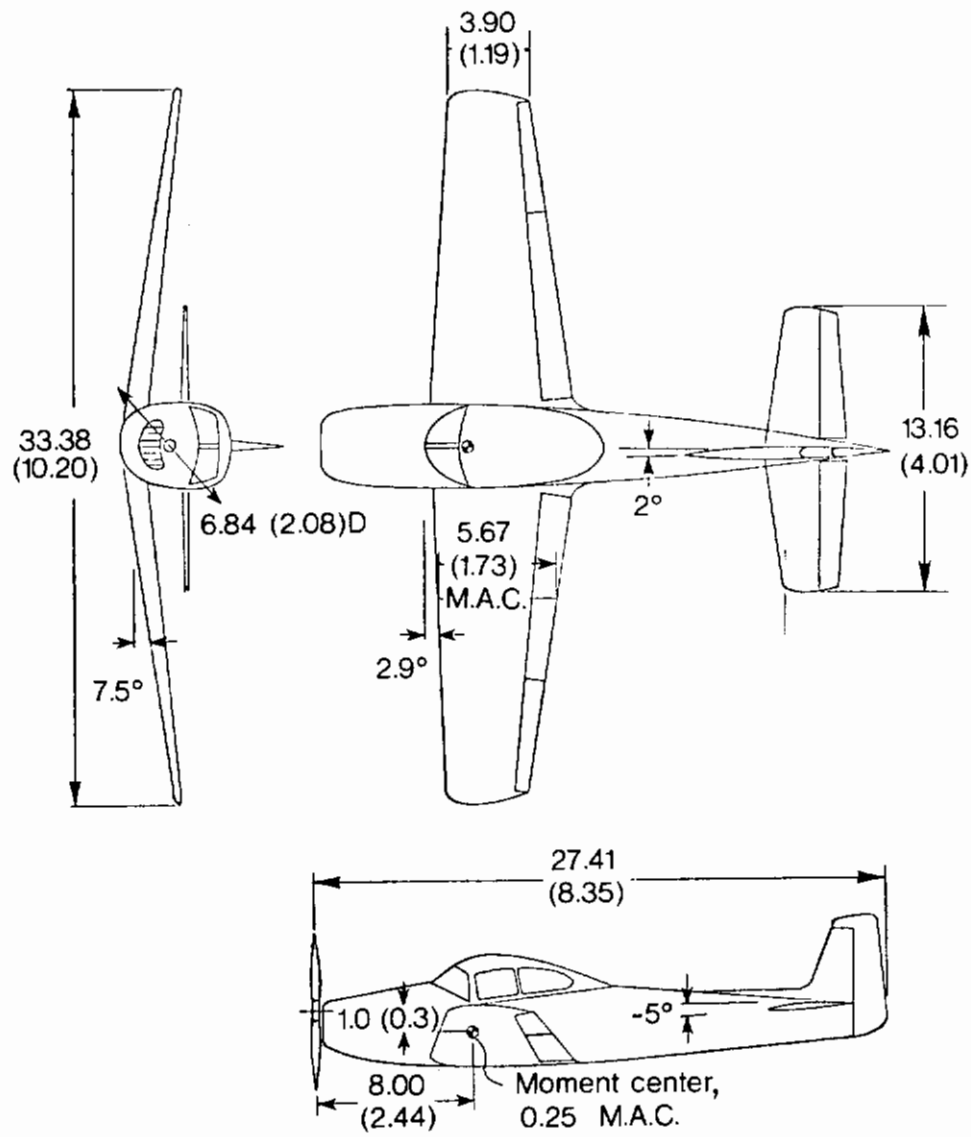


Figure 7. Simplified Vortex System Used in Calculation of Tail Lift



All dimensions are in feet (meters).

Figure 8. Sketch of Navion Aircraft

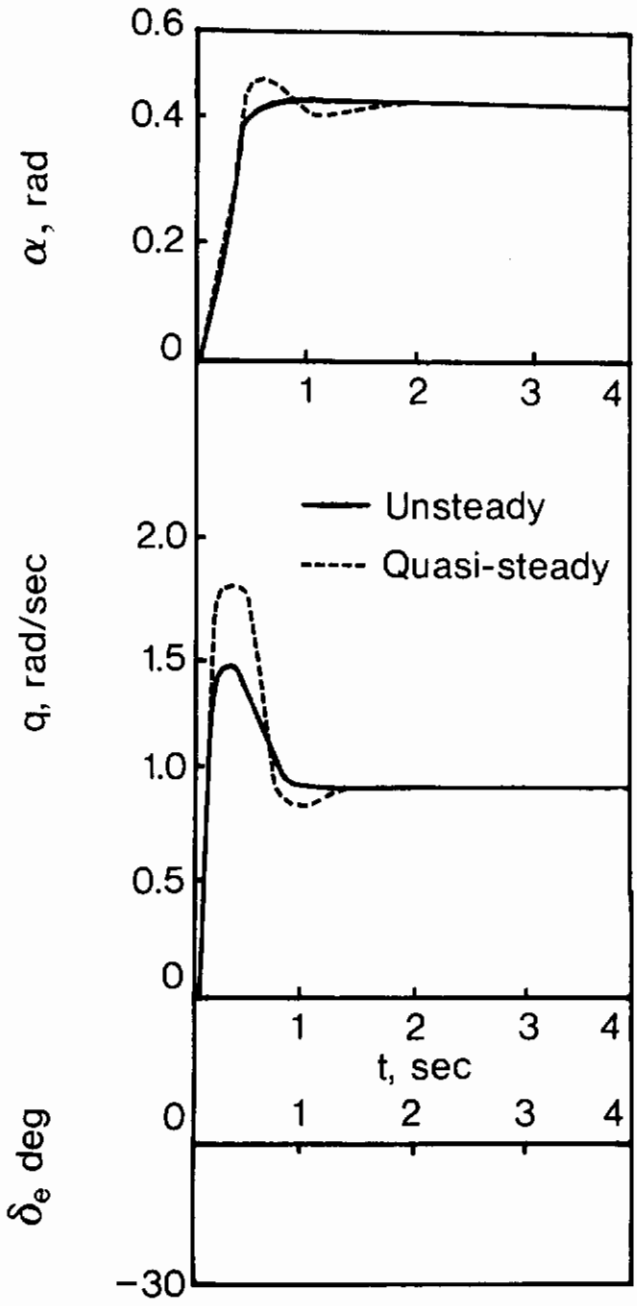


Figure 9. Angle of Attack and Pitch Rate Response to Elevator Step Input

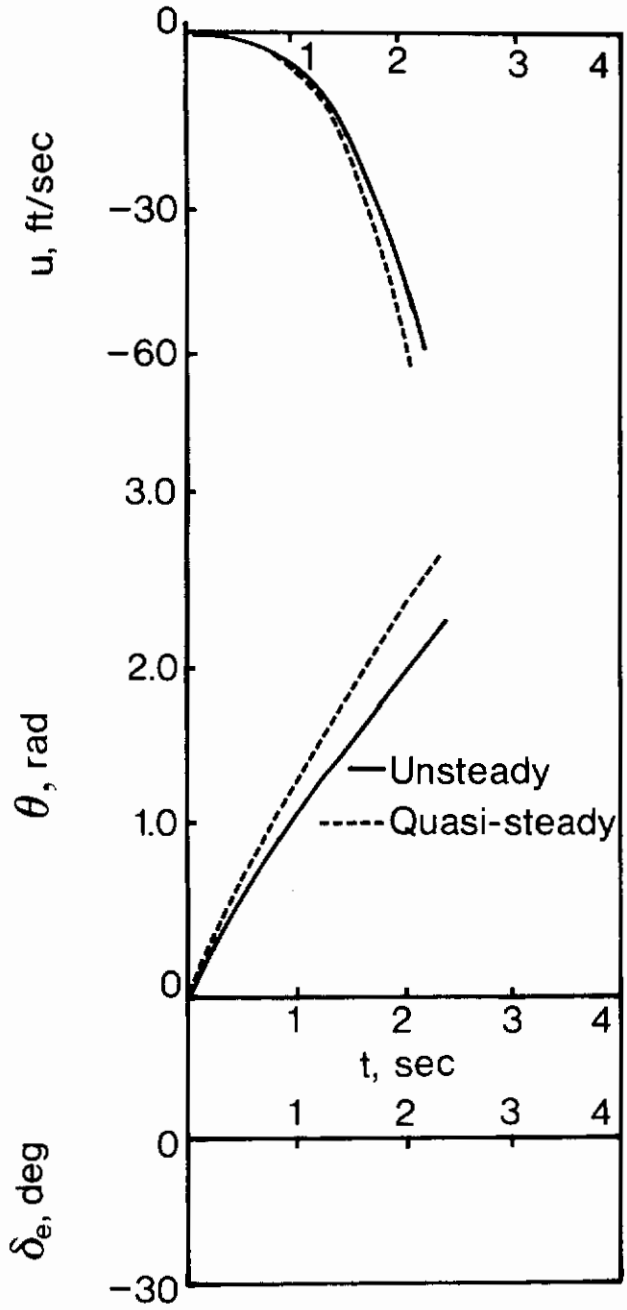


Figure 10. Velocity and Pitch Response to Elevator Step Input

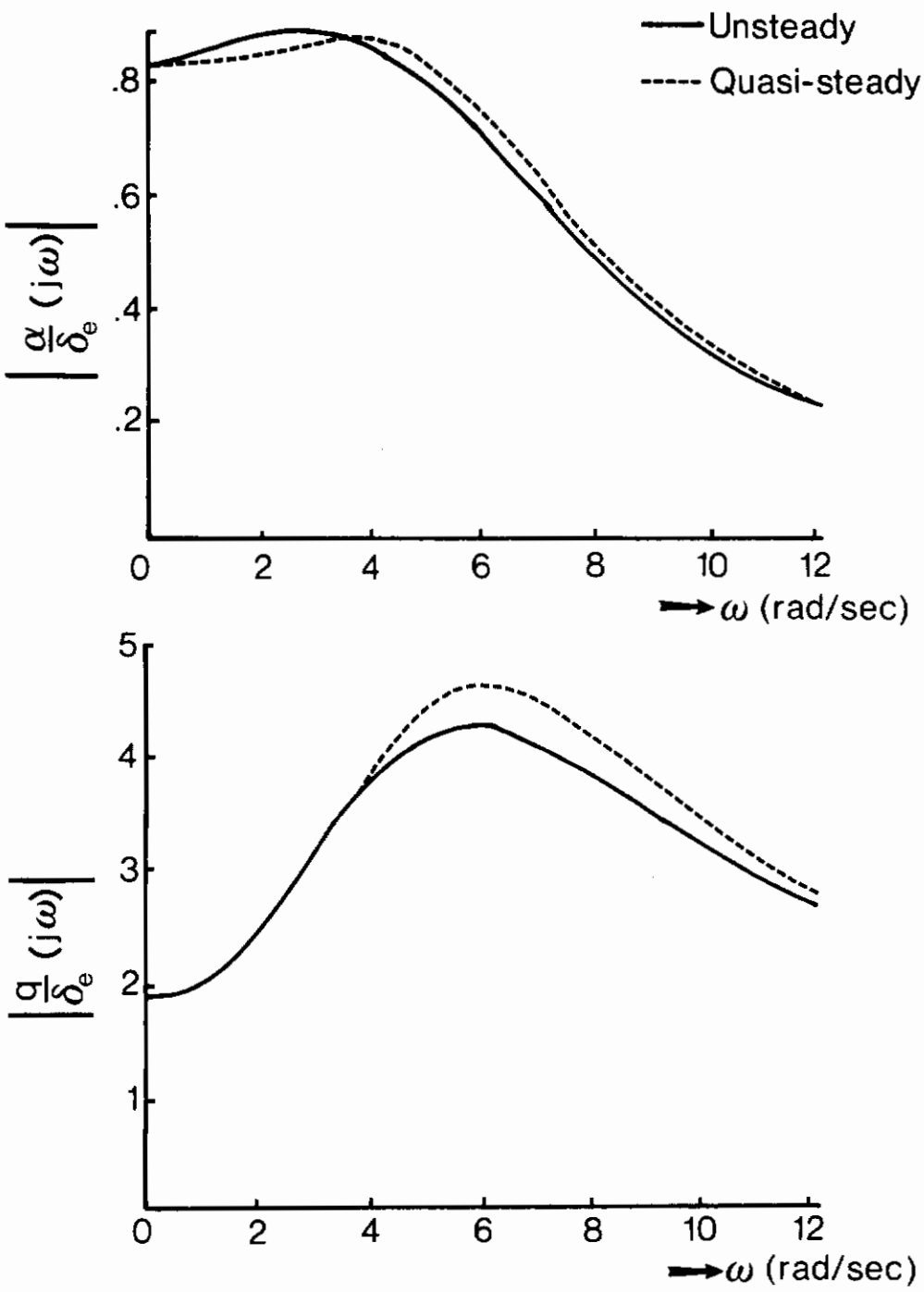


Figure 11. Frequency Response of Navion Aircraft

APPENDIX

VON KÁRMÁN - SEARS SOLUTION TO THE WAGNER PROBLEM

The problem Wagner [4] solved was the transient lift generated on a flat plate airfoil inclined to the freestream at an angle of attack α started impulsely from rest as shown in Figure A1. The circulation around the airfoil changes for unsteadiness in either the velocity or the angle of attack of the airfoil. Consequently, an equal amount of vorticity of opposite sign forms in the wake and remains fixed relative to the airfoil. First we consider the effect of an isolated free vortex of strength $\gamma(\xi)d\xi$ located at a distance ξ from the origin. The total circulation Γ about the airfoil is made up of two parts: that due to motion with no wake (quasi-steady motion) with circulation Γ_0 and the motion induced by the wake which creates a circulation Γ_1 around the airfoil.

Then

$$\Gamma = \Gamma_0 + \Gamma_1 = \int_{-1}^{+1} \gamma(x)dx \qquad (A1)$$

The value of Γ_0 (quasi-steady theory) is given as

$$\Gamma_0 = \rho U c \qquad (A2)$$

To obtain Γ_1 we must first determine $\gamma(x)$, the vorticity distribution along the plate. This is obtained easiest from methods of conformal mapping according to Figure A2. In the z'- plane, the vortex pair Γ' and $-\Gamma'$ located at η and $1/\eta$ respectively will create a stream function given by

$$\Psi(r, \beta) = \frac{\Gamma'}{2\pi} \ell n\, r_2 - \frac{\Gamma'}{2\pi} \ell n\, r_1 \qquad (A3)$$

where

$$r_1 = [r^2 + (1/\eta)^2 - 2r(1/\eta)\cos\beta]^{1/2} \tag{A4}$$

$$r_2 = [r^2 + \eta^2 - 2r\eta\cos\beta]^{1/2}$$

This results in a velocity field on the surface of the cylinder given by

$$u_r = o$$
$$u_\beta = \frac{\Gamma'}{2\pi}\sqrt{\frac{\xi^2-1}{\xi-\cos\beta}} \tag{A5}$$

At the trailing edge of the plate ($\beta = 0$) we have

$$u_\beta = \frac{\Gamma'}{2\pi}\sqrt{\frac{\xi+1}{\xi-1}} \tag{A6}$$

Since we seek a physically realistic flow satisfying Kutta's condition at the trailing edge ($\beta = 0$) we must distribute along the plate a vorticity of strength $\gamma(x)$ which will create, on the cylinder, a tangential velocity $-\frac{\Gamma'}{2\pi}\sqrt{\frac{\xi+1}{\xi-1}}$ and radial velocity zero so that the sum of this value and that from equation (A5) results in a surface tangential flow field satisfying the Kutta condition at the trailing edge. Then the resultant tangential velocity at the surface is given as

$$u_\beta = \frac{\Gamma'}{2\pi}\left[\sqrt{\frac{\xi^2-1}{\xi-\cos\beta}} - \sqrt{\frac{\xi+1}{\xi-1}}\right] \tag{A7}$$

The corresponding point on the surface of the flat plate has the velocity.

$$q' = \frac{u_\beta}{\sin\beta} = \frac{\Gamma'}{2\pi\sin\beta}\left[\frac{\sqrt{\xi^2-1}}{\xi-\cos\beta} - \sqrt{\frac{\xi+1}{\xi-1}}\right] \tag{A8}$$

Note that on the surface of the plate, $x = \cos\beta$. Also note that on the upper surface $\sin\beta > o$ and on the lower surface $\sin\beta < o$. This results in

$$q'_u = \frac{1}{2\pi}\frac{\Gamma'}{\xi-x}\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{\xi+1}{\xi-1}} \tag{A9}$$

$$q'_{\ell} = \frac{1}{2\pi} \frac{\Gamma'}{\xi - x} \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{\xi+1}{\xi-1}} \tag{A10}$$

Since $\gamma(x)$ equals the jump in tangential velocity across the plate we have

$$\gamma(x) = q'_u - q'_{\ell} = \frac{1}{\pi} \frac{\Gamma'}{\xi - x} \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{\xi+1}{\xi-1}} \tag{A11}$$

Then the circulation around the plate due to the free vortex Γ' located at ξ is

$$\int_{-1}^{+1} \gamma(x)dx = \frac{\Gamma'}{\pi} \sqrt{\frac{\xi+1}{\xi-1}} \int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}} \frac{dx}{\xi - x} \tag{A12}$$

$$= \Gamma' \left[\sqrt{\frac{\xi+1}{\xi-1}} - 1\right]$$

For a continuous distribution $\Gamma' = \gamma(\xi)d\xi$, the total circulation about the plate due to the wake is obtained by integrating over the length of the wake ℓ_w, i.e.,

$$\Gamma_1 = \int_1^{\ell_w} \gamma(\xi) \left(\sqrt{\frac{\xi+1}{\xi-1}} - 1\right) d\xi \tag{A13}$$

Then the total circulation about the plate is

$$\Gamma = \pi U c + \int_1^{\ell_w} \gamma(\xi) \left(\sqrt{\frac{\xi+1}{\xi-1}} - 1\right) d\xi \tag{A14}$$

The remaining unknown is the strength of the vorticity in the wake $\gamma(\xi)$. We find this by use of the fact that the total circulation of the whole system (plate and wake) is zero, i.e.,

$$\Gamma + \int_1^{\ell_w} \gamma(\xi)d\xi = 0 \tag{A15}$$

or

$$\pi U c + \int_1^{\ell_w} \gamma(\xi) \sqrt{\frac{\xi+1}{\xi-1}} \, d\xi = 0 \tag{A16}$$

For the impulsively started plate $\ell_w = 1+Ut$ and $\gamma(\xi)$ is determined from the integral equation

$$1 + \int_{1}^{1+Ut} \frac{\gamma(\xi)}{\pi Uc} \sqrt{\frac{\xi+1}{\xi-1}}\, d\xi = 0 \tag{A17}$$

Wagner solved this numerically for $\gamma(\xi)/\pi Uc$.

The forces on the plate can be found from the unsteady form of Bernoulli's equation

$$p = -1/2\ \rho q'^2 - \frac{\partial \phi}{\partial t} \tag{A18}$$

where

$$\phi = -Ux\cos\alpha - Uy\sin\alpha + \frac{\Gamma\beta}{2\pi} \tag{A19}$$

The forces in the x and y directions are then

$$X = -\int_{c} p\ell ds$$

$$Y = -\int_{c} pmds \tag{A20}$$

where ℓ and m are direction cosines of the normal to the surface and the integration is taken from the trailing edge over the plate and back to the trailing edge. Glauert [5] has shown that the integration reduces to the following form:

$$X = -\rho U\Gamma\sin\alpha$$

$$Y = \rho U\Gamma\cos\alpha + \rho\frac{d\Gamma}{dt} + \rho\frac{\partial}{\partial t}\int_{c} q'xdx \tag{A21}$$

Von Kármán and Sears [9] compute the lift directly by relating it to the momentum of the fluid expressed as

$$L = -\frac{dI}{dt} \tag{A22}$$

where

$$I = \rho\int_{-1}^{1} \gamma(x)xdx + \rho\int_{1}^{\infty} \gamma(\xi)\xi d\xi \tag{A23}$$

This results in a sectional lift coefficient given by ($\alpha = 1$)

$$c_\ell = 2\pi \left[1 + \int_1^{1+Ut} \frac{\gamma(\xi)}{\pi Uc} \frac{d\xi}{\sqrt{\xi^2-1}}\right] \tag{A24}$$

The quantity

$$k_1(t) = 1 + \int_1^{1+Ut} \frac{\gamma(\xi)}{\pi Uc} \frac{d\xi}{\sqrt{\xi^2-1}} \tag{A25}$$

is called the Wagner function.

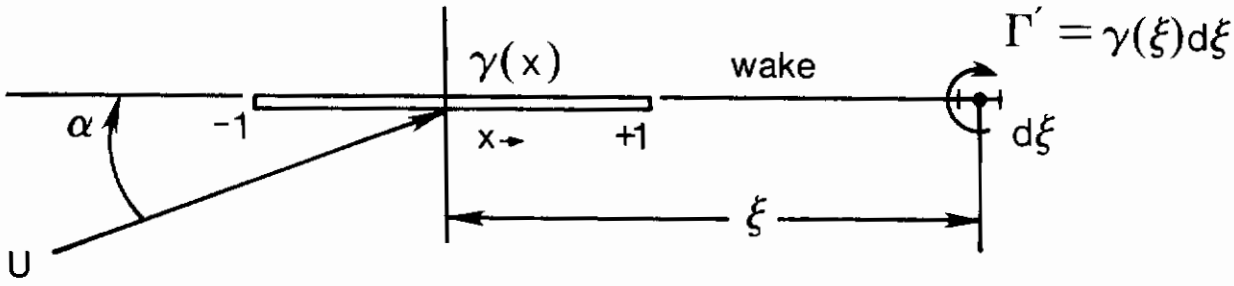


Figure A1. Flat Plate Started Impulsively From Rest

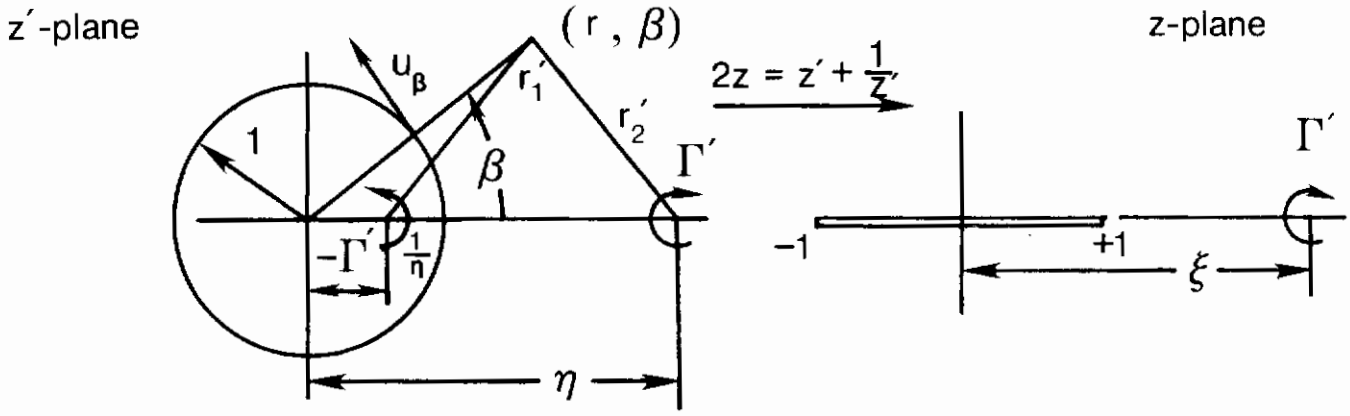


Figure A2. Conformal Mapping Used in the Wagner Problem

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