THE EQUIVALENCE OF FORCE AND DISPLACEMENT METHODS IN THE MATRIX ANALYSIS OF ELASTIC STRUCTURES

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The four sets of variables occurring in the linear theory of structures, and the basic relations between them are defined in terms of the theory of finite-dimensional vector spaces. The two types of method in practical use, mentioned in the title, are derived in a uniform manner, and the equivalence of these two methods is established by comparing the resultant matrices with those obtained by inverting the so-called structural matrix, suitably partitioned in each case. Discussion is limited to problems where loads are applied at joints only, but should be applicable to wider class. Conclusion: the two methods are not dual to each other, as sometimes stated, but variables are. Vector space approach is found more general and better suited to structure analysis than systems or network theory.

1.0 INTRODUCTION

The analysis of complex structures by means of matrix methods has reached a stage of development where it is appropriate to stop and re-examine the premises on which the theory is founded. As a small contribution toward this goal we shall try to compare the two methods of solution currently in use, and see how they can be derived from first principles in a uniform way.

Our concern will be with the linear theory of structures, in which proportionality between loads and deflections is assumed. The deflections must then be sufficiently small so as not to affect the equilibrium of the forces. Under these conditions, the principle of superposition is applicable, and this in a natural way leads to the use of the concepts of linear algebra, in particular of the more elementary parts, using matrices to represent relations between the variables; this in turn leads to a direct application of digital computer techniques.

In order to concentrate on essentials, we discuss mainly the relatively simple problem in which loads are applied directly to the joints, and no initial stresses, thermal effects, etc., are present. This is called the restricted structural problem. We first give the necessary definitions and explain our notations, and then give the problem a fairly general formulation. This is done in terms of quantities familiar to every structural engineer, namely forces, stresses, displacements and strains, and the derivations are based solely on static and kinematic relations, without recourse to energy considerations. In fact, the various theorems and so-called principles which involve work or strain energy appear as corollaries to our basic relations. In order to present a coherent picture of the two methods now in general use it has been necessary to repeat much that is not new. In this we have been following the classical presentations of Argyris (Reference 1) and of Hall and Woodward (Reference 2), although using our own notations. The algebraic topological aspects of the theory have been put forward by Langefors (Reference 3), although not in the language of vector spaces adopted in this paper. The unified approach has enabled us to identify the various matrices involved in the solution of the problem, and also to derive some identities and properties believed to be new. In this form the equivalence of the two methods is an obvious result.

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It will not be possible in this paper to discuss a number of topics important in setting up and in solving structural problems. One of these is the question of idealizing an elastic body, that is, of representing it adequately by a mathematical model; this may not be too difficult for relatively simple frameworks, but it becomes a much disputed question when the structure is more complex, as shown by Gallagher et al (Reference 4) on the example of multi-web wings. The difficulties are still greater if it is desired to represent in some detail continuous elastic bodies such as plates, shells, etc., as structures composed of discrete elements. Closely associated with idealization is the question of proper interpretation of the results of the analysis of the ideal model in terms of behaviour of the original body. The choice of idealized structure will largely depend on the type of result desired (stress analysis or calculation of deflections to be used in dynamic or aeroelastic work), on facilities available for computing, and even on the method of analysis to be adopted. For our part we shall consider the idealized structure as given. A further point, recently much debated, is that of assigning adequate and consistent elastic properties to the members, as represented by the member stiffness or flexibility matrices. As pointed out by Melosh (Reference 5), the stiffness matrix, especially in the discrete representation of a continuous body, must satisfy a criterion of monotonic convergence as the size of representative element is decreased.

Among the other items that will have to be left out of this discussion are comparisons of the methods as to their speed, accuracy, and overall efficiency. These things depend too much on the type and size of computer, on the accuracy desired, and on the type and size of structure, to be evaluated in a general way. In some cases it would be possible to obtain an estimate of the volume of work to be performed by counting operations, in particular multiplications.

2.0 MATRIX FORMULATION OF THE STRUCTURAL PROBLEM

2.1. Some Definitions

We define an elastic structure as an assemblage of finite number of interconnected elastic members. Each member is an idealization of an elastic body, usually of simple shape such as a rod, a beam, a plane panel, a portion of plate or shell; in a more general way, a statically indeterminate ring or frame, an entire component or substructure could be regarded as a member of the structure. The latter approach has been taken by Przemieniecki (Reference 6) and by other authors.

Definition and Behaviour of a Member

An elastic member is characterized by a number of terminals through which the member may be connected to adjoining members and through which forces may be applied to the member. In this paper no other forces will be taken into consideration, since we deal here with a restricted class of structural problems. A terminal need not be a single point, but may include two or more points or, in the case of a continuous attachment, a line segment or a portion of area.

In contrast to the theory of elastic continua, out of the infinity of ways in which loads can be applied to body, we choose to consider only a finite number, the number of degrees of static freedom assigned to the member. Similarly, we consider only the displacements of the member terminals, while the motion of other points of the body is of no interest from the point of view of the structure as a whole; thus kinematic freedom is also limited to a finite number of degrees.

When removed from the structure and from possible supports, that is, as a free body, a member is in equilibrium under the forces applied to it through the terminals. We use the term "force" in its generalized sense, including any convenient stress resultants such as couples, bimoments, shear flows, and the like; or even coefficients of a polynomial or of a

Fourier expansion of a distributed loading. After deducting from the number of degrees of static freedom the number of conditions necessary for the equilibrium of forces (three in the plane case, six in the three-dimensional case) we are left with the number of independent parameters that will be called the number of degrees of elastic freedom of the member; for member "i" let that number be denoted by m_i. These parameters will in general be of the nature of stress resultants acting on some chosen cross sections, and will be denoted by $\underline{s}^{(i)} = \{s_1, s_2, \ldots, s_{m_i}\}$, a column vector.

If a system of coordinate axes be attached to a material particle belonging to the member, the absolute displacements of the terminals can be described by the components of displacement of the coordinate system (three in the plane case, six in the three-dimensional case), plus a number of parameters related to the elastic distortion or deformation of the member, defined as displacements of the terminals with respect to the coordinate system. It is always possible to make those deformation components correspond with our generalized force components in the Lagrangian sense; these deformations are the ones on which the force components do positive work. The number of independent parameters necessary to completely describe the member deformation in our limited way is thus equal to the number m_i of elastic degrees of freedom; again this is the total number of kinematic freedoms considered minus the three or six necessary to describe a rigid-body motion. For a rod having ball joints at both ends we have $m_i = 1$; for a beam forming part of a plane framework it is 2 when axial loads and elongations are not considered, or 3 when they must be taken into account; for a beam in the general three-dimensional case the number is 6, and so on.

Summing up, the static condition of a member is described by an m_i-dimensional vector $\mathbf{s}^{(i)} = \{s_1, s_2, \ldots, s_{mi}\}_i$ with reference to a particular mode of support: the stress at any cross section and in any direction can in theory be obtained as a linear combination of these stress defining components. Similarly, the condition of deformation of the member is described by the corresponding vector of deformation components $\mathbf{e}^{(i)} = \{e_1, e_2, \ldots, e_{mi}\}_i$. Both these vectors are to some extent arbitrary. In particular, their components depend on the choice of the coordinate system, i.e. of the mode of support, on the choice of the stress resultants and distortions used for description, and also on the sequence in which they are written down, as well as on the sign convention and on the units of measurement. Altogether these factors constitute a system of reference more complete than the coordinate system alone; in the language of vector spaces such a system of reference is called a basis.

For instance, a beam segment AB may be regarded either as a cantilever with end A fixed, or with end B fixed, or as a beam simply supported at A and B. In the first and second cases the couple and shear force acting at the free end may be chosen as the components of the stress-resultant vector, with slope and transverse deflection at the free end as corresponding components of the deformation vector. In the case of the simply-supported beam, the stress resultants would normally be the end couples, with corresponding rotations or slopes as deformation components.

In linear algebra (Reference 7) a set of quantities that depend linearly on a finite number of parameters and satisfy certain axioms is said to form a finite-dimensional vector space. Thus, all the possible stress conditions of a member form an m_i-dimensional vector space, denoted by $S^{(i)}$, over the field of real numbers (since the vector components are real numbers): this space is said to be generated or spanned by the set of unit vectors $\mathbf{s}_1 = \{1, 0, \ldots, 0\}, \mathbf{s}_2 = \{0, 1, \ldots, 0\}, \ldots, \mathbf{s}_{mi} = \{0, 0, \ldots, 1\}$. Similarly, the condition of elastic deformation of the member is represented by a vector $\mathbf{e}^{(i)}$ of an m_i-dimensional vector space denoted by $E^{(i)}$ that is spanned by the vectors $\mathbf{e}_1 = \{1, 0, \ldots, 0\}, \mathbf{e}_2 = \{0, 1, \ldots, 0\}, \ldots, \mathbf{e}_{mi} = \{0, 0, \ldots, 1\}$. In both cases the spanning vectors form a basis for their respective space, i.e., any vector of the space can be expressed as a linear combination of the basis vectors. It is easy to verify that the axioms of vector addition and of multiplication by a scalar are satisfied by these spaces.

An important operation in a vector space is a change of basis. Since every vector of a new basis (b) is a linear combination of the vectors of basis (a), and vice-versa, the stresses relative to basis (b) can be obtained from those relative to basis (a) by means of a reversible, i.e., nonsingular linear transformation. Specifically, if a stress resultant at some cross section Q is represented in reference to basis (a) as a linear combination $c_1 s_1 + \ldots + c_m s_m$, and in reference to basis (b) as a linear combination $c'_1 s'_1 + \ldots + c'_m s'_m$, then the column vector $\{c_1, \ldots, c_m\}$ of the c-coefficients is obtained from the column vector of the c'-coefficients by premultiplying the latter by the transformation matrix N. The k-th column of the matrix N is formed by the components of the k-th vector of the basis (b) with reference to basis (a). Concrete examples of these transformations can be found in Hall and Woodhead (Reference 2), although without mention of vector spaces. A change of coordinate system is seen to be a special case of a change of basis.

So far nothing was said about relations between stress resultants and deformations, except that the work done by a set of loads identified relative to basis (a) by the vector $s = \{s_1, \ldots, s_m\}$ on a set of deformations identified relative to the corresponding basis by the vector $e = \{e_1, \ldots, e_m\}$ is given by the sum $W = \frac{1}{2}(e_1 s_1 + \ldots + e_m s_m) = \frac{1}{2} e^T s$, where e^T is the row vector transpose of e. This quantity W is a scalar, independent of the system of reference in which the components are expressed, provided that in either system the supports are fixed; this is a corollary to Betti's theorem of reciprocity. If now we express the loads relative to basis (b) using the transformation just mentioned, $s = N s'$, the work will be given by $W = \frac{1}{2} e^T (N s')$ and this is seen to be equal to $W = \frac{1}{2} s'^T N^T e = \frac{1}{2} s'^T e'$, where $e' = N^T e$ are the deformation components relative to basis (b), i.e., corresponding to the load components s'. We see that the transformation of the deformation components is given by the inverse transpose of the matrix that transforms the load components. Such two sets of variables are said to be contragredient, and their vector spaces are said to be dual or conjugate to each other. Langefors (Reference 3) calls this the principle of co-transference.

The Assembled Structure

In the assembled structure the members are numbered in an arbitrary sequence, $i = 1, 2, \ldots, M$, where M is the number of members. The total number of elastic degrees of freedom for the structure is thus equal to the sum $m = m_1 + m_2 + \cdots + m_M$. It is convenient to re-number the stress resultant components from 1 to m and to form an m-dimensional vector $s = \{s_1^{(1)}, \ldots, s_{m_1}^{(1)}, s_1^{(2)}, \ldots, s_{m_2}^{(2)}, \ldots, s_1^{(M)}, \ldots, s_{m_M}^{(M)}\} = \{s_1, \ldots, s_m\}$; the same is done with the deformation components, writing $e = \{e_1, \ldots, e_m\}$. The space S of the vectors s is the direct sum of the member subspaces, meaning that every vector s has one and only one expression as a sum of subvectors $s = s^{(1)} + \cdots + s^{(M)}$; the union of any bases for the $S^{(i)}$ is then a basis for the space S. The same will hold for the space E, the direct sum of the member subspaces $E^{(i)}$.

Nothing prevents us, however, from taking linear combinations of stress resultant vectors pertaining to various members and thus forming a new, perhaps more convenient basis for the vector space S. This is done, for instance, in the case of a box beam, when instead of the axial loads and axial deformations of the four corner flanges it is often preferable to take such combinations as: total axial load and average axial deformation, bending moments and rotations in two planes, and a warping group as four equivalent sets of variables. More complex groups have been used by Argyris in analysis of fuselages. Such group stress resultants and group deformations may again be taken as new bases obtained by linear transformation from the member bases. Under these transformations the dimension remains invariant, and again the deformations transform in a contragredient way to the stress resultants.

To assemble the members into a complete structure it may be necessary to use some constraints in order to make them fit together; or uneven thermal expansion would produce initial stresses in the structures, even in the absence of external loads. In this discussion we shall assume that no such initial stresses are present.

Joint Loads and Deformations

Again, the load condition of the structure may be described by a set of load components, with the term "load" taken in the sense of generalized force. As stated before, we assume the loads to be applied only to joints. The problem of determining stresses in the members and displacements of the joints under such a restricted type of loading and with no initial or thermal stresses will be called the restricted structural problem. For the description of these loads a coordinate system is introduced, called the global system. A unit set of these load components form a basis for a vector space P of n dimensions, where n is the number of independent load parameters or degrees of freedom (this excludes the 3 or 6 conditions necessary to establish equilibrium). It is naturally possible to adopt a different basis for this space, by changing the system of coordinates, by taking the loads in a modified sequence, by combining them linearly to form more convenient unit vectors. Such a change of basis will always be expressed by a linear transformation or numerically by a transformation matrix. This transformation must naturally be reversible, i.e., the matrix must be non-singular, and the dimension n will remain invariant under such a transformation, in accordance with a known theorem of linear algebra.

The kinematic condition of the complete structure will be described by a set of displacements in the generalized sense, corresponding to the external loads. There will thus be n independent displacement parameters. In a way entirely analogous to member distortions the joint displacements u form an n dimensional vector space U dual to the vector space of the loads.

Altogether, the external behaviour of the structure is characterized by an n dimensional load vector denoted by $p = \{p_1, \ldots, p_n\}$ and the corresponding dual displacement vector denoted by $u = \{u_1, \ldots, u_n\}$. We may not be interested in the displacements of some of the joints at which no loads are to be applied; the degrees of freedom pertaining to that joint may then be ignored and will not contribute to the total n. We may assign n_j degrees of freedom to a joint j, where the index j covers the whole range of joints, except those that are fixed relative to the supporting medium or to the global system of coordinates. The effective total number of degrees of elastic freedom of the complete structure is thus obtained by adding the degrees of freedom of all the joints and subtracting the 3 or 6 degrees necessary for equilibrium of the loads.

2.2 The Stress-to-Load Relations of Equilibrium

Having described the four sets of variables we turn our attention to the relations that exist between these sets. Since we are considering the linear problem, these relations will naturally take the form of matrix equations.

At each joint, considered as a material point or line, there are two or more members meeting and several components of applied joint loads. The conditions of equilibrium for each joint can be written down without much difficulty in a way similar to that used in the analysis of pin-jointed trusses, except that the contribution of each incident member extends in general to all the defining stress resultants of that member. The number of equations of joint equilibrium is equal to the total number of joint degrees of freedom. If the structure is a free body, an additional 3 or 6 equations will express the fact that the external joint loads are dependent; in order not to have a singular system of equations, these additional equations may be used to eliminate a corresponding number of loads, leaving a number of equations equal to the number of elastic degrees of freedom of the structure. If the structure is supported on the ground, the same may be done by regarding the ground as a rigid member and applying to the support joints a system of reacting forces.

The coefficients of the member stress resultants occurring in any one of the equilibrium equations will form one row of a matrix which we call the <u>geometric assembly matrix</u>, denoted by $\mathbf{A}$. This is a rectangular matrix with n rows and m columns. The complete set of equilibrium equations is then written as a single matrix equation.

The matrix $\mathbf{A}$ expresses the combined effects of geometrical position of the members with respect to the global coordinate system and of the incidence of the members at the joints; it thus represents a linear transformation of the basis vectors of the member stress resultant space S into the basis of the vector space P of the joint-applied loads.

$$\mathbf{A}\,\mathbf{s} = \mathbf{p} \qquad (1)$$

The case $m < n$ is that where there are not enough member stress resultants to transmit the loads through the structure; the structure is not a rigid one, and the members will be able to move as rigid bodies, forming a kinematic chain, i.e., a mechanism. It is also possible that the number of member freedoms is sufficient, but the arrangement is such that some part of the structure will form a mechanism, while the remaining part is quite rigid. Such a situation will find its expression in a matrix of rank lower than n; in any case the <u>rank criterion</u> is a necessary and sufficient one for an assembly to form a structure, while the simple inequality $m \geq n$ is merely a necessary, but not sufficient condition.

The next possibility is that $m = n$; we then say the structure is exactly determinate statically. The matrix $\mathbf{A}$ is square of order and rank n and therefore possesses a reciprocal. Calculating the latter we solve the problem of determining all the stress resultants in the members. There is no need to further discuss these two cases.

Regarding Equation 1 as a system of nonhomogeneous linear equations to be solved for the unknown stress resultants $\mathbf{s}$ we know that such a system has a solution if and only if the rank of the matrix $\mathbf{A}$ equals the rank of the augmented matrix $[\mathbf{A}, \mathbf{p}]$. Also the theorem states that if $\mathbf{s} = \mathbf{s}_0$ be a particular solution of Equation 1 then every solution $\mathbf{s}$ can be written as the sum $\mathbf{s} = \mathbf{s}_0 + \mathbf{q}$ where $\mathbf{q}$ is a solution of the homogeneous system $\mathbf{A}\mathbf{q} = 0$.

In the theory of linear transformations of vector spaces the set of all column vectors that satisfy the homogeneous linear equations $\mathbf{A}\mathbf{q} = 0$ forms a subspace Q of the domain of the transformation (i.e., of the space S). This subspace Q is called the null-space or kernel of the transformation matrix, and its dimension, called the nullity of Q, is equal to the difference $r = m - n$ where n is the rank of $\mathbf{A}$. In structural terms the homogeneous equation represents a condition for the stress-resultant vector to be self-equilibrating, and consequently the set of all such vectors forms a subspace of dimension r; or every self-equilibrating stress-resultant vector can be represented as a linear combination of any set of r independent such vectors, the set forming a basis for the subspace. Following custom we call those vectors <u>redundant</u> or <u>hyperstatic</u>; the term "statically indeterminate" seems to be misleading, since there is actually a statical overdetermination. The number r is called degree of redundancy or of statical overdetermination of the structure.

2.3 The Displacement-to-Distortion Conditions of Compatibility

In view of the contragredient character of the transformations of the force and displacement vectors on one side, and of the stress-resultant and distortion vectors on the other it is easy to see that the correspondence between the displacement vector $\mathbf{u}$ and the distortion vector $\mathbf{e}$ is given by the equation

$$\mathbf{e} = \mathbf{A}^T \mathbf{u} \qquad (2)$$

where $\mathbf{A}^T$ is the transpose of the matrix $\mathbf{A}$. No more needs to be said about this important relation, dual to Equation 1.

2.4 Member Flexibility and Member Stiffness Matrices

The assumption of linearly elastic members means that for each small element a generalized Hooke's law holds, establishing proportionality between stresses and strains. As a result, the distortions of each member are linear functions of the force system that defines the stress condition in the member. This will be true whether the material is isotropic or anisotropic, and whether the stress or strain is distributed uniformly, linearly, or in some other way. The mathematical expression of this linear relationship is the equation

$$e^{(i)} = F^{(i)} s^{(i)} \tag{3}$$

valid for each member 'i', where $s^{(i)}$ is the column vector of the stress resultants defining the stress condition in the member, $e^{(i)}$ is the column vector of the corresponding distortions defining the kinematic condition of the member, and $F^{(i)}$ is called the member flexibility matrix. $F^{(i)}$ is a square matrix of order m_i, its elements are symmetrical with respect to the principal diagonal, in accordance with Maxwell's reciprocity relation. Moreover we know the matrix $F^{(i)}$ to be positive definite and hence nonsingular.

We shall not go into the details of how the entries of the member flexibility matrix can be calculated; usually some approximate stress distribution is assumed under each unit loading condition, and the corresponding row of the flexibility matrix is found as the set of influence coefficients. For members of simple shape, such as rods, beams, etc., standard expressions for these coefficients are available. For our purposes we consider the member flexibilities as well as geometric assembly matrix as basic data of the structural problem.

The rule of transformation of the member flexibility matrix under change of basis is as follows. If the 'old' components of the stress resultant vector are expressed in terms of the 'new' ones by means of the transformation $s = Hs'$ where H is a nonsingular transformation matrix, and hence the corresponding distortions transform as $e' = H^T e$, we may express the strain energy either in terms of the 'old' or of the 'new' components, and we should in both cases find the same invariant quantity $V = \frac{1}{2} e^T s = \frac{1}{2} s^T F s = \frac{1}{2} s'^T H^T F H s$, and $V = \frac{1}{2} e'^T s' = \frac{1}{2} s'^T F' s'$. Since this relation must hold for any vector s , we must have $F' = H^T F H$. The flexibility matrix for components referred to the new basis is obtained by a congruence transformation from the old one and is symmetric.

The converse deformation-to-stress resultant relation for a member is simply given by the matrix equation

$$s^{(i)} = K^{(i)} e^{(i)}$$

where $K^{(i)}$ is the member stiffness matrix. Each column of this matrix represents the reactions in direction of the basis components when a unit displacement is applied in the direction of a particular component, all other deformations being zero. We have $K^{(i)} = (F^{(i)})^{-1}$, also a square positive definite matrix. In a variant of the displacement method, the so-called direct stiffness method, the kinematic condition of the member is left indeterminate at the outset, admitting the total number of degrees of freedom (elastic plus rigid body). The stiffness matrix is then singular, since by virtue of the equilibrium conditions some of the reactions are linear combinations of the remaining ones. The rank of the expanded stiffness matrix is thus still equal to the number of elastic degrees of freedom m_i of the member. The superfluous rows and columns are deleted according to the actual conditions of support of the member.

Since the stiffness matrix is the inverse of the flexibility matrix, the rule of transformation under a change of basis is $K' = H^{-1} K \; (H^T)^{-1}$, easily verified when expressing the strain energy in terms of the deformation components.

2.5 Structure Flexibility and Structure Stiffness Matrices

In the assembled structure where the space of the stress resultant vectors is the direct sum of the subspaces of the single members, the same goes for the deformations, and the relation between the complete vectors is given in matrix form by

$$v = F\,s \qquad (3')$$

where F is a supermatrix having in its diagonal the blocks $F^{(i)}$ and zeros elsewhere, thus

$$F = \operatorname{diag}\left\lceil F^{(1)}, F^{(2)}, \ldots, F^{(M)} \right\rfloor \qquad (4)$$

This diagonal supermatrix is easily inverted by inversion of the diagonal blocks,

$$K = F^{-1} = \operatorname{diag}\left\lceil K^{(1)}, K^{(2)}, \ldots, K^{(M)} \right\rfloor \qquad (4')$$

But if the basis for the vector space S involves vectors composed of stress resultants belonging to more than one member, there will be coupling flexibility and stiffness terms at the intersections of rows and columns pertaining to different members, and some of the off-diagonal blocks will be nonzero. The inversion then becomes more complicated.

3.0 SOLUTION OF THE RESTRICTED STRUCTURAL PROBLEM

Having thus defined and reviewed the four basic sets of variables intervening in the restricted structural problem, and having discussed the three types of relations that are the consequences of the basic assumptions, we now proceed to describe the methods used to solve the problem.

3.1 The Displacement Method

The situation may be visualized by the schematic diagram of Figure 1. The full-line arrows indicate the Relations 1, 2 and 3, we now regard as given; the fourth, symbolized by the dotted arrow, may be obtained either directly or by inverting the member flexibility matrix F. Our goal is to obtain a load-to-stress and a load-to-displacement relation. This cannot be done directly, so first an inverse to the latter is found by calculating the triple matrix product

$$\Gamma = A K A^T \qquad (5)$$

This represents a stiffness matrix for the assembled structure, a symmetric matrix of order n, positive definite since congruent to the positive definite K matrix. This matrix may be inverted to obtain the flexibility matrix Φ for the complete structure,

$$\Phi = \Gamma^{-1} \qquad (6)$$

The entries of Φ are known under the name of deflection influence coefficients; in dynamic and aeroelastic work the determination of Φ is one of the first steps. These and the following operations are schematically represented by the lower diagram of Figure 2. By means of the matrix $\mathbf{\Phi}$ we now are in a position to calculate the joint displacements in terms of the given loads,

$$\mathbf{u} = \mathbf{\Phi}\,\mathbf{p} \tag{7}$$

as a first part of our solution. From previous relations we had expressed the stress resultants in terms of displacements,

$$\mathbf{s} = \mathbf{K}\,\mathbf{A}^T\mathbf{u}$$

or, in terms of applied loads,

$$\mathbf{s} = \mathbf{C}\,\mathbf{p} \tag{8}$$

where

$$\mathbf{C} = \mathbf{K}\mathbf{A}^T\,\mathbf{\Phi}$$

is the desired second part of the solution. It is useful to check these calculations by verifying that the conditions of equilibrium at the joints (Equation 1) are satisfied. For a given load system premultiplying the calculated stress resultants by the geometric matrix $\mathbf{A}$ should reproduce the given loads. When operating with unit loads to cover a large number of load cases, the product $\mathbf{AC}$ should be equal to the unit matrix of order n,

$$\mathbf{AC} = \mathbf{I}_n \tag{9}$$

Any departure from equality will be a measure of the inaccuracy of our results. If desired the calculated basic stress resultants may now serve as a point of departure for a more detailed stress analysis.

3.2 The Force Method

This method derives its validity from the theorem which states that, for $m > n$, every solution of the system of unhomogeneous linear Equations 1 can be written as the sum of a particular solution and of a solution of the homogeneous system. The latter solution, called the redundant stress system, forms a self-equilibrating set of stresses depending linearly on $r = m - n$ arbitrary constants. In the structural problem these constants are determined from supplementary conditions of compatibility between the member deformations and the joint displacement (or, equivalently, from a variational principle). Cuts or releases are often introduced into the structure to make the reasoning physically plausible.

The particular solution is any stress vector that satisfies the Equilibrium Conditions 1; it may be either a statically determinate or indeterminate stress system, or even the actual final stress vector. Holding to the first of these possibilities we shall assume that the numbering of the stress resultants has been re-shuffled in such a way that the first n columns of matrix $\mathbf{A}$ correspond to the components of the statically determinate stress vector $\mathbf{s}_0$. These first columns then form a square submatrix denoted by $\mathbf{A}_0$, which is by definition nonsingular. The matrix $\mathbf{A}$ is thus partitioned as $\mathbf{A} = [\mathbf{A}_0 \,\vdots\, \mathbf{A}_1]$, where the complementary submatrix $\mathbf{A}_1$ has n rows and r columns. The submatrix $\mathbf{A}_1$ is of rank r, since its columns are independent. Equation 1 is now partitioned correspondingly,

$$\left[A_0 \,\vdots\, A_1 \right] \begin{bmatrix} s_0 \\ \hline q \end{bmatrix} = p \tag{10}$$

The statically determinate stress resultants, numbered 1, 2, ... ,n, are obtained in terms of the applied joint loads as

$$s_0 = A_0^{-1} \; p \tag{11}$$

Owing to the special character of the geometric matrix A the inversion of A_0 is in most cases a very simple task that can be performed directly without computer help.

The complete stress vector constituting the particular solution may be written

$$\begin{bmatrix} s_0 \\ \hline O \end{bmatrix} = C_0 \, p \qquad \text{with} \qquad C_0 = \begin{bmatrix} A_0^{-1} \\ \hline O \end{bmatrix}$$

while the remaining stress resultants, numbered $n+1, \ldots, m$, are all zero.

Regarding the subspace Q of the redundant stress vectors, it is natural to take the columns of submatrix A_1 as a basis. Then the components numbered $n+1, \ldots, m$ are simply equal to these columns, multiplied by constant scalars $q_1, \ldots, q_r$, respectively, while the stress resultants 1 to n are found from the conditions of zero equilibrium at the joints,

$$A \begin{bmatrix} s_q \\ q \end{bmatrix} = O, \qquad \text{or} \qquad A_0 s_q + A_1 q = O \,;$$

the solution is

$$s_q = -B_0 \, q \qquad \text{with} \qquad B_0 = A_0^{-1} \; A_1 \tag{12}$$

The complete stress vector that represents the self-equilibrating systems has the form

$$\begin{bmatrix} s_q \\ q \end{bmatrix} = B \, q \qquad \text{with} \qquad B = \begin{bmatrix} -B_0 \\ I \end{bmatrix}$$

the r arbitrary constants q_i forming a column vector and I being a unit matrix of order r.

The final or resultant stress vector may now be written in the form

$$s = C_0 \, p + B \, q = \left[C_0 \,\vdots\, B \right] \begin{bmatrix} p \\ \hline q \end{bmatrix} \tag{13}$$

The statically indeterminate vector q will now be obtained from the condition that the member deformations e derived from the Stress Resultants 13 form a compatible system. The deformation vector is, by Equation 3′, $e = F s$. We now require that the displacements that correspond to the complementary solution form a zero vector; in the interpretation of cuts or releases this condition means that continuity at the cuts must be restored by the redundant systems. This condition may be written

$$u_q = B^T e = 0 \tag{14}$$

Or substituting for e, then for s,

$$u_q = B^T F C_0 p + B^T F B q = 0$$

or with the notations $B^T F B = D$, $B^T F C_0 = G$, for the triple product matrices,

$$G p + D q = 0. \tag{14'}$$

D is a square matrix of order r; moreover, D is symmetric and positive definite, since it is obtained by a congruent matrix transformation from the positive definite matrix F. The matrix G has r rows and n columns.

Equation (14) may be solved for the redundant systems

$$q = -D^{-1} G p \tag{15}$$

Substituting in Equation 13 for the vector q we obtain the final stresses

$$s = C_0 p - B D^{-1} G p = C p \tag{16}$$

in terms of the applied loads; spelling out the G matrix, write the load-to-stress matrix C,

$$C = C_0 - B D^{-1} B^T F C_0 = (I - \Delta) C_0$$

we may obtain the final stresses in terms of the initially assumed particular solution

$$s_0 = C_0 p$$

$$s = s_0 - \Delta s_0 = (I - \Delta) s_0$$

The second term in this expression represents a <u>correction</u>, symbolized by the matrix

$$\Delta = B D^{-1} B^T F.$$

This is a square matrix of order m that has some interesting properties hitherto not mentioned in the literature. Δ is singular, for its rank cannot exceed the lowest among the ranks of its factor matrices, which is r. Moreover, the square Δ^2 is equal to the matrix Δ itself, for

$$\Delta^2 = (B D^{-1} B^T F)(B D^{-1} B^T F) = B D^{-1} (B^T F B) D^{-1} B^T F$$
$$= B D^{-1} B^T F = \Delta$$

Thus the matrix Δ is as mentioned and so is its complement to the unit matrix of order m namely $(I - \Delta)^2 = I - 2\Delta + \Delta^2 = I - \Delta$. Although actual computation of Δ is not necessary, these properties are important in that they explain why, when applying the procedure of correcting the stress system s taken as the initial system one arrives at an identical result.

We are now in a position to evaluate the displacements at the joints or, more generally, the flexibility matrix Φ for the complete structure. Regarding the load-to-stress relation $s = Cp$ as a linear transformation, we may expect the deformation-to-displacement relation to be given by the contragredient transformation, $u = C^T$. Substituting for e and then for s we find

$$u = \phi p, \tag{17}$$

where $\Phi = C^T F C$. This matrix product may be modified in two ways. First, express C as $(I - \Delta) C_0$; then

$$\begin{aligned} \Phi &= C_0^T (I - \Delta^T) F (I - \Delta) C_0 \\ &= C_0^T (F - F\Delta - \Delta^T F + \Delta^T F \Delta) C_0 \end{aligned}$$

and the products

$$\Delta^T F \Delta = (F B D^{-1} B^T) F (B D^{-1} B^T F) = F B D^{-1} B^T F = \Delta^T F = F\Delta$$

all reduce to the same matrix $F\Delta$. We thus may write

$$\Phi = C_0^T (F - F\Delta) C_0 = C_0^T F (I - \Delta) C_0 = C_0^T F C. \tag{18}$$

This proves the known theorem that to calculate deflections as linear functions of stress resultants it is sufficient to take either the stress resultants or the deformations corresponding to a stress system statically equivalent to the applied loads, while the other vector is that of the actual system.

The alternative expression for the flexibility matrix is obtained by observing that the term $C_0^T F C_0 = \Phi_0$ represents the flexibility of the structure without consideration of its statically indeterminate character, i.e. with $q = 0$. The second term, $-C_0^T F \Delta C_0$ represents a correction to that initial flexibility equal to $-G^T D^{-1} G$, so that

$$\Phi = \Phi_0 - G^T D^{-1} G \tag{18'}$$

The correction is of rank at most equal to r, since this is the rank of its factors; the matrix $G^T D^{-1} G$ is positive definite, since it is congruent with the positive definite matrix D^{-1}. Thus the work of the external loads on the joint displacements, $W = \frac{1}{2} u^T p = \frac{1}{2} p^T \Phi p$, will be less in the statically indeterminate structures than in the statically determinate one. The effect of the redundant stress systems is to reduce its flexibility on the whole, i.e. to increase its stiffness.

Having presented an outline of the two methods we shall now try to establish a common basis for them with a view to demonstrating the necessity for the results to be identical.

3.3 The Structural Matrix and Its Inversion

We combine the three matrices A, A^T and F so that the first two, which are rectangular matrices, border the square matrix F, and fill the vacant $n \times n$ square by zeros. This supermatrix, denoted by M, will be given the name structural matrix.

$$\mathbf{M} = \begin{bmatrix} \mathbf{F} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \tag{19}$$

We propose to show that, by partitioning **M** in different ways and inverting the partitioned matrix we repeat the steps described above for solving the restricted structural problem by the displacement and force methods. Since the inverse of a nonsingular matrix over the field of real numbers exists and is unique, the result of inversion will not be affected by the way in which the matrix has been partitioned, and the equivalence of the two methods will thus be established.

According to Kosko (Reference 8) a very useful device for inverting a partitioned matrix is to reduce it to <u>super-diagonal</u> form. A diagonal supermatrix is inverted simply by inverting the diagonal blocks, and the inverse of the original matrix is found from this diagonal inverse by reversing the operation of reduction. For a symmetric matrix the reduction to diagonal form is obtained by a congruence transformation, using as postmultiplier the transpose of the premultiplying nonsingular matrix transformation. We shall proceed according to this program using two different reduction-to-diagonal schemes; the first will correspond to the displacement method, and the second to the force method.

"Natural" Diagonalization of the Structural Matrix

To obtain a diagonal form of the bordered matrix **M** it is necessary to premultiply it by the transformation matrix

$$\mathbf{X} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}\mathbf{F}^{-1} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}\mathbf{K} & \mathbf{I} \end{bmatrix}$$

that is lower-triangular, and postmultiply by the transpose $\mathbf{X}^T$. The reduced matrix is

$$\bar{\mathbf{M}}^{-1} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}\mathbf{K}\mathbf{A}^T \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\Gamma} \end{bmatrix}$$

recalling earlier notations. This diagonal supermatrix is readily inverted, with the diagonal blocks known from previous work,

$$\bar{\mathbf{M}}^{-1} = \begin{bmatrix} \mathbf{F}^{-1} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\Gamma}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\Phi} \end{bmatrix}$$

Now the operations of reduction to diagonal are performed in reverse-order, i.e. **X** is used as postmultiplier and $\mathbf{X}^T$ as premultiplier. The resulting inverse of the structural matrix is

$$\bar{\mathbf{M}}^{-1} = \mathbf{X}^T \bar{\mathbf{M}}^{-1} \mathbf{X} = \begin{bmatrix} \mathbf{K} - \mathbf{K}\mathbf{A}^T\boldsymbol{\Phi}\mathbf{A}\mathbf{K} & \mathbf{K}\mathbf{A}^T\boldsymbol{\Phi} \\ \boldsymbol{\Phi}\mathbf{A}\mathbf{K} & -\boldsymbol{\Phi} \end{bmatrix} \tag{20}$$

We note that the two submatrices appearing in the last column are precisely the load-to-stress matrix $\mathbf{C} = \mathbf{K}\mathbf{A}^T\mathbf{\Phi}$ (see Equation 8), and the negative flexibility (load-to-displacement) matrix $-\mathbf{\Phi}$ which constitute the solution of the structural problem by the displacement method.

Reduction Along The Secondary Diagonal

The structural matrix $\mathbf{M}$ is now partitioned in a way that corresponds to the partition of the geometric matrix $\mathbf{A}$ in the force method. The member flexibilities are then separated into those of the statically determinate stress components $\mathbf{F}_0$ and those of the redundants $\mathbf{F}_1$. In general, there may be some cross-coupling between these two groups, represented by the rectangular submatrix $\mathbf{F}_{10}$ and its transpose $\mathbf{F}_{10}^T$. The inversion of a triply partitioned matrix has been described by Kosko (Reference 9). In our case we observe that the sub matrix $\mathbf{A}_0$ and its transpose are easy to invert and for this reason we shall try to obtain a diagonal supermatrix in which the nonzero blocks follow the secondary diagonal. In this case the pre-multiplier happens to be

$$\mathbf{Y} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & -\frac{1}{2}\mathbf{A}_0^{-1}\mathbf{F}_0 \\ -\mathbf{B}_0^T & \mathbf{D} & -\mathbf{G} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

where again previous notations are recalled. The reduced diagonal matrix and its inverse are

$$\overline{\overline{\mathbf{M}}} = \mathbf{Y}\mathbf{M}\mathbf{Y}^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{A}_0^T \\ \mathbf{0} & \mathbf{D} & \mathbf{0} \\ \mathbf{A}_0 & \mathbf{0} & \mathbf{0} \end{bmatrix} \; ; \qquad \overline{\overline{\mathbf{M}}}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{A}_0^{-1} \\ \mathbf{0} & \mathbf{D}^{-1} & \mathbf{0} \\ (\mathbf{A}_0^T)^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and the inverse of the structural matrix, partitioned 3x3, is

$$\mathbf{M}^{-1} = \mathbf{Y}^T\overline{\overline{\mathbf{M}}}^{-1}\mathbf{Y} = \begin{bmatrix} \mathbf{B}_0\mathbf{D}^{-1}\mathbf{B}_0^T & -\mathbf{B}_0\mathbf{D}^{-1} & -\mathbf{A}_0^{-1} + \mathbf{B}_0\mathbf{D}^{-1}\mathbf{G} \\ -\mathbf{D}^{-1}\mathbf{B}_0^T & \mathbf{D}^{-1} & -\mathbf{D}^{-1}\mathbf{G} \\ (\mathbf{A}_0^T)^{-1} + \mathbf{G}^T\mathbf{D}^{-1}\mathbf{B}_0^T & -\mathbf{G}^T\mathbf{D}^{-1} & -\mathbf{\Phi}_0 + \mathbf{G}^T\mathbf{D}^{-1}\mathbf{G} \end{bmatrix} \tag{21}$$

This may be written as the sum of a matrix involving only statically determinate elements and of a singular matrix being a congruent transform of the redundant kernel matrix

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{A}_0^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (\mathbf{A}_0^T)^{-1} & \mathbf{0} & -\mathbf{\Phi}_0 \end{bmatrix} + \begin{bmatrix} -\mathbf{B}_0 \\ \mathbf{I} \\ -\mathbf{G}^T \end{bmatrix} \mathbf{D}^{-1} \begin{bmatrix} -\mathbf{B}_0 & \mathbf{I} & -\mathbf{G} \end{bmatrix} \tag{21'}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{C}_0 \\ \mathbf{C}_0^T & -\mathbf{\Phi}_0 \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ -\mathbf{G}^T \end{bmatrix} \mathbf{D}^{-1} \begin{bmatrix} \mathbf{B}^T & -\mathbf{G} \end{bmatrix}$$

We now see that in the position of the load-to-stress matrix $\mathbf{C} = \mathbf{K}\mathbf{A}^T\mathbf{\Phi}$ we now have $\mathbf{C} = \mathbf{C}_0 - \mathbf{B}\mathbf{D}^{-1}\mathbf{G}$ as obtained by the force method. Also, in the position of the structure flexibility matrix $\mathbf{\Phi}$ we now have $\mathbf{\Phi}_0 - \mathbf{G}^T\mathbf{D}^{-1}\mathbf{G}$ which checks with Equation 18. The other submatrices occurring in $\mathbf{M}^{-1}$ are of interest in the wider structural problem where effect of loads applied directly to members and of initial stresses are considered. The demonstration of the equivalence of the two methods would then proceed in much the same way as was done here for the restricted problem.

4.0 CONCLUDING REMARKS

On the margin of the foregoing discussion several questions arise which are believed to be of general interest. We turn our attention to them.

4.1 Duality of Variables, but Not of Methods

We have noted the complete duality that exists between the stress resultants and corresponding member deformations, and also between loads applied at the joints and corresponding displacements of these joints. This duality, well known since Maxwell and Betti, has important practical consequences, permitting to reduce the number of required data and facilitating calculations and checks in many ways. Some authors, like Argyris (Reference 1), have even suggested a dictionary by means of which any statement concerning one type of variable can immediately be translated into a dual statement concerning the other type. It has also been implied that this duality extends to methods of solution of structural problems. This may be true if dual problems are considered, e.g. a problem in which joint displacements are the given quantities may be solved by a method which is dual to that used in a problem in which the loads on the joints are given. But if it comes to solve one and the same problem, as was the case in the present paper, the displacement method and the force method as usually defined are <u>not dual</u> at all. A glance at the relevant equations should be enough to convince us of the difference, the deeper reason for it being that each corresponds to a different type of partitioning of the structural matrix.

4.2 Analogy with the Theory of Elasticity

The concept of a structure may be regarded as derived from that of a continuous elastic body by a process of abstraction or simplification. The capacity of the continuum to be subjected to an infinite variety of loads and to deform into an infinity of patterns is reduced in the structure to a representation by a finite number of members, each of them capable of sustaining only a finite number of load types and of deforming in only a finite number of ways. The question arises: is it allowable to translate statements concerning the behaviour of the continuous body into analogous statements concerning the discrete structure and, if so, what are the quantities to be considered as analogous? In a most interesting recent paper (Reference 10), Besseling has tried to establish such an analogy. To this author, however, the line of argument taken there is not entirely convincing. True, some of the analogies are self-evident, like that of continuous stresses to our stress resultants and of strains to member deformations; the duality between these two sets of variables is also preserved. True also, some energy theorems and derived variational methods may be directly translated but the difficulty appears when considering the two configurations as a whole. In the continuous body, a distinction is made between conditions existing <u>within the boundary</u>, having their expression in the field equations, and conditions obtaining <u>at the boundary</u>. In the structure, on the other hand, such a distinction cannot be clearly drawn; the identification of a complete or partial set of joints with the boundaries is quite arbitrary and rather artificial and does not find any confirmation in the way the corresponding equations are written. Also, if one looks at the four basic equations of the theory of elasticity, namely the equations of equilibrium and of compatibility, in terms of stresses and of strains, one immediately perceives a lack of symmetry between these two sets of variables, even though their tensor components vary in contragredient fashion; we have here the Cauchy-Navier and Beltrami-Michell equations. For these reasons the methods of solution using displacements or stresses as unknown functions are of a widely different nature.

In some idealizations of elastic bodies, as encountered in engineering beam theory, plate and shell theory, it is found possible to achieve perfect symmetry between the two sets of variables and also in the governing equations, although these idealizations do not meet our definition of structure, not being truly a discrete representation, but rather continuous simplifications of the three dimensional elastic body.

It would be interesting to study the reasons for which a perfect duality exists between variables in these simplified and also in the discrete representations, while it is lacking in the full body. However, these differences are mentioned here in order to show that it is not safe to derive equations or prove theorems by direct analogy with the theory of elasticity. Hypothetically one may have deduced the equivalence of the methods of structural analysis from the unicity of solutions of the first boundary-value problem of the theory of elasticity. We found it more prudent to prove this equivalence by purely algebraic means.

4.3 Linear Algebra vs Network, Systems, or Graph Theory

Some recent papers - too many to quote here - have attempted to reduce structural analysis to a branch of one of the theories mentioned in the caption. It is true that certain types of structures, especially those whose members are one dimensional, like pin- or ball- jointed trusses, rigid-jointed frameworks, exhibit properties that make it possible to establish some analogies between these skeletal structures and networks of other one-dimensional elements. But even then, a graph or network does not permit to distinguish between a plane and a three-dimensional configuration, a distinction that is essential in the analysis of a structure, but only incidental in network analysis. Moreover, unless all of the joints are of the same kind (rigid or articulated), the graph in itself does not convey the necessary information without an additional specification of releases, etc. Finally, those structures that are the most difficult to treat, comprising two- and three-dimensional elements, do not seem amenable to treatment by graph-theoretical means in a general way. In other words, a one-to-one representation of a structure as a network is possible only for limited classes of structures. Furthermore, even where this possibility exists, the usefulness of such representation is questionable and of rather academic character. For instance, the classical reduction of a network to a system of trees and of linking strings or meshes has its counterpart in the reduction of a rigid-jointed framework to tree-like structures and rings; this procedure is useful in establishing the degree of statical overdetermination of the structure. But from the point of view of analysis it is preferable to make each ring statically determinate by inserting an appropriate number of pins rather than by making cuts; the first alternative, however, does not seem to have an analogue in the network. On the other hand, for a pin- or ball-jointed truss, reduction to a system of trees would transform the structure into a collapsible mechanism. Another objection is the introduction of terminology foreign to the structural field, entirely unnecessary.

In conclusion it appears that network theory, in spite of claims to the contrary, is not general enough to permit an adequate representation or interpretation of the theory of structures. On the other hand, linear algebra or theory of vector spaces provide the ideal mathematical tool for structural analysis, in that every topological, geometrical and elastic property and every relation in the structure find their true representation in the language of matrices.

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VARIABLES	INTENSIVE	EXTENSIVE
JOINT COMPONENTS	LOADS $p = \{p_1 \text{-----} p_n\}$	DISPLACEMENTS $u = \{u_1 \text{----} u_n\}$
MEMBER COMPONENTS	STRESS RESULTANTS $s = \{s_1 \cdots s_m\}$	DISTORTIONS $e = \{e_1 \text{-----} e_m\}$

BASIC MATRICES A = GEOMETRIC ASSEMBLY MATRIX (CONNECTIVITY + TRANSFORMATION FROM MEMBER REFERENCE SYSTEMS TO GLOBAL SYSTEM)

$F = \text{DIAG.} \left\langle F^{(1)} \text{---} F^{(M)} \right\rangle$ = MEMBER FLEXIBILITIES IN LOCAL SYSTEMS OF REF.

BASIC RELATIONS

JOINT EQUILIBRIUM		$As = p$	(1)
COMPATIBILITY		$A^T u = e$	(2)
MEMBER FLEXIBILITY (GEN. HOOKE'S LAW)		$F^{(i)} s^{(i)} = e^{(i)}$	(3)
	OR	$Fs = e$	(3')

Figure 1. Formulation of Structural Problem

<u>GIVEN</u>: LOADS p; MATRICES A, F

<u>REQUIRED</u>: STRESSES s; DISPLACEMENTS u

a. OBTAIN MEMBER STIFFNESS MATRIX

$$K = \text{DIAG.} \left\langle K^{(1)} \cdots K^{(M)} \right\rangle = F^{-1} \tag{4}$$

EITHER DIRECTLY OR BY INVERTING F.

$$\begin{tikzcd} p & u \arrow[d, "A^T"] \\ s \arrow[u, "A"] \arrow[r, "F"'] & e \arrow[l, dashed, bend right, "K = F^{-1}"'] \end{tikzcd}$$

b. CALCULATE STRUCTURE STIFFNESS MATRIX

$$\Gamma = AKA^T \tag{5}$$

c. OBTAIN STRUCTURE FLEXIBILITY MATRIX

$$\Phi = \Gamma^{-1} \tag{6}$$

d. DISPLACEMENTS

$$u = \Phi p \tag{7}$$

e. STRESSES

$$s = KA^T u = KA^T \Phi p \tag{8}$$

f. CHECK EQUILIBRIUM

$$p = As, \quad \text{OR} \quad AKA^T \Phi = I \tag{9}$$

$$\begin{tikzcd} p \arrow[r, dashed, "\Phi"'] \arrow[d, dashed, "KA^T\Phi"'] & u \arrow[l, bend right, "\Gamma"'] \arrow[d, "A^T"] \\ s & e \arrow[l, "K"'] \end{tikzcd}$$

Figure 2. Displacement Method

a. EQUILIBRIUM RELATION (1) PARTITIONED (A$_0$ NON-SINGULAR; q SELF-EQUILIBRATING) $$\left[A_0 \,\vdots\, A_1\right]\begin{bmatrix} s_0 \\ \hdashline q \end{bmatrix} = p \qquad (10)$$

b. STATICALLY DETERMINATE STRESSES $$s_0 = A_0^{-1}p \qquad (11)$$

c. REDUNDANT STRESS SYSTEMS $$s_q = -A_0^{-1}A_1 q \qquad (12)$$

d. RESULTANT STRESSES $C_0 = \begin{bmatrix} A_0^{-1} \\ \hdashline 0 \end{bmatrix}$; $B = \begin{bmatrix} -A_0^{-1}A_1 \\ \hdashline I \end{bmatrix}$ $$s = \left[C_0 \,\vdots\, B\right]\begin{bmatrix} p \\ q \end{bmatrix} \qquad (13)$$

e. CONTINUITY AT CUTS OR RELEASES

OR WITH $B^TFB = D$; $B^TFC_0 = G$

$$u_q = B^TFs = 0 \qquad (14)$$

OR $$B^TFC_0p + B^TFBq = 0 \qquad (14')$$

$$Gp + Dq = 0 \qquad (14'')$$

f. SOLVE FOR REDUNDANT SYSTEMS $$q = -D^{-1}Gp \qquad (15)$$

g. FINAL STRESSES $$s = C_0p - BD^{-1}Gp = (C_0 - BD^{-1}B^{-T}FC_0)p = (I-\Delta)C_0p = Cp \qquad (16)$$

WHERE $\Delta = BD^{-1}B^TF$; $C = (I-\Delta)C_0$

h. DISPLACEMENTS $$u = \Phi p \qquad (17)$$

WHERE $\Phi = C^TFC = C_0^T(I-\Delta^T)F(I-\Delta)C_0 = C_0^TFC$

OR $$\Phi = \Phi_0 - G^TD^{-1}G \qquad \text{WITH } \Phi_0 = C_0^TFC_0 \qquad (18)$$

Figure 3. Force Method

THE STRUCTURAL MATRIX

$$M = \begin{bmatrix} F & A^T \\ A & 0 \end{bmatrix} = \begin{bmatrix} F_0 & F_{10}^T & A_0^T \\ F_{10} & F_1 & A_1^T \\ A_0 & A_1 & 0 \end{bmatrix} \quad (19)$$

"NATURAL" DIAGONALIZATION

$$\bar{M} = X M X^T$$

WITH $X = \begin{bmatrix} I & 0 \\ -AF^{-1} & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -AK & I \end{bmatrix}$

$$\bar{M} = \begin{bmatrix} F & 0 \\ 0 & -AF^{-1}A^T \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & -\Gamma \end{bmatrix}$$

INVERSION $\bar{M}^{-1} = \begin{bmatrix} F^{-1} & 0 \\ 0 & -\Gamma^{-1} \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & -\Phi \end{bmatrix}$; $M^{-1} = X^T \bar{M}^{-1} X = \begin{bmatrix} K - KA^T\Phi AK & KA^T\Phi \\ & \Phi AK & -\Phi \end{bmatrix} \quad (20)$

REDUCTION ALONG SECONDARY DIAGONAL $\bar{\bar{M}} = YMY^T$; $M^{-1} = Y^T \bar{\bar{M}}^{-1} Y$

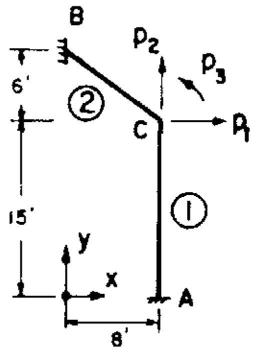
WITH $Y = \begin{bmatrix} I & 0 & -\frac{1}{2}A_0^{-1}F_0 \\ -B_0^T & I & -G \\ 0 & 0 & I \end{bmatrix}$ $\quad \bar{\bar{M}} = \begin{bmatrix} 0 & 0 & A_0^T \\ 0 & D & 0 \\ A_0 & 0 & 0 \end{bmatrix}$ $\quad \bar{\bar{M}}^{-1} = \begin{bmatrix} 0 & 0 & A_0^{-1} \\ 0 & D^{-1} & 0 \\ (A_0^T)^{-1} & 0 & 0 \end{bmatrix}$

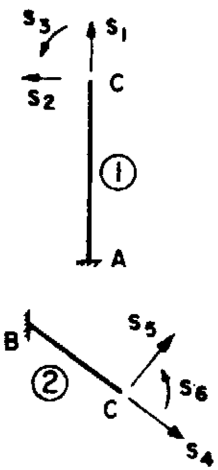
WHERE $B_0 = A_0^{-1}A_1$ $\quad D = F_1 - F_{10}B_0 - B_0^T F_{10}^T + B_0^T F_0 B_0 = B^T F B$

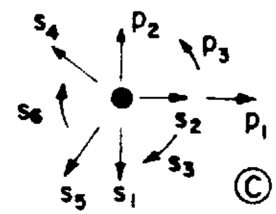
$\Phi_0 = (A_0^T)^{-1} F_0 A_0^{-1} = C_0^T F C_0$ $\quad G = (F_{10} - B_0^T F_0) A_0^{-1} = B^T F C_0$

$$M^{-1} = \left[\begin{array}{c:c:c} B_0 D^{-1} B_0^T & -B_0 D^{-1} & A_0^{-1} + B_0 D^{-1} G \\ -D^{-1}B_0^T & D^{-1} & -D^{-1}G \\ (A_0^T)^{-1} + G^T D^{-1} B_0^T & -G^T D^{-1} & -\Phi_0 + G^T D^{-1} G \end{array}\right] = \begin{bmatrix} 0 & 0 & A_0^{-1} \\ 0 & 0 & 0 \\ (A_0^T)^{-1} & 0 & -\Phi_0 \end{bmatrix} + \begin{bmatrix} -B_0 \\ I \\ -G^T \end{bmatrix} D^{-1} \begin{bmatrix} -B_0^T & I & -G \end{bmatrix} = \begin{bmatrix} 0 & C_0 \\ C_0^T & -\Phi_0 \end{bmatrix} + \begin{bmatrix} B \\ -G^T \end{bmatrix} D^{-1} \begin{bmatrix} B^T & -G \end{bmatrix} \quad (21)$$

Figure 4. Inversion of Structural Matrix by Partioning







Equilibrium at Joint C

$$A = \begin{bmatrix} \cdot & -1 & \cdot & 0.8 & 0.6 & \cdot \\ 1 & \cdot & \cdot & -0.6 & 0.8 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \end{bmatrix}$$

DATA

Member	1 =AC	2 =BC
ℓ	15	10
λ	0	0.8
μ	1	-0.6
EA	1500	1000
EI	15	20

Member Flexibilities

$$F^{(1)} = \begin{bmatrix} 0.01 & \cdot & \cdot \\ \cdot & 75 & 7.5 \\ \cdot & 7.5 & 1.0 \end{bmatrix}; \quad F^{(2)} = \begin{bmatrix} 0.01 & \cdot & \cdot \\ \cdot & 16.67 & 2.5 \\ \cdot & 2.5 & 0.5 \end{bmatrix}$$

Structure Stiffness

$$\Gamma = AKA^T = \begin{bmatrix} 64.14 & -47.88 & -0.32 \\ -47.88 & 136.15 & -0.96 \\ -0.32 & -0.96 & 12.0 \end{bmatrix}$$

Structure Flexibility

$$\Phi = \Gamma^{-1} = \begin{bmatrix} 21.16 & 7.45 & 1.16 \\ 7.45 & 9.97 & 1.00 \\ 1.16 & 1.00 & 83.44 \end{bmatrix} \times 10^{-3}$$

Member Stiffnesses

$$K^{(1)} = \begin{bmatrix} 100 & \cdot & \cdot \\ \cdot & 0.0533 & -0.4 \\ \cdot & -0.4 & 4.0 \end{bmatrix}$$

$$K^{(2)} = \begin{bmatrix} 100 & 0.24 & \cdot \\ \cdot & 0.24 & -1.2 \\ \cdot & -1.2 & 8.0 \end{bmatrix}$$

Load-to-distortion matrix

$$A^T\Phi = \begin{bmatrix} 7.45 & 9.97 & 1.00 \\ -21.16 & -7.45 & -1.16 \\ 1.16 & 1.00 & 83.44 \\ 12.46 & -0.02 & 0.33 \\ 18.65 & 12.45 & 1.49 \\ 1.16 & 1.00 & 83.44 \end{bmatrix} \times 10^{-3}$$

Load-to-stress matrix

$$C = KA^T\Phi = \begin{bmatrix} 0.7450 & 0.9972 & 0.0996 \\ -0.0016 & -0.0008 & -0.0334 \\ 0.0131 & 0.0070 & 0.3342 \\ 1.2457 & -0.0023 & 0.0330 \\ 0.0031 & 0.0018 & -0.0998 \\ -0.0131 & -0.0070 & 0.6658 \end{bmatrix}$$

Figure 5. Built-In Two-Member Frame - Displacement Method

$$A_0 = \begin{bmatrix} \cdot & -1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{bmatrix};\quad A_1 = \begin{bmatrix} 0.8 & 0.6 & \cdot \\ -0.6 & 0.8 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix};\quad C_0 = \begin{bmatrix} \cdot & 1 & \cdot \\ -1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix};\quad B = \begin{bmatrix} 0.6 & -0.8 & \cdot \\ 0.8 & 0.6 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix};\quad FB = \begin{bmatrix} 0.006 & -0.008 & 0 \\ 60 & 45 & -7.5 \\ 6.0 & 4.5 & -1 \\ 0.010 & 0 & 0 \\ 0 & 16.67 & 2.5 \\ 0 & 2.5 & 0.5 \end{bmatrix}$$

Auxiliary Matrices

$$D = \begin{bmatrix} 48.0136 & 35.9952 & -6.0 \\ 35.9952 & 43.6731 & -2.0 \\ -6.0 & -2.0 & 1.5 \end{bmatrix};\quad D^{-1} = \begin{bmatrix} 0.20403 & -0.13929 & 0.63040 \\ -0.13929 & 0.11948 & -0.39786 \\ 0.63040 & -0.39786 & 2.65781 \end{bmatrix};\quad G = \begin{bmatrix} -60 & 0.006 & 6.0 \\ -45 & 0.008 & 4.5 \\ 7.5 & 0 & -1 \end{bmatrix}$$

Stress Correction / **Load-to-stress Matrix**

$$\Delta C_0 = \begin{bmatrix} -0.74493 & 0.00284 & -0.09965 \\ -0.99842 & 0.00080 & 0.03344 \\ -0.01310 & -0.00697 & 0.66578 \\ -1.24569 & 0.00234 & -0.03304 \\ -0.00314 & -0.00179 & 0.09978 \\ 0.01310 & 0.00697 & -0.99578 \end{bmatrix};\qquad C = \begin{bmatrix} 0.74493 & 0.99716 & 0.09965 \\ -0.00158 & -0.00080 & -0.03344 \\ 0.01310 & 0.00697 & 0.33422 \\ -1.24569 & -0.00234 & 0.03304 \\ 0.00314 & 0.00179 & -0.09978 \\ -0.01310 & -0.00697 & 0.66578 \end{bmatrix}$$

Structure Flexibility

Static. determ. / Correction / Resultant

$$\Phi_0 = \begin{bmatrix} 75 & 0 & -7.5 \\ 0 & 0.01 & 0 \\ -7.5 & 0 & 1.0 \end{bmatrix};\quad G^T D^{-1} G = \begin{bmatrix} 74.98157 & -0.00745 & -7.50144 \\ -0.00745 & 0.00003 & -0.00100 \\ -7.50144 & -0.00100 & 0.91658 \end{bmatrix}\qquad \Phi = \begin{bmatrix} 18.43 & 7.45 & 1.44 \\ 7.45 & 9.97 & 1.00 \\ 1.44 & 1.00 & 83.42 \end{bmatrix} \times 10^{-3}$$

Alternative Calculations

$$C^T F C = \begin{bmatrix} 21.16 & 7.45 & 1.23 \\ 7.45 & 9.97 & 1.00 \\ 1.23 & 1.00 & 83.43 \end{bmatrix} \times 10^{-3};\quad C_0^T F C = \begin{bmatrix} 20.25 & 7.45 & 1.25 \\ 7.45 & 9.97 & 1.00 \\ 1.25 & 1.00 & 83.42 \end{bmatrix}$$

Figure 6. Built-In Two-Member Frame - Force Method