MATRIX ANALYSIS OF CREEP AND PLASTICITY PROBLEMS

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Matrix analysis of structures is an algebraic analogy of the continuous field theory for the description of the mechanical behaviour of solid bodies. Fields are replaced by vectors with a finite number of elements. The displacement vector defines the generalized strains of the various structural elements and these generalized strains may be considered as a superposition of elastic-, thermal- and inelastic-components. For the inelastic-components of the generalized strains a theory of creep and plasticity is developed that is fully analogous to the continuous field theory of creep and plasticity. In the case of a structure, composed of elements subject to creep, the deformation problem is solved by integration of a system of first order differential equations. Plastic deformation in a number of structural elements, on the other hand, gives rise to a step by step modification of the elastic solution of the deformation problem.

MATRIX FORMULATION OF THE ELASTIC DEFORMATION PROBLEM

In the matrix formulation of structural analysis the continuous field concept from the classical description of the mechanical behaviour of structures is replaced by an algebraic analogy in the form of vectors with a finite number of elements, representing lumped field quantities. These lumped field quantities, though admittedly permitting a less detailed description, may comprise, if properly chosen, sufficient information for an adequate analysis of complex structural problems, whereby all the necessary numerical operations can be programmed directly for the digital computer.

In order to arrive at the displacement vector, that shall replace the continuous displacement field in a structure, we conceive this structure with the aid of imaginary cuts as a composition of structural elements, which may greatly differ in size and shape. Continuity of the structure will be ensured if the displacement fields in the individual structural elements are compatible with continuous displacement functions, defined along the imaginary cuts that separate the elements. These functions may be chosen such as to contain the displacements in a finite number of discrete points as degrees of freedom. Additional degrees of freedom may be defined in terms of displacements in a finite number of discrete points on the natural boundary, which represent parameters in displacement functions along this natural boundary. The vector containing all degrees of freedom, thus defined, is the displacement vector.

From the displacement vector the boundary displacements of the individual structural elements are obtained by elementary matrix transformations. Each degree of freedom for a structural element in excess of its six degrees of freedom with respect to rigid body motions corresponds to a deformation mode. The linearly independent deformation modes of the

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individual elements will be denoted as generalized strains. The matrix equation relating the vector of generalized strains to the vector of boundary displacements replaces the expressions for the strain components in terms of the displacement field in the continuous field theory.

Some simple examples may illustrate the procedure. In the portal frame of Figure 1 the structural elements are straight rods. For loading in its plane continuity of the structure in accordance with elastic line theory will be ensured if the deformation of each rod is compatible with two displacements and one rotation in each cut, that is necessary to free the individual elements. All these displacements and rotations are now arranged in the displacement vector $\{u\, u^\circ\}$, where the subvector u° contains the prescribed displacements and rotations and where the elements of u are kinematically free. The vector of six displacements and rotations for the end points of the k^{th} element will now be denoted by u^k. The vector u^k for each element is derived by multiplication of the displacement vector for the structure as a whole with the so-called "location matrix" $[L^k\ L^{ko}]$, with zero and unit elements, that effects the necessary interchange and elimination of elements in the displacement vector

$$u^k = \begin{bmatrix} L^k & L^{ko} \end{bmatrix} \begin{bmatrix} u \\ u^o \end{bmatrix} \tag{1}$$

Since in the discrete analysis the mechanical behaviour of the structural elements will be fixed for each element independently, it is desirable to define displacement parameters and the deformation modes to be derived therefrom in one, for the element under consideration suitably chosen coordinate system, which in general will not coincide with the coordinate system in which the displacement parameters $\{u\, u^\circ\}$ are defined. The transformation of the displacement parameters u^k, defined with respect to the general coordinate system, to the vector of displacement parameters in the proper coordinate system of the k^{th} element, to be denoted by $\bar{u}^k$, is effected by a transformation matrix T .

$$\bar{u}^k = T^k u^k \tag{2}$$

For the structural elements of the portal frame of Figure 1 the matrix T^k implies at most a simple rotation of the coordinate axes.

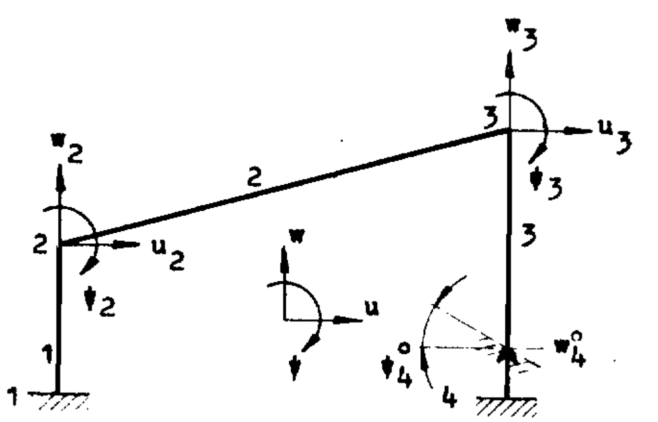

Figure 1. Kinematical Degrees of Freedom in Portal Frame

As each element of the portal frame possesses three degrees of freedom with respect to rigid body motion in a plane, the six displacements and rotations of the vector $\bar{u}^k$ define 6 - 3 = 3 linearly independent deformation modes. The vector of deformation modes ϵ^k is obtained by premultiplication of the vector u^{-k} with a combination matrix C^k, which is characteristic for all straight rods (see Figure 2)

$$\epsilon^k = C^k \, u^{-k} \tag{3}$$

$$u^{-k} = \{ u_1 \;\; w_1 \;\; \psi_1 \;\; \ell_0 \;\; u_2 \;\; w_2 \;\; \psi_2 \;\; \ell_0 \}$$

$$\epsilon^k = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}; \; C^k = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\lambda & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & \lambda \end{bmatrix}, \; \lambda = \ell / \ell_0$$

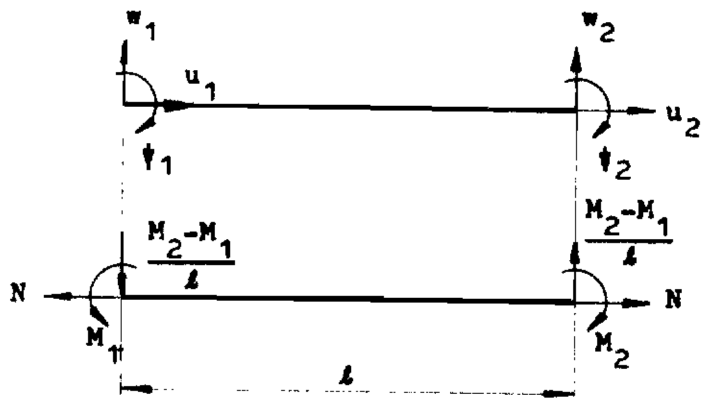

Figure 2. Displacement Parameters and Generalized Stresses of Rod

The product matrix $C^k T^k$ will be denoted by D^k and $D^k = C^k T^k$ is called the "finite-difference matrix" of the element k.

If the vectors of deformation modes ϵ^k of all N elements of the structure are combined into one vector ϵ,

$$\epsilon = \{ \epsilon^1, \;\; \epsilon^2, \;\; \epsilon^3, \ldots \epsilon^N \}, \tag{4}$$

and if the total finite-difference and location matrices of the structure are defined by

$$D = \left[\begin{array}{cc|c} D^1 & 0 & 0 \\ 0 & D^2 & 0 \\ \hline 0 & 0 & D^N \end{array}\right], \; \left[L \;\; L^0 \right] = \left[\begin{array}{cc} L^1 & L^{10} \\ L^2 & L^{20} \\ \hline L^N & L^{N0} \end{array}\right], \tag{5}$$

then the following vector equation is obtained as the algebraic analogy of the expressions for the strain components in terms of the displacement field in the continuous field theory.

$$\epsilon = D \left[L \ L^{o} \right] \begin{bmatrix} u \\ u^{o} \end{bmatrix}$$

To illustrate the generality of the procedure described above, we consider the thick-walled tube under internal pressure of Figure 3. In view of the axial symmetry of the load a subdivision into concentric cylinders is indicated. If we restrict ourselves to the deformation problem in sections sufficiently far away from the end-sections of the tube, the degrees of freedom with respect to rigid body motions may be eliminated from the start by considering a displacement vector, composed of the radial displacements u_n along the concentric, cylindrical cuts and the natural boundaries, and a constant specific axial elongation ϵ_a. If all elements of the displacement vector u are given the same dimension by dividing the displacements u_n by the local radius r_n, the finite-difference matrix D^k for one tubular element degenerates in this case to the unit matrix.

$$u^{-k} = u^{k} = \left\{ \frac{u_n}{r_n}, \frac{u_{n+1}}{r_{n+1}}, \epsilon_a \right\}$$

$$\epsilon^{k} = \begin{bmatrix} \frac{u_n}{r_n} \\ \frac{u_{n+1}}{r_{n+1}} \\ \epsilon_a \end{bmatrix}, \quad D^{k} = C^{k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

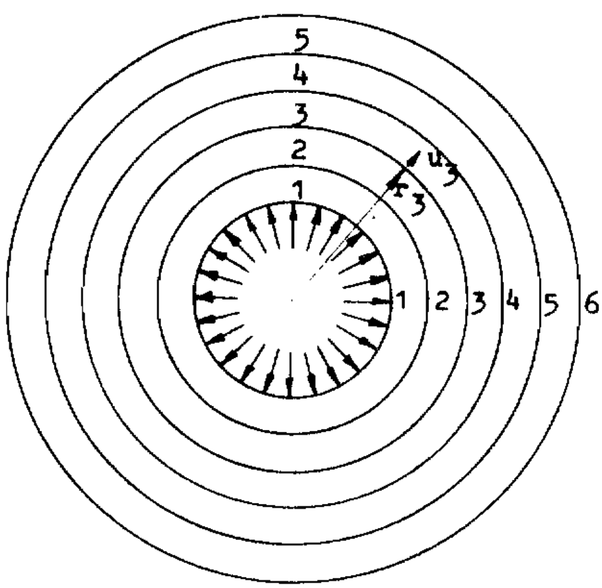

Figure 3. Structural Elements in Tube Under Internal Pressure

Finite-difference matrices for other types of structural elements may for instance be found in Reference 1.

It should be noted here that so far only displacements along the artificial and natural boundaries of the structural elements have been considered, whilst the displacement field inside an element corresponding to any one deformation mode has still been left undefined. For rectangular Cartesian coordinates a displacement field u_i, restricted by

$$\left| \frac{\partial u_i}{\partial x} \right| << 1$$

defines six independent components of the strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{7}$$

For an isotropic elastic solid the static linear theory of elasticity can be summarized in terms of the principle of minimum potential energy by the condition that the expression

$$P = \int_V \left[\frac{1}{2} \left\{ 2G\left(\delta_{i\alpha}\, \delta_{j\beta} - \frac{1}{3}\delta_{ij}\, \delta_{\alpha\beta}\right) + C\delta_{ij}\, \delta_{\alpha\beta} \right\} \epsilon_{ij}\, \epsilon_{\alpha\beta} - f_i\, u_i \right] dV - \int_{A^p} p_i^o\, u_i\, dA \tag{8}$$

shall have minimum value with respect to all displacement fields u_i subject to the subsidiary conditions (Equation 7) and that

$$u_i = u_i^o \text{ on } A^u .$$

Here G and C are the elastic constants and f_i^o and p_i^o represent prescribed external loads. The Euler-Lagrange equations and natural boundary conditions of this variational problem are the familiar Cauchy equations from the theory of elasticity with the appropriate boundary conditions.

In order to arrive at an expression for the elastic potential of a structural element in terms of the boundary displacements u^k along the boundary defining the structural element, it is necessary to know the displacement field and thereby the strain tensor field inside the element corresponding to these boundary displacements.

The simplest way to proceed is to approximate the displacement field inside an element k by suitably chosen functions $\bar{u}_i$, that are compatible with the boundary displacements and hence contain the elements of the vector $\bar{u}^k$ as parameters. This leads to a strain tensor field $\bar{\epsilon}_{ij}$ that is determined by the elements of the vector ϵ^k. If the integrations in Expression 8 are carried out over the element k, then the potential energy of this element is approximated by

$$P^k = \frac{1}{2}(\epsilon^k)^T S^k \epsilon^k \quad (u^{-k}) f^{ko} \tag{9}$$

Here the vector f^{ko} is determined by the prescribed external loads. Since the elastic potential must be positive definite in terms of the linearly independent deformation modes of ϵ^k the symmetric stiffness matrix S^k is nonsingular.

In the case of the structural elements considered above a more accurate procedure is possible. The elasticity equations both for the tubular elements can be solved for boundary conditions fixed by the elements of the vectors $\bar{u}^k$.

If shear deformation is neglected the differential equations for the displacements of the elastic line element read

$$EA\frac{d^2u}{dx^2} = 0, \quad EI\frac{d^4w}{dx^4} = 0,$$

where E is the elastic modulus and A and I are the area and the moment of inertia of the cross-section of the rod. For boundary conditions fixed by the elements of the vector $\bar{u}^k$ they have the following solutions $(0 \leq \xi \leq 1)$

$$w = \begin{bmatrix} w_1 & \psi_1 \ell_0 & w_2 & \psi_2 \ell_0 \end{bmatrix} \begin{bmatrix} 2\xi^3 - 3\xi^2 + 1 \\ \lambda(-\xi^3 + 2\xi^2 - \xi) \\ -2\xi^3 + 3\xi^2 \\ \lambda(-\xi^3 + \xi^2) \end{bmatrix}; \quad u = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix}$$

The corresponding deformations $\frac{d^2w}{dx^2}$ and $\frac{du}{dx}$ contain as parameters the deformation modes defined by Equation 3 and evaluation of the integral for the elastic potential

$$\int_0^{\ell} \left[\frac{1}{2} EA \left(\frac{du}{dx}\right)^2 + \frac{1}{2} EI \left(\frac{d^2w}{dx^2}\right)^2 \right] dx$$

leads to a quadratic form

$$\frac{1}{2} (\epsilon^k)^T S^k \epsilon^k$$

with the following stiffness matrix

$$S^k = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & 4S_2 & -2S_2 \\ 0 & -2S_2 & 4S_2 \end{bmatrix}; \quad S_1 = \frac{EA}{\ell}; \quad S_2 = \frac{EI}{\ell^3}.$$

For the tubular elements we have for the radial displacement u the well-known elasticity equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

with the following solution in terms of the elements $\frac{u_n}{r_n}$ and $\frac{u_{n+1}}{r_{n+1}}$ of $\bar{u}^k$:

$$u = \frac{u_{n+1}}{r_{n+1}}\left[\frac{r_{n+1}^2\left(r - \frac{r_n^2}{r}\right)}{r_{n+1}^2 - r_n^2}\right] - \frac{u_n}{r_n}\left[\frac{r_n^2\left(r - \frac{r_{n+1}^2}{r}\right)}{r_{n+1}^2 - r_n^2}\right]$$

The stiffness matrix from the quadratic form in ϵ^k, defining the elastic potential, is given by

$$S^k = \frac{\pi}{r_{n+1}^2 - r_n^2}\begin{bmatrix} 4r_n^4\left\{C+G\left(\frac{1}{3}+\frac{r_{n+1}^2}{r_n^2}\right)\right\} & -4r_n^2 r_{n+1}^2\left(C+\frac{4}{3}G\right) & -2r_n^2\left(r_{n+1}^2-r_n^2\right)\left(C-\frac{2}{3}G\right) \\ -4r_n^2 r_{n+1}^2\left(C+\frac{4}{3}G\right) & 4r_{n+1}^4\left\{C+G\left(\frac{1}{3}+\frac{r_n^2}{r_{n+1}^2}\right)\right\} & 2r_{n+1}^2\left(r_{n+1}^2-r_n^2\right)\left(C-\frac{2}{3}G\right) \\ -2r_n^2\left(r_{n+1}^2-r_n^2\right)\left(C-\frac{2}{3}G\right) & 2r_{n+1}^2\left(r_{n+1}^2-r_n^2\right)\left(C-\frac{2}{3}G\right) & \left(r_{n+1}^2-r_n^2\right)^2\left(C+\frac{4}{3}G\right) \end{bmatrix}$$

In view of the creep and plasitcity problem, to be treated later on, it is useful to observe here that the terms in the elements of this stiffness matrix with the shear modulus G determine the distortional energy and the terms with the bulk modulus C the volume strain energy.

As soon as the stiffness matrices S^k and the load vectors f^{ko} for the individual elements of the structure have been determined, the total stiffness matrix and the total load vector of the structure is defined by

$$S = \begin{bmatrix} S^1 & 0 & 0 \\ 0 & S^2 & 0 \\ 0 & 0 & S^N \end{bmatrix}, \quad f^{\circ} = \begin{vmatrix} (T^1)^T f^{10} \\ (T^2)^T f^{20} \\ (T^N)^T f^{N0} \end{vmatrix} \tag{10}$$

For the total potential energy of the structure may now be written

$$P = \frac{1}{2}\epsilon^T S \epsilon - \left[u^T (u^{\circ})^T\right]\begin{bmatrix} L^T \\ (L^{\circ})^T \end{bmatrix} f^{\circ} \tag{11}$$

This expression has to be minimized with the subsidiary conditions (Equation 6). The condition for a stationary value of P reads then

$$L^T D^T S D L u = -L^T D^T S D L^{\circ} u^{\circ} + L^T f^{\circ} \tag{12}$$

The solution of these n equations with n unknowns provides the solution of the deformation problem. This method of solution is known as the "displacement method".

In a treatment based upon the principle of minimum potential energy the concept of stress arises if the subsidiary conditions of the variational problem are taken into account with the aid of the Lagrangian multiplier method. So it can be stated that the expression

$$F = \int_V \Big[\tfrac{1}{2}\{2G(\delta_{i\alpha}\delta_{j\beta} - \tfrac{1}{3}\delta_{ij}\,\delta_{\alpha\beta}) + C\delta_{ij}\,\delta_{\alpha\beta}\}\epsilon_{ij}\epsilon_{\alpha\beta} - f_i^\circ u_i +$$

$$+ \Big\{\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) - \epsilon_{ij}\Big\}\sigma_{ij}\Big] dV - \int_{A^p} p_i^\circ u_i\, dA + \int_{A^u} (u_i^\circ - u_i) p_i\, dA \tag{13}$$

shall have a stationary value with respect to ϵ_{ij}, u_i and with respect to the multipliers σ_{ij} and p_i, and that this stationary value is equal to the required minimum value of the potential energy. The proof of this statement follows immediately from the fact that this stationary value principle leads to differential equations and boundary conditions that are identical with the Cauchy equations with their boundary conditions, while the stationary value of F is numerically equal to the value of P. In addition however this stationary value principle defines quantities σ_{ij} and p_i, which are to be identified as the components of the stress tensor and as the surface tractions respectively.

Similarly in the matrix formulation of the deformation problem a vector of generalized stresses σ can be defined, if the subsidiary conditions (Equation 6) of the minimum value problem for Expression 11 are taken into account by the multiplier method. The requirement that the expression

$$F = \frac{1}{2}\,\boldsymbol{\epsilon}^T \mathbf{S}\boldsymbol{\epsilon} - \big[\mathbf{u}^T\ \mathbf{u}^T\big]\begin{bmatrix}\mathbf{L}^T \\ (\mathbf{L}^\circ)^T\end{bmatrix}\mathbf{f}^\circ + \Big(\big[\mathbf{u}^T(\mathbf{u}^\circ)^T\big]\begin{bmatrix}\mathbf{L}^T \\ (\mathbf{L}^\circ)^T\end{bmatrix}\mathbf{D}^T - \boldsymbol{\epsilon}^T\Big)\boldsymbol{\sigma} \tag{14}$$

shall have a stationary value with respect to the elements of $\boldsymbol{\epsilon}, \mathbf{u}$, and with respect to the multipliers contained in the vector $\boldsymbol{\sigma}$, leads to the following equations

$$\boldsymbol{\sigma} = \mathbf{S}\,\boldsymbol{\epsilon}, \tag{15}$$

$$\mathbf{L}^T\,\mathbf{D}^T\boldsymbol{\sigma} = \mathbf{L}^T\,\mathbf{f}^\circ, \tag{16}$$

$$\boldsymbol{\epsilon} = \mathbf{D}\big[\mathbf{L}\mathbf{L}^\circ\big]\begin{bmatrix}\mathbf{u} \\ \mathbf{u}^\circ\end{bmatrix}.$$

Elimination of the vector σ immediately reduces the system to Equation 12.

It can be observed that by virtue of the equations (15) the elastic potential can be written as

$$\frac{1}{2}\,\boldsymbol{\epsilon}^T\mathbf{S}\boldsymbol{\epsilon} = \frac{1}{2}\,\boldsymbol{\sigma}^T\boldsymbol{\epsilon} \tag{17}$$

and the vector $\boldsymbol{\sigma}$ may appropriately be denoted as the vector of generalized stresses. Equation 15 replaces the stress-strain relations, that define the material properties in the continuous field theory of elasticity. The generalized stresses of the structural elements have to satisfy Equations 16, which are analogous to the equilibrium equations for the stress tensor.

The mechanical interpretation of the generalized stresses is best obtained by equating Expression 17 to the elastic potential expressed in terms of the boundary tractions and displacements. For the elastic line element holds (Figure 2):

$$\frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\epsilon} = \frac{1}{2}\Big[N(u_2 - u_1) + \frac{M_1}{\ell}(w_1 - w_2 - \psi_1\,\ell) + \frac{M_2}{\ell}(-w_1 + w_2 + \psi_2\,\ell)\Big]$$

$$= \frac{1}{2}\Big[N\epsilon_1 + \frac{M_1}{\ell}\epsilon_2 + \frac{M_2}{\ell}\epsilon_3\Big]$$

or

$$\sigma_1 = N, \quad \sigma_2 = \frac{M_1}{\ell}, \quad \sigma_3 = \frac{M_2}{\ell}.$$

Thus the generalized stresses are seen to be simply the stress resultants acting on the rod.

For the tubular element the generalized stresses can be expressed in terms of the radial stresses σ_r and the axial stress σ_a

$$\frac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{\epsilon} = \frac{1}{2}\left[-2\pi r_n \sigma_{r,n}\, u_n + 2\pi r_{n+1}\, \sigma_{r,n+1}\, u_{n+1} + \epsilon_a 2\pi \int_{r_n}^{r_{n+1}} \sigma_a\, r dr\right]$$

or

$$\sigma_1 = -2\pi r_n^2 \sigma_{r,n}, \sigma_2 = 2\pi r_{n+1}^2\, \sigma_{r,n+1}, \sigma_3 = 2\pi \int_{r_n}^{r_{n+1}} \sigma_a\, r dr.$$

The determination of the stationary value of F (Equation 14) can be performed in steps such that an extremum principle is obtained complementary to the principle of minimum potential energy. In this procedure, due to K.O. Friedrichs, we first make F stationary with respect to $\boldsymbol{\epsilon}$ and $\mathbf{u}$ while keeping the multipliers, contained in $\boldsymbol{\sigma}$, at a fixed value. This leads to the Equations 15 and 16. These equations substituted into F transform F into the expression

$$-C = -\frac{1}{2}\boldsymbol{\sigma} \mathbf{S}^{-1} \boldsymbol{\sigma} + (u^o)^T (L^o)^T D^T \boldsymbol{\sigma}. \tag{18}$$

Since the determination of the stationary value of -C with respect to variations of $\boldsymbol{\sigma}$, that satisfy the equilibrium equations (16), is the last step in the determination of the stationary value of F with respect to all variables, this stationary value of -C is equal to P_{min}. Moreover it is easily seen that this stationary value of -C is a maximum, demonstrating the complementary nature of the variational principle for C.

It should be mentioned here that an approximate solution of a structural problem by means of the principle of minimum complementary energy in matrix form, as formulated above, does not concur in general with a solution by the so-called "force method" of matrix analysis despite the formal similarity of the equations. In fact the solution would still be identical with a solution obtained with the displacement-method, since the flexibilities $\mathbf{S}^{-1}$ and the generalized stresses $\boldsymbol{\sigma}$ are derived quantities. For true complementary solutions the reader be referred to the literature (References 1 and 2).

BEHAVIOR OF STRUCTURAL ELEMENTS SUBJECT TO CREEP AND PLASTICITY

The behaviour of structural elements subject to creep and plasticity is characterized by a deformation that depends on the loading history. We observe that the vector of generalized strains $\boldsymbol{\epsilon}$, introduced by (Equation 3), is a measure of deformation for a structural element as a whole, whilst the strain tensor (Equation 7) in a point of the body is a measure of deformation for the surrounding atomic structure. For an element of finite size the deformation pattern may still vary strongly at a fixed deformation vector ϵ, but on a microscopic scale also the strain tensor gives but an average of the deformation in the neighbourhood of each point. This indeterminacy in the description of the deformation does not prevent the existence of unique relations between deformations and stresses, both in the discrete and the continuous description, in the case of reversible, purely elastic deformation. In the discrete matrix formulation these are relations between generalized stresses and generalized strains, which read according to Equation 15

$$\sigma = S\epsilon \quad \text{or} \quad \epsilon = S^{-1} \sigma .$$

If in a structural element inelastic deformation occurs an elastic deformation vector can be defined by

$$\epsilon^{T} = S^{-1} \sigma \quad \text{or} \quad \sigma \quad S \epsilon^{T} \tag{19}$$

The connection with the total deformation vector ϵ is obtained by the introduction of a plastic deformation vector $\bar{\epsilon}$

$$\epsilon' = \epsilon - \bar{\epsilon} \tag{20}$$

In general the elements of the vector of generalized strains ϵ may be considered as a superposition of elastic-, inelastic- and thermal components. The latter are zero in the case of isothermal deformation, to which this paper will be restricted.

We now have to deal with the basic problem of the theory of creep and plasticity. Which are the laws that govern the development of the inelastic strains? It is noted that in inelastic-deformation part of the work exerted on the body is dissipated and cannot be recovered as mechanical energy. In the discrete analysis, like in the continuous field theory, the author found it a fruitful starting point to postulate that the rate of energy dissipation at any moment is a positive semi-definite function of the state variables and that this function uniquely determines the dissipation process, and hence the dissipation vector $\frac{d\bar{\epsilon}}{dt}$.

An approximate description of the creep phenomenon is obtained, if the rate of dissipation is considered to be a function of the mechanical state variables in the discrete analysis, i.e. the vector σ or the vector ϵ'. Let the dissipation function be given by $g(\epsilon')$ or $f(\sigma)$. Then we have the following equalities

$$\sigma^{T} \frac{d\bar{\epsilon}}{dt} = f(\sigma) \, , \quad (\epsilon')^{T} S \frac{d\bar{\epsilon}}{dt} = g(\epsilon') . \tag{21}$$

According to our postulate the functions $f(\sigma)$ and $g(\epsilon')$ shall fix direction and magnitude of the vectors $\frac{d\bar{\epsilon}}{dt}$ and $S \frac{d\bar{\epsilon}}{dt}$ in the σ- and the ϵ'-vector space respectively. Since a scalar function of a vector can only determine a direction in the corresponding vector space by means of its derivatives with respect to the elements of this vector, we have

$$\frac{d\bar{\epsilon}}{dt} = \mu \frac{\partial f}{\partial \sigma} \, , \quad S \frac{d\bar{\epsilon}}{dt} = \lambda \frac{\partial g}{\partial \epsilon'} \tag{22}$$

These are vectors normal to a surface of constant rate of energy dissipation. The equalities (Equation 21) furnish finally the magnitude of the positive scalar factors μ and λ. The expressions for $\frac{d\bar{\epsilon}}{dt}$ then read

$$\frac{d\bar{\epsilon}}{dt} = \left[\sigma^{T} \frac{\partial f}{\partial \sigma} \right]^{-1} f(\sigma) \frac{\partial f}{\partial \sigma}$$

or

$$\frac{d\bar{\epsilon}}{dt} = S^{-1} \left[(\epsilon')^{T} \frac{\partial g}{\partial \epsilon'} \right]^{-1} g(\epsilon') \frac{\partial g}{\partial \epsilon'} \tag{23}$$

As soon as one of the dissipation functions $f(\sigma)$ or $g(\epsilon')$ is known, the creep deformation problem is fully determined. Before we shall consider some explicit forms for these dissipation functions we shall first discuss the phenomenon of time independent plastic deformation.

When we plot along a radius from the origin in the σ- or ϵ'-space the rate of energy dissipation, then necessarily a highly nonlinear increase must show up. From the fact that the stresses cannot exceed certain values, depending on the material, we may conclude that with increasing stresses the rate of energy dissipation tends to infinity (Figure 4). As the relation between rate of dissipation and stress is more strongly nonlinear, surfaces for equal difference in rate of dissipation will lie closer together in the σ- or ϵ'-space. In the limit a surface of indefinite rate of energy dissipation may be conceived, that separates the region of rate of dissipation equal to zero from the region of rate of dissipation equal to infinity. It is the yield surface of the so-called elastic-ideally plastic material. It can be defined by equations of the form

$$\phi\ (\sigma) - k^2 = 0 \qquad \text{or} \qquad \chi(\epsilon') - k^2 = 0 \tag{24}$$

For values of σ or ϵ' inside the yield surface the structural element is purely elastic. At the yield surface the direction of the vectors $\frac{d\bar{\epsilon}}{dt}$ and $S\ \frac{d\bar{\epsilon}}{dt}$ concides with the normal to the yield surface as a surface of indefinite rate of energy dissipation. Hence

$$\frac{d\bar{\epsilon}}{dt} = \mu \frac{\partial \phi}{\partial \sigma},\ S\ \frac{d\bar{\epsilon}}{\partial \epsilon}, = \lambda \frac{d\chi}{\partial \epsilon'} \tag{25}$$

The positive scalar factors λ and μ, which determine the magnitude of the rate of energy dissipation, depend on the rate of total deformation or follow from the rate of stress. In order that the state of stress does not leave the yield surface, which would imply $\frac{d\bar{\epsilon}}{dt} = 0$, the following conditions must be satisfied

$$\frac{d\phi}{dt} = \left(\frac{\partial \phi^T}{\partial \sigma}\right)\frac{d\sigma}{dt} = \left(\frac{\partial \phi^T}{\partial \sigma}\right) S \left(\frac{d\epsilon}{dt} - \frac{d\bar{\epsilon}}{dt}\right) = 0 \tag{26}$$

or

$$\frac{d\chi}{dt} = \left(\frac{\partial \chi^T}{\partial \epsilon'}\right)^T\left(\frac{d\epsilon}{dt} - \frac{d\bar{\epsilon}}{dt}\right) = \left(\frac{\partial \chi^T}{\partial \epsilon'}\right) S^{-1}\ \frac{d\sigma}{dt} = 0 \tag{27}$$

By substitution of expressions 25 into 26 and 27 respectively we are lead to

$$\mu = \frac{\left(\frac{\partial \phi^T}{\partial \sigma}\right) S\ \frac{d\epsilon}{dt}}{\left(\frac{\partial \phi^T}{\partial \sigma}\right) S\ \frac{\partial \phi}{\partial \sigma}} \tag{28}$$

and

$$\lambda = \frac{\left(\frac{\partial \chi^T}{\partial \epsilon'}\right)\ \frac{d\epsilon}{dt}}{\left(\frac{\partial \chi^T}{\partial \epsilon'}\right) S^{-1}\ \frac{\partial \chi}{\partial \epsilon'}} \tag{29}$$

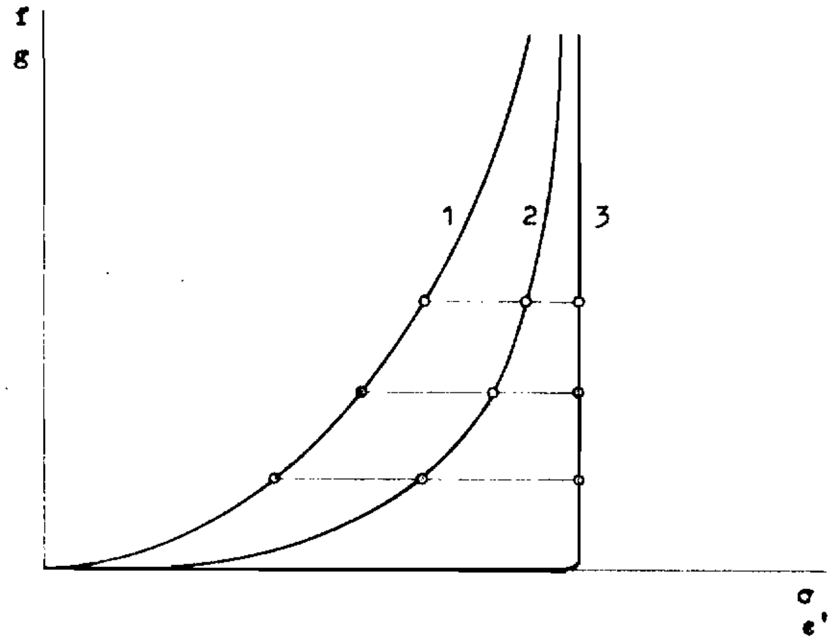

Figure 4. Rate of Energy Dissipation as a Function of Stress Intensity

By the relations and conditions derived above the phenomenon of time-independent plastic deformation is fully determined as soon as the yield surface is known.

A creep phenomenon that depends only on the mechanical state variables and the temperature may be denoted as ideal creep. We have seen that ideal plasticity can be considered as a limiting case of ideal creep. Now it is known from experiments that a real material approximates at most asymptotically to the ideal creep or to ideal plastic behaviour. Phenomena like primary creep and recovery, or hardening and Bauschinger effects escape the description considered here. The explanation for this discrepancy, if we retain the postulate concerning the dissipation function, may be sought in the indeterminacy in the description of the deformation and the ensuing drastic reduction of the number of state variables.

In the discrete analysis it is clear from the start that the state variables, contained in the vector σ, do not exhaustively describe all possible stress distributions in the structural elements. It may be shown that precisely the redundant stress distributions in these elements give rise to the type of deviations from the ideal creep and plastic behaviour as is found experimentally.

If the deformation or stress parameters in the discrete analysis refer to a homogeneous state of stress and deformation (as is for instance the case in prismatic tensile bars), then the degree of approximation in the discrete analysis is the same as in the continuous description. But also in the continuous description it may be argued that the stress tensor defines insufficiently the state of the complicated atomic structure surrounding a point, and indeed with the aid of the concept of hidden microstresses the deviations from the ideal creep and plastic behaviour can be explained, at least qualitatively (Reference 3).

Though the creep and plasticity relations derived above are admittedly of an approximate nature, they enable us nevertheless to carry out an overall analysis of the inelastic behaviour of a structure, particularly with respect to possible deformation patterns and the ultimate load for structures, fabricated from a ductile material.

We shall now consider the form that a dissipation function may assume for a particular structural element. For a beam element, loaded in bending by a shear force in a plane of symmetry, the generalized stresses are the bending moments at the outer ends (Figure 2). The bending moment in a point x is given by

$$M = M_1 + (M_2 - M_1)\xi \,, \quad \xi = \frac{x}{\ell}.$$

From creep theory it is known that the rate of energy dissipation in a point of the beam element depends mainly on the value of the bending moment, whilst the influence of the shear force may in general be neglected. If the dissipation process in a point of the beam is described by a process of ideal creep a possible form of the dissipation function for an element dx is given by

$$df = \frac{\gamma}{\ell}\left(\frac{M}{M_0}\right)^{2n} dx, \quad n = \text{positive integer}.$$

Here γ is a temperature dependent factor with the dimension of work per unit of item. For a beam element of uniform temperature the dissipation function then reads

$$f = \int_0^{\ell} df = \frac{\gamma}{2n+1} \; \frac{M_2^{2n+1} - M_1^{2n+1}}{M_0^{2n}(M_2 - M_1)}$$

According to (Equation 23) the rate of change of the inelastic bending modes is then given by

$$\frac{d\bar{\epsilon}_2}{dt} = \frac{\gamma}{2n+1} \; \frac{2nM_1^{2n+1} - (2n+1)M_1^{2n}M_2 + M_2^{2n+1}}{2nM_0^{2n}(M_2 - M_1)^2}$$

$$\frac{d\bar{\epsilon}_3}{dt} = \frac{\gamma}{2n+1} \; \frac{M_1^{2n+1} - (2n+1)M_1M_2^{2n} + 2nM_2^{2n+1}}{2nM_0^{2n}(M_2 - M_1)^2}$$

The yield surface for an elastic-ideally plastic beam element is obtained from the dissipation function given above if we let the exponent n approach infinity. The region of rate of dissipation equal to zero is then separated from the region with rate of dissipation equal to infinity by a surface defined by (Figure 5).

$$\left.\begin{matrix} M_1 - M_0 = 0 \\ M_1 + M_0 = 0 \end{matrix}\right\} -M_0 \le M_2 \le M_0 \,, \quad \left.\begin{matrix} M_2 - M_0 = 0 \\ M_2 + M_0 = 0 \end{matrix}\right\} -M_0 \le M_1 \le M_0$$

The yield function $\phi(M_1, M_2) - k^2$ is not a continuously differentiable function of the state variables M_1 and M_2. The region of rate of dissipation equal to zero in the $M_1 - M_2$ - plane is bounded by four continuously differentiable functions, ϕ_1 through ϕ_4. In the corners of the yield curve (p.e. $M_1 = M_2 = M_0$) two yield functions are simultaneously equal to zero and the direction of $\frac{d\bar{\epsilon}}{dt}$ in the $M_1 - M_2$ - plane is then situated in between the two normals to the yield curve (Figure 5).

$$\frac{d\bar{\epsilon}}{dt} = \mu_\alpha \frac{\partial \phi_\alpha}{\partial \sigma} + \mu_\beta \frac{\partial \phi_\beta}{\partial \sigma} \,, \quad \mu_\alpha \ge 0, \quad \mu_\beta \ge 0$$

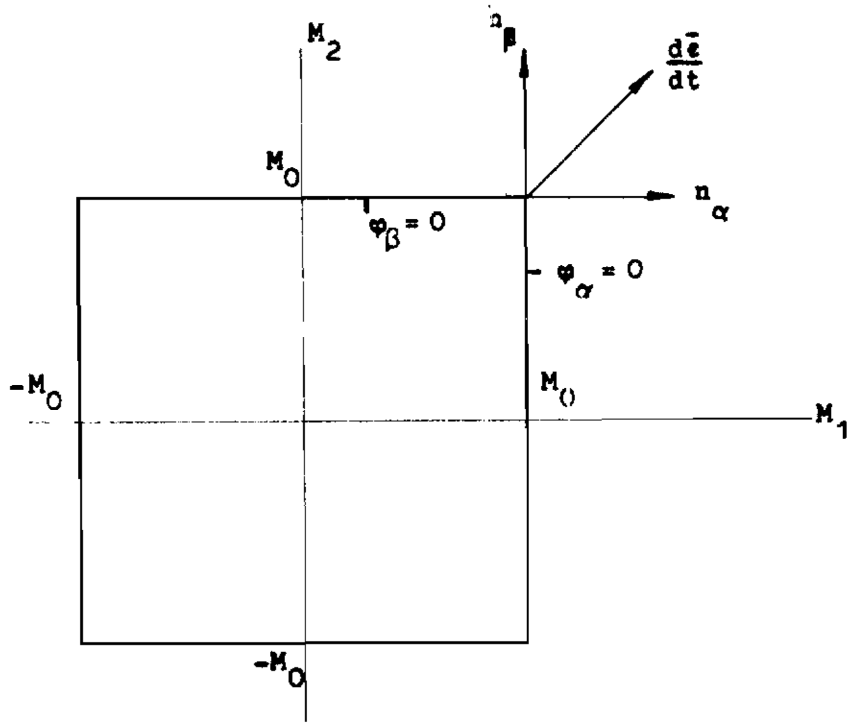

Figure 5. Yield Curve of Beam Element

The values of the scalar factors μ_α and μ_β follow from the conditions

$$\frac{d\phi_\alpha}{dt} = \left(\frac{d\phi_\alpha^T}{\partial\sigma}\right) S \left(\frac{d\epsilon}{dt} - \frac{d\bar{\epsilon}}{dt}\right) = 0$$

$$\frac{d\phi_\beta}{dt} = \left(\frac{\partial\phi_\beta^T}{\partial\sigma}\right) S \left(\frac{d\epsilon}{dt} - \frac{d\epsilon}{dt}\right) = 0$$

For the beam element under consideration the following simple expressions are found in the case $M_1 = M_2 = M_0$.

$$\frac{d\bar{\epsilon}_2}{dt} = \frac{d\epsilon_2}{dt} \quad \text{provided} \quad \frac{d\epsilon_2}{dt} \geq 0,$$

$$\frac{d\bar{\epsilon}_3}{dt} = \frac{d\epsilon_3}{dt} \quad \text{provided} \quad \frac{d\epsilon_3}{dt} \geq 0.$$

These results agree with the concept of plastic hinges in limit load analysis of beam structures.

Next we consider the dissipation function for a tubular element. We have seen that the elastic potential

$$E = \frac{1}{2} (\epsilon')^T S \epsilon'$$

is the sum of the distortional energy, defined by a stiffness matrix $\mathbf{S}_D$,

$$E_D = \frac{1}{2} (\epsilon')^T \mathbf{S}_D \, \epsilon' \, ,$$

and the volume strain energy, defined by a stiffness matrix $\mathbf{S}_V$,

$$E_V = \frac{1}{2} (\epsilon')^T \mathbf{S}_V \, \epsilon' \, .$$

The matrices $\mathbf{S}_D$ and $\mathbf{S}_V$ are formed from the matrix $\mathbf{S}$ by the terms with the shear modulus G and by the terms with the bulk modulus C respectively. Both matrices are singular; only their sum $\mathbf{S} = \mathbf{S}_D + \mathbf{S}_V$ has an inverse.

As is known from experiments change of volume is always a thermo-elastic phenomenon, and hydrostatic pressure does not influence the inelastic phenomena in metals. In an isotropic material the rate of energy dissipation may therefore be a function of the distortional elastic energy and the temperature. A possible dissipation function for a tubular element is then given by

$$g = \gamma \left(\frac{(\epsilon')^T \mathbf{S}_D \, \epsilon'}{k^2} \right)^n , \quad n = \text{positive integer.}$$

Again γ represents a temperature dependent factor. According to relations shown in (Equation 23) this dissipation function gives for ideal creep of a tubular element

$$\frac{d\bar{\epsilon}}{dt} = \frac{\gamma}{k^2} \, \mathbf{S}^{-1} \, \mathbf{S}_D \, \epsilon \, \left(\frac{(\epsilon')^T \mathbf{S}_D \, \epsilon'}{k^2} \right)^{n-1}$$

Time independent plastic behaviour is again obtained in the limiting case $n \to \infty$. Plastic behaviour of a tubular element is then characterized by a yield surface in ϵ'-space, given by

$$\chi = (\epsilon')^T \mathbf{S}_D \, \epsilon - k^2 = 0$$

This is the analogy in finite element analysis of the Von Mises criterion in the continuous field theory of plasticity. Instead of a bound on the local value of the distortional energy, now an upper bound, the "elastic limit" is assigned to this energy, integrated over a small, but finite structural element. The expression for the rate of change of the plastic deformation vector $\bar{\epsilon}$, as derived from Equations 25 and 29 reads

$$\frac{d\bar{\epsilon}}{dt} = \frac{\mathbf{S}^{-1} \mathbf{S}_D \, \epsilon' \, (\epsilon')^T \, \mathbf{S}_D \, \frac{d\epsilon}{dt}}{(\epsilon')^T \, \mathbf{S}_D \, \mathbf{S}^{-1} \, \mathbf{S}_D \, \epsilon'} \, , \quad \text{provided} \begin{cases} (\epsilon')^T \mathbf{S}_D \frac{d\epsilon}{dt} \geq 0 , \\ (\epsilon')^T \mathbf{S}_D \, \epsilon' - k^2 = 0 \end{cases}$$

otherwise $\frac{d\bar{\epsilon}}{dt} = 0$.

In Reference 1 a slightly different expression was given for the dissipation vector, which however was not in agreement with the creep and plasticity postulate, put forward in this paper.

To which degree of accuracy the relations presented here give a description of the inelastic behaviour of structural elements depends on the proper choice of these elements and their dissipation function. Lack of experience in this matter may be compensated by taking smaller elements and consequently accepting a larger number of unknowns.

STRUCTURAL ANALYSIS OF CREEP AND PLASTICITY PROBLEMS

So far only creep and plasticity relations have been given for one single structural element. For the k^{th} element out of a structure, composed of N elements, all quantities pertaining to this element have to be labeled with a superscript k.

The Expressions 23 for the rate of change of the inelastic components of the generalized strains ϵ^k may be summarized by a vector

$$\frac{d\epsilon^{-k}}{dt} = c^k , \tag{30}$$

where the components of the creep vector c^k are in general nonlinear expressions in the state variables ϵ'^k or σ^k . The creep vector $\mathbf{c}$ for the structure as a whole is composed of the creep vectors for all individual structural elements as follows

$$\mathbf{c} = \left\{ c^1 , c^2 , c^3 , \ldots c^N \right\} \quad \frac{d\bar{\epsilon}}{dt} = \mathbf{c} \tag{31}$$

The relations for time-independent plasticity, Equation 25, with 28 and 29, may be written in the incremental form

$$\Delta\epsilon^{-k} = Y^k \quad \Delta\epsilon^k \tag{32}$$

The elements of the "yield matrix" Y^k are again general non-linear expressions in the state variables, ϵ'^k or σ^k. If in q elements from a structure, composed of N elements, the following plasticity conditions are satisfied

$$\chi(\epsilon'^k) - k^2 = 0, \quad \left(\frac{\partial \chi^T}{\partial \epsilon'}\right) \Delta\epsilon^k \geq 0$$

or

$$\phi(\sigma^k) - k^2 = 0, \quad \left(\frac{\partial \phi^T}{\partial \sigma^k}\right) S^k \quad \Delta\epsilon^k \geq 0 , \tag{33}$$

then the yield matrix for the structure as a whole is composed of the yield matrices for the q individual elements.

$$Y = \left[\begin{array}{cc|c} Y^1 & 0 & 0 \\ 0 & Y^2 & 0 \\ \hline 0 & 0 & Y^N \end{array}\right] , \Delta\epsilon^{-y} = Y \quad \Delta\epsilon^y \tag{34}$$

If the plasticity problem is analysed in terms of the stresses, then the calculations are governed by the Conditions 26 or 27, which for the k^{th} element may be written in the form

$$(p^k)^T \Delta\sigma^k = 0 \tag{35}$$

provided the yield Conditions 33 are satisfied. For q elements yielding the "plasticity vectors" p^k for these elements may be collected in the plasticity matrix P for the structure as a whole.

$$\mathbf{P} = \left[\begin{array}{cc|c} p^1 & 0 & 0 \\ 0 & p^2 & 0 \\ \hline 0 & 0 & p^q \end{array}\right], \quad \mathbf{P}^T \Delta \boldsymbol{\sigma}^y = \mathbf{0} \tag{35}$$

It may be added that in a singular point of the yield surface of a structural element in general more than one condition will apply to the generalized stress-increments. The number of conditions will namely equal the number of yield functions, for which the normal to the yield surface has a non-negative projection on the increments of the inelastic generalized strains, $\Delta\bar{\boldsymbol{\epsilon}}$. Thus for the beam element of Figure 1 in the point $M_1 = M_2 = M_0$ of the yield curve (Figure 5) two conditions have to be satisfied

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta M_1 \\ \Delta M_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ provided } \begin{cases} \Delta\bar{\epsilon}_2 \geq 0 \\ \Delta\bar{\epsilon}_3 \geq 0 \end{cases}$$

We now come to the analysis of structures subject to creep and plasticity. In the presence of inelastic strains the principle of minimum potential energy still applies, provided the inelastic strains themselves are given and are treated as fixed quantities in the variational process. So it can be stated that

$$P = \frac{1}{2}(\boldsymbol{\epsilon}')^T \mathbf{S} \boldsymbol{\epsilon}' - \left[\mathbf{u}^T (\mathbf{u}^0)^T\right] \begin{bmatrix} \mathbf{L}^T \\ (\mathbf{L}^0)^T \end{bmatrix} \mathbf{f}^0 = \text{minimum} \tag{37}$$

with respect to variations of $\mathbf{u}$, subject to the subsidiary conditions

$$\boldsymbol{\epsilon}' = \mathbf{D}\left[\mathbf{L} \ \mathbf{L}^0\right] \begin{bmatrix} \mathbf{u} \\ \mathbf{u}^0 \end{bmatrix} - \bar{\boldsymbol{\epsilon}} \tag{38}$$

Application of this variational principle at times t and $t + \Delta t$ furnishes the following Equations

$$\mathbf{L}^T \mathbf{D}^T \mathbf{S} \mathbf{D} \mathbf{L} \mathbf{u} = -\mathbf{L}^T \mathbf{D}^T \mathbf{S}(\mathbf{D} \mathbf{L}^0 \mathbf{u}^0 - \bar{\boldsymbol{\epsilon}}) + \mathbf{L}^T \mathbf{f}^0, \tag{39}$$

and

$$\mathbf{L}^T \mathbf{D}^T \mathbf{S} \mathbf{D} \mathbf{L}(\mathbf{u} + \Delta\mathbf{u}) = -\mathbf{L}^T \mathbf{D}^T \mathbf{S}(\mathbf{D}\mathbf{L}^0 (\mathbf{u}^0 + \Delta\mathbf{u}^0) - (\bar{\boldsymbol{\epsilon}} + \Delta\bar{\boldsymbol{\epsilon}})) + \mathbf{L}^T (\mathbf{f}^0 + \Delta \mathbf{f}^0). \tag{39}$$

We consider a state of deformation, in which Equations 39 are satisfied. The increments $\Delta\mathbf{u}$ must consequently obey the equations

$$\mathbf{L}^T \mathbf{D}^T \mathbf{S} \mathbf{D} \mathbf{L} \Delta\mathbf{u} = -\mathbf{L}^T \mathbf{D}^T \mathbf{S}\left[\mathbf{D} \mathbf{L}^0 \Delta\mathbf{u}^0 - \Delta\bar{\boldsymbol{\epsilon}}\right] + \mathbf{L}^T \Delta \mathbf{f}^0 \tag{40}$$

In the case of creep, Equation 41 can be written as differential equations for $\mathbf{u}$ with the aid of the creep vector $\mathbf{c}$.

$$L^T D^T S D L \frac{du}{dt} = - L^T D^T S \left[D L^o \frac{du^o}{dt} - c \right] + L^T \frac{df^o}{dt} \tag{42}$$

Substitution into

$$\frac{d\epsilon'}{dt} = \frac{d\epsilon}{dt} - \frac{d\bar{\epsilon}}{dt} = D \left[L \; L^o \right] \begin{bmatrix} \frac{du}{dt} \\ \frac{du^o}{dt} \end{bmatrix} - c \tag{43}$$

leads to a system of ordinary differential equations for the state variables ϵ' or σ

$$\frac{d\epsilon'}{dt} + c + D L \left[L^T D^T S D L \right]^{-1} L^T D^T S \; c =$$

$$= \left[I - D L \left[L^T D^T S D L \right]^{-1} L^T D^T S \right] D L^o \frac{du^o}{dt} + \left[L^T D^T S D L \right]^{-1} L^T \frac{df^o}{dt} \tag{44}$$

These equations can be integrated for given initial conditions and given $u^o = u^o(t)$ and $f^o(t)$. The displacements follow by simultaneous integration of Equations 42. Since the equations are nonlinear the integration has to be carried out numerically.

If the mechanical behaviour of the structural elements is approximated by the elastic-ideally plastic model, and if in q elements the plasticity conditions are satisfied, Equations 41 may with the aid of Expressions 34, be written in the form

$$L^T D^T S D L \; \Delta u = - L^T D^T S D L^o \; \Delta u^o + (L^y)^T (D^y)^T S^y \, Y \, D^y \left[L^y \; L^{yo} \right] \begin{bmatrix} \Delta u \\ \Delta u^o \end{bmatrix} + L^T \Delta f^o$$

After multiplication with the matrix $[L^T D^T S D L]^{-1}$, supposed to be known, we have

$$I - \left[L^T D^T S D L \right]^{-1} (L^y)^T (D^y)^T S^y \, Y \, D^y L^y \Big] \Delta u = \Delta u^e$$

$$+ \left[L^T D^T S D L \right]^{-1} (L^y)^T (D^y)^T S^y Y D^y L^{yo} \Delta u^o, \tag{45}$$

where Δu^e denotes the displacement increments, that would occur in the case of purely elastic behaviour.

$$\Delta u^e = - \left[L^T D^T S D L \right]^{-1} \left[L^T D^T S D L^o \;\; \Delta u^o - L^T \Delta f^o \right] \tag{46}$$

The elements of the vector Δu can be arranged such, that the location matrix L^y falls apart into a null matrix and a matrix, that refers only to the plastically deforming elements in the structure. Then it may be written

$$(L^y)(D^y) S^y Y D^y L^y = \begin{bmatrix} 0 & 0 \\ 0 & (L^{yy})^T (D^y)^T S^y \, Y \, D^y L^{yy} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}$$

If $[L^T D^T S T L]^{-1}$ is denoted by P. it follows

$$\left[I - \left[L^T D^T S\, D\, L\right] (L^y)^T (D^y)^T S^y Y\, D^y L^y\right]^{-1} = \begin{bmatrix} I & P^{12} Q \left[I - P^{22} Q\right]^{-1} \\ O & \left[I - P^{22} Q\right]^{-1} \end{bmatrix}, \quad P = \begin{bmatrix} P^{11} & P^{12} \\ P^{21} & P^{22} \end{bmatrix}.$$

Apparently for the calculation of the displacement increments Δu from Equation 45, besides the vector Δu^e only those columns of the matrix $[L^T D^T S\, D L]^{-1}$ are required, that refer to the degrees of freedom of the plastically deforming elements in the structure, whilst the number of these degrees of freedom determines the order of the matrix to be inverted for the matrix to be inverted for the solution of Δ ..

An alternative method of calculation for the plastic behaviour of structures bears a close resemblance to the modification technique, introduced by Argyris with the aid of fictitious initial stresses (Reference 4). Here a more formal and less intuitive approach will be followed.

For Equations 45 we try a solution of the form

$$\Delta u = \Delta u^e + \left[L^T D^T S\, D\, L\right]^{-1} (L^y)^T (D^y)^T X\; D^y \left[L^y \quad L^{yo}\right] \begin{bmatrix} \Delta u^e \\ \Delta u^o \end{bmatrix} \tag{47}$$

where X represents some unknown square auxiliary matrix. Substitution of Equation 47 into 45 should now furnish an identity

$$\left[L^T D^T S\, D\, L\right]^{-1} (L^y)^T (D^y)^T \left[I - S^y\, Y\, D^y\, L^y \left[L^T D^T S\, D\, L\right]^{-1} (L^y)^T (D^y)^T\right] X\, D^y \left[L^y \quad L^{yo}\right] \begin{bmatrix} \Delta u^e \\ \Delta u^o \end{bmatrix}$$

$$= \left[L^T D^T S\, D\, L\right]^{-1} (L^y)^T (D^y)^T S^y Y\; D^y \left[L^y \quad L^{yo}\right] \begin{bmatrix} \Delta u^e \\ \Delta u^o \end{bmatrix}$$

or

$$X = \left[I - S^y\, Y\, D^y\, L^y \left[L^T D^T S\, D\, L\right]^{-1} (L^y)^T (D^y)^T\right]^{-1} S^y\; Y \tag{48}$$

The order of the matrix that is to be inverted, is now determined by the number of deformation modes of the plastically deforming elements.

Equations 45 and 46, and Equations 47 and 48 enable us to analyse growing plastic deformation in a structure by a step by step numerical procudure. After each step the plasticity conditions for the structural elements have to be checked. At the collapse load the matrix to be inverted, both in Equations 45 and 48, become singular.

In the presence of inelastic strains of a given magnitude the principle of minimum complementary energy assumes the form

$$C = \frac{1}{2}\, \sigma^T S^{-1}\, \sigma - \sigma^T \left[D\, L^o\, u^o - \bar{\epsilon}\right] = \text{minimum} \tag{49}$$

with respect to variations of $\boldsymbol{\sigma}$, obeying the subsidiary conditions

$$L^T D^T \boldsymbol{\sigma} = L^T f^o . \tag{50}$$

The equilibrium equations 49 are in general n equations for the p unknown elements of $\boldsymbol{\sigma}$, where $n \leq p$. Since the deformation modes, defined by Equation 38, are linearly independent, the rank of the matrix DL is equal to n and any solution of Equations 50 may be written in the form

$$\boldsymbol{\sigma} = B^o \ddot{L} f^o + B\bar{\bar{\boldsymbol{\sigma}}} , \tag{51}$$

where $\bar{\bar{\sigma}}$ is the vector of redundant generalized stresses. If we minimize Expression 49 with respect to $\bar{\bar{\sigma}}$ at times t and $t + \Delta t$, we are lead to the following equations

$$B^T S^{-1} B\bar{\bar{\boldsymbol{\sigma}}} = - B^T \left[S^{-1} B^o L^T f^o + \bar{\boldsymbol{\epsilon}} - D L^o u^o \right] \tag{52}$$

and

$$B^T S^{-1} B\left(\bar{\bar{\sigma}} + \Delta\bar{\bar{\sigma}}\right) = -B^T\left[S^{-1} B^o L^T\left(f^o + \Delta f^o\right) + \bar{\boldsymbol{\epsilon}} + \Delta\bar{\boldsymbol{\epsilon}} - D L^o\left(u^o + \Delta u^o\right)\right] . \tag{53}$$

When we start out from a state of deformation, in which Equations 52 are satisfied, the increments $\Delta\bar{\bar{\boldsymbol{\sigma}}}$ must apparently obey the equations

$$B^T S^{-1} B \, \Delta\bar{\bar{\sigma}} = - B^T \left[S^{-1} B^o L^T \Delta f^o + \Delta\bar{\boldsymbol{\epsilon}} - D L^o \Delta u^o \right] . \tag{54}$$

In the case of creep these equations can be written as differential equations and lead together with Equations 31 and 51 to a set of differential equations for the state variables σ or ϵ'.

$$\frac{d\boldsymbol{\sigma}}{dt} + B\left[B^T S^{-1} B\right]^{-1} B^T c = \left[I - B\left[B^T S^{-1} B\right]^{-1} B^T S^{-1}\right] B^o L^T \frac{df^o}{dt} + B\left[B^T S^{-1} B\right]^{-1} D L^o \frac{du}{dt} . \tag{55}$$

For given initial conditions and given $f^o = f^o(t)$ and $u^o = u^o(t)$ these equations can be integrated numerically. With the aid of the principle of virtual work, or equivalently by the unit force method, it can be shown that the displacement vector $\mathbf{u}$ at any instant follows by simultaneous integration of the equations

$$\frac{du}{dt} = \left(B^o\right)^T\left[I - S^{-1} B\left[B^T S^{-1} B\right]^{-1} B^T\right]\left[S^{-1} B^o L^T \frac{df^o}{dt} + c - D L^o \frac{du^o}{dt}\right] \tag{56}$$

In the case of the elastic-ideally plastic model of a structural element the increments $\Delta\bar{\boldsymbol{\epsilon}}$ cannot be expressed in terms of the generalized stress increments $\Delta\boldsymbol{\sigma}$. Therefore Equations 54 cannot be transposed directly into equations for the unknown increments of the redundant generalized stresses, $\Delta\bar{\bar{\sigma}}$. We can observe, however, that Condition 35 implies according to the Relations 25, 26 and 27, that for each plastically deforming structural element must hold

$$\left(\Delta\bar{\boldsymbol{\epsilon}}^k\right)^T \Delta\boldsymbol{\sigma}^k = 0 \tag{57}$$

Now two methods of calculation can be given.

In the first place in the application of the minimum principle (Equation 49) the vector $\Delta\bar{\epsilon}$ can be eliminated with the aid of Conditions 36 and 57. Condition 36 may be written in the form

$$P^T \Delta \sigma^y = P^T (B^{yo})^T L^T \Delta f^o + P^T B^y \Delta \bar{\bar{\sigma}} = 0 \qquad (58)$$

In order that a solution for $\Delta\bar{\bar{\sigma}}$ exists the matrix $P^T B^y$ must contain a square non-singular matrix $P^T \bar{B}^y$ and a rest matrix $P^T \bar{\bar{B}}^y$. If the number of redundant stress parameters $\bar{\bar{\sigma}}$ is equal to p-n and if the matrix $P^T B^y$ is of order (rx(p-n)), then the vector $\Delta\bar{\bar{\sigma}}$ can be expressed in terms of a vector $\Delta\bar{\bar{\sigma}}^*$ with p-n-r redundant generalized stresses.

$$\Delta\bar{\bar{\sigma}} = R^o L^T \Delta f^o + R \Delta \bar{\bar{\sigma}}^* , \qquad (59)$$

where

$$R^o = \begin{bmatrix} -[P^T \bar{B}^y]^{-1} P^T B^{yo} \\ 0 \end{bmatrix}, \quad R = \begin{bmatrix} -[P^T B^y]^{-1} P^T \bar{\bar{B}}^y \\ I \end{bmatrix}$$

In a state of stress, satisfying Equations 52, the following expression has to be minimized with respect to variations of $\Delta\bar{\bar{\sigma}}^*$

$$\frac{1}{2} (\sigma^T + \Delta\sigma^T) S^{-1} (\sigma + \Delta\sigma) - (\sigma^T + \Delta\sigma^T) [DL^o(u^o + \Delta u^o) - \bar{\epsilon} - \Delta\bar{\epsilon}]$$

after substitution of

$$\Delta\sigma = [B^o + BR^o] L^T \Delta f^o + BR \Delta\bar{\bar{\sigma}}^* , \qquad (60)$$

$$\Delta\sigma^T \Delta\bar{\epsilon} = 0 .$$

This variational process furnishes the following equation:

$$R^T B^T S^{-1} B^T S^{-1} BR \Delta\bar{\bar{\sigma}}^* = - R^T B^T \left[S^{-1} [B^o + BR^o] L^T \Delta f^o - DL^o \Delta u^o \right] -$$
$$- R^T [B^T S^{-1} B\bar{\bar{\sigma}} + B^T S^{-1} B^o L^T f^o - B^T DL^o u^o + B^T \bar{\epsilon}] .$$

Since the Equation 52 are supposed to be satisfied, the increments $\Delta\bar{\bar{\sigma}}^*$ in Equation 60 are determined by the equations

$$R^T B^T S^{-1} BR \Delta\bar{\bar{\sigma}}^* = - R^T B^T \left[S^{-1} B^o + BR^o L^T \Delta f^o - DL^o u^o \right] \qquad (61)$$

In this first method of stress analysis in the plastic range the order of the matrix to be inverted is equal to the number of redundant generalized stresses, diminished with the number of columns r of the plasticity matrix P, i.e. the number of plasticity conditions for the strincrements. The collapse load of the structure is reached as soon as the rank of the matrix $P^T B^y$ becomes less than r.

In the second method for the calculation of the stress increments we observe that, by virtue of Equations 54, for the plastically deforming elements holds

$$\Delta \sigma^y = \Delta \sigma^{ye} - B^y [B^T S^{-1} B]^{-1} (B^y)^T \Delta \bar{\epsilon} \quad , \tag{62}$$

where $\Delta \sigma^{ye}$ are the stress increments, that would occur if all structural elements behaved purely elastically.

$$\Delta \sigma^{ye} = B^{yo} L^T \Delta f^o - B^y (B^T S^{-1} B)^{-1} B^T \left(S^{-1} B^o (L^o)^T \Delta f^o - D L^o \Delta u^o \right) \quad . \tag{63}$$

Application of the plasticity Conditions 36 to Expressions 62 after substitution of

$$\Delta \bar{\epsilon} = P X P^T \Delta \sigma^{ye} \tag{64}$$

leads to equations, from which the auxiliary matrix X can be solved:

$$P^T \Delta \sigma^y = P^T \Delta \sigma^{ye} - P^T B^y (B^T S^{-1} B)^{-1} (B^y)^T P X P^T \Delta \sigma^{ye} = 0 \ ,$$

or

$$X = [P^T B^y [B^T S^{-1} B]^{-1} (B^y)^T P]^{-1} \quad . \tag{65}$$

The solution for the stress increments then reads

$$\Delta \sigma = (B^o - B (B^T S^{-1} B)^{-1} (B^y)^T P X P^T B^{yo}) L^T \Delta f^o - (B - B (B^T S B)^{-1} (B^y)^T P X P^T B^y)$$

$$(B^T S^{-1} B)^{-1} B^T \ (S^{-1} B^o L^T \Delta f^o - D L^o \Delta u^o) \ . \tag{66}$$

Collapse of the structure takes place at the load for which the matrix $P^T B^y (B^T S^{-1} B)^{-1} (B^y)^T P$ becomes singular. The order of this matrix, which has to be inverted for a solution, is equal to the number of generalized stresses in the plastically deforming elements.

CONCLUDING REMARKS

For the analysis of creep and plasticity problems various sets of equations have been given. In the case of a structure, composed of elements subject to creep, the deformation problem is solved by numerical integration of a system of nonlinear, first order differential equations. Plastic deformation, on the other hand, gives raise to a step by step modification of the elastic solution of the deformation problem, that is closely related to the modification technique of Argyris.

At the moment there is still little experience with the various sets of equations. The author so far has applied successfully Equations 42 and 43 for the solution of the creep problem of Figure 3, Equations 45 and 46 to the corresponding plasticity problem, and finally Equations 60 and 61 for the elastic-plastic analysis of portal frames. The rapid development of the computation technique makes it possible, however, to tackle more ambitious projects, like the elastic-plastic analysis of the deformation problem of a railhead under the action of a rolling load, that is now being carried out.

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