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STRUCTURAL INFLUENCE COEFFICIENTS FOR A REDUNDANT SYSTEM INCLUDING BEAM-COLUMN EFFECTS*

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A Force Method is presented for the solution of the structural influence coefficients (SICs), internal load distribution, and buckling characteristics of a redundant elastic beam system including beam-column effects. The formulation is general to the extent of considering each structural element to have both variable bending and shear flexibilities; the extension to include torsion and axial flexibilities is indicated. An iterative procedure is developed for the analysis of the most general redundant case in which beam-column effects cause interaction between the external loading and the redundant internal reactions. The method of solution is illustrated by three examples. The first two examples deal with a doubly redundant two member frame under two types of loading: a panel point loading, and a transverse loading at an intermediate point on one of its members. The first example leads to a straight forward redundant beam-column problem since the axial loading of the two members is known at the outset. The second example leads to a non-linear beam-column problem requiring the iterative solution and provides an illustration of the general procedure. The third example is a case of non-conservative loading, a clamped column tangentially loaded at its free end. The derivation of the static SICs is shown, and the non-conservative buckling load is determined by utilizing the unsymmetrical SICs in vibration analyses up to the point of frequency coalescence.

NOMENCLATURE

A	Element of matrix defined in Equation 31
A_v	Effective cross-section area for transverse shear stress
a	Element of structural influence coefficient matrix
a_r	Element of redundant structural flexibility matrix without beam-column effects
a_s	Element of statically determinate flexibility matrix without beam-column effects
B	Element of matrix defined in Equation 32

*This paper is based on a computer program developed at the Aerospace Corporation by the author and his colleagues Edith F. Farkas, Gerard L. Commerford and Heather A. Malcom. The derivation of equations, example problems, and programming logic and listing are given in Reference 1. The present paper outlines the derivation of the equations and illustrates some wider applications of the method.

**Research and Development Scientist

b_r	Element of matrix of beam-column effects of redundant structure
b_s	Element of matrix of beam-column effects of statically determinate structure
C	Element of matrix defined in Equation 33
c	Scale factor for beam-column axial loading
E	Young's modulus of elasticity
G	Modulus of rigidity
I	Effective cross-section moment of inertia in bending
J	Number of external loads inducing beam-column effects
K_b	Bending flexibility constant
K_v	Transverse shear flexibility constant
k	Coefficient of transverse shear stress at neutral axis
ℓ	Length of structural element
M	Bending moment
m	Bending moment caused by virtual loading
N	Number of structural elements
P	External load
p	Virtual external load
Q	External load that induces beam-column effects
R	Number of redundants
V	Transverse shear
v	Transverse shear caused by virtual loading
X	Redundant internal reaction
x	Redundant reaction caused by virtual loading; axial coordinate of structural element
y	Deflection along structural element
δ	Deflection of structural panel or control points at which virtual loads are applied
λ	Buckling eigenvalue

Subscripts

cr	Critical
f	Final
i	Inboard
j	Dummy subscript denoting number of external loads inducing beam-column effects
n	Iteration number; dummy subscript denoting structural element number
o	Outboard
p	Reaction in cut structure to virtual external loading
r	Dummy subscript denoting redundant number
x	Reaction in cut structure to virtual internal redundant loading

Matrix Notation*

[ ]	Square matrix
$[\ ]^T$	Transposed matrix
$[\ ]^{-1}$	Inverse of matrix
{ }	Column matrix
⌊ ⌋	Row matrix
⌈ ⌋	Diagonal matrix
⌈I⌋	Unit diagonal matrix

Note: Additional symbols are defined as needed in the example problems.

INTRODUCTION

For many heavily loaded structural configurations, a significant increase in transverse flexibility may result from beam-column effects. This in turn will affect the transverse vibration characteristics that can be determined by collocation (i.e., lumped parameter) methods using static structural influence coefficients (SICs). This paper, therefore, develops the SICs and related data on internal load distributions for the general problem of a built-up redundant structure with variable bending and shear flexibilities, subjected to sufficiently large external loading that beam-column effects must be accounted for. The buckling characteristics of the structure are also determined. The effects of temperature are considered to the extent that they determine the material properties of each structural element; no thermally induced stresses are considered.

*Ed. Because of the unique symbolisms in this paper bold face type is not used to represent matrices

The present development is a further extension of the method of Ogness (Reference 2). Earlier extensions have included redundant box beams with varying depth, spar cap areas, and shear web thicknesses (Reference 3), the addition of thermal loads (Reference 4), and statically determinate beams with beam-column effects (Reference 5). The method is classified as a Force Method according to Pestel and Leckie (Reference 6). The method of Ogness was based on an application of Castigliano's First Theorem. However, the First Theorem cannot be applied directly to the beam-column problem,* although it can be modified appropriately as shown by Gallagher and Padlog (Reference 7). Rather than utilize the modified First Theorem in the present derivation we turn to the method of virtual work.

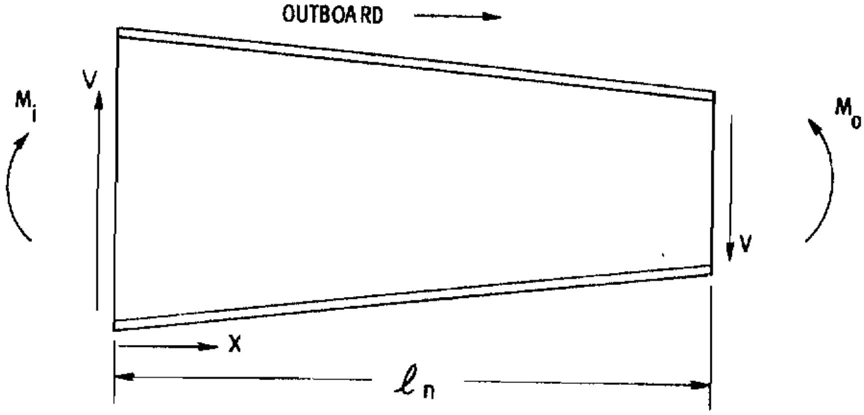

Figure 1. Geometry and Loading of nth Structural Element

GENERAL EQUATIONS AND SOLUTION OF THE LINEAR CASE

Consider a redundant structure composed of N structural elements having bending and shear flexibilities, and assume for each element that the moment varies linearly from its inboard value M_i to its outboard value M_o and that the shear V is constant along the length of the element**. The geometry and loading of a typical element are shown in Figure 1. The internal reactions in a given R-degree redundant elastic structure to a system of external concentrated loads can be expressed as linear combinations of the known external loads and a set of arbitrarily chosen internal redundant loads equal in number to the degree of redundancy. In addition, the internal reactions induced by beam-column effects can be expressed as linear combinations of the deflections of the points of application of the external loads. The basic conditions that the external loads, the internal redundants, and the deflections must satisfy are the conditions of static equilibrium. By imagining the structure to be "cut" in R places such that it is capable of sustaining loads only in a statically determinate manner, and by introducing each of the external loads, each of the redundants at its corresponding cut, and each of the deflections in turn, one can determine the reactions of each structural element by applying only the conditions of static equilibrium. The combined effects of the external loads, the internal redundants, and the deflections then define the total internal reactions of each

*The derivation of Reference 5 therefore is incorrect even though its end results are correct as will be shown. The reason for the rather fortuitous results is evident in the equation for virtual work, Equation (28).

**Reference 1 also considers torsion and axial flexibilities and assumes not only that the shear is constant along each element but also that the torque and axial force are constant. Accordingly, the shear terms presented here may be regarded as typical of the three loadings and flexibilities. The limitation here to bending and shear flexibilities is only for the sake of brevity.

structural element. If we denote the set of external loads* by $\{P\}$, the deflections of each of their points of application parallel to their lines of action by $\{\delta\}$, and the set of redundants by $\{X\}$, the total internal reactions of each structural element may be summarized in the following matrix expressions.

$$\{M_i\} = [M_i/P]\{P\} + [M_i/X]\{X\} + [M_i/\delta]\{\delta\} \tag{1}$$

$$\{M_o\} = [M_o/P]\{P\} + [M_o/X]\{X\} + [M_o/\delta]\{\delta\} \tag{2}$$

$$\{V\} = [V/P]\{P\} + [V/X]\{X\} + [V/\delta]\{\delta\} \tag{3}$$

The elements of the various coefficient matrices are determined by principles of statics; that is, the first element of the matrix $[M_i/P]$ is the coefficient (a length if the load is a force) that specifies the moment on the inboard end of the first structural element due to the first external load, the first element of the matrix $[M_i/X]$ is the coefficient that specifies the moment on the inboard end of the first structural element due to the first redundant, and the first element of the matrix $[M_i/\delta]$ is the coefficient that specifies the moment on the inboard end of the first structural element induced through beam-column effects by the first displacement. The calculation of the elements of the matrices of $\{P\}$ and $\{X\}$ has been discussed in References 2 to 4. The calculation of the elements of the matrices of coefficients of $\{\delta\}$ has been discussed in Reference 5 for the case of a statically determinate beam. A general discussion with illustrations by several examples of the calculation of the various matrix elements will be given in a later section. However, the relationship between the coefficients of $\{\delta\}$ and the external loading warrants some additional remarks at this point. It is necessary to make a distinction between two types of external loading: We distinguish between the set $\{P\}$ that causes the deflections $\{\delta\}$ and another set of external loads $\{Q\}$ that introduces the beam-column effects. The two sets may be independent or they may be identical; the relationship in any instance is determined by the structural configuration. The beam-column effects, i.e., the coefficients of $\{\delta\}$ in Equations 1 to 3, depend on the distributed reactions (including redundants) to $\{Q\}$ throughout the structure. Therefore, in general for a redundant structure, each coefficient of $\{\delta\}$ is expressed as a linear combination of the effects of each load Q acting upon the statically determinate structure and each redundant reaction X. The necessary linear combinations of matrices may be indicated as follows.

$$[M_i/\delta] = \sum_{j=1}^{J} Q_j\,[M_i/\delta Q_j] + \sum_{r=1}^{R} X_r\,[M_i/\delta X_r] \tag{4}$$

$$[M_o/\delta] = \sum_{j=1}^{J} Q_j\,[M_o/\delta Q_j] + \sum_{r=1}^{R} X_r\,[M_o/\delta X_r] \tag{5}$$

$$[V/\delta] = \sum_{j=1}^{J} Q_j\,[V/\delta Q_j] + \sum_{r=1}^{R} X_r\,[V/\delta X_r] \tag{6}$$

where J is the number of loads Q, and R is the number of redundants. It is apparent from Equations 1 to 6 that the most general problem in which the redundants and the deflections are

*The general nature of the external loads should be emphasized. If a load P is a force, the consequent deflection δ is a translation, whereas if P is a moment, the deflection δ becomes a rotation. The redundants may also be either forces or moments, but of course, no relative deflection results from their action.

interdependent is nonlinear and requires an approximate solution. However, for a wide class of problems, either there is no interaction between the redundants and the deflections or, at least, any interaction may be neglected. Then the first terms of Equations 4 to 6 are sufficient to account for beam-column effects, and the determination of redundants and deflections becomes a linear problem. We consider the solution of this linear problem in the present section, and then consider modifications of the linear solution to obtain an iterative solution to the nonlinear problem in the next section.

We begin by considering a virtual load (or unit dummy load) applied to the statically determinate structure at the point of application and in the direction of one of the external loads but in the absence of all external loading. We denote its reacting moment and shear throughout the structure by m_p and v_p, respectively. Then the deflection in the direction of this virtual load, when all external loads are applied, is the work done by the virtual load,

$$\delta = \sum_{n=1}^{N} \int_{0}^{\ell_n} \left(Mm_p/EI + Vv_p/kA_vG \right) dx \tag{7}$$

where M and V are, respectively, the total moment and shear in the structure caused by the external loading, including the secondary effects of deflection as well as the effects of all of the redundant reactions. The condition that determines the redundant reactions is that a virtual load applied at the point of action and in the direction of a redundant can do no work. If we denote the reacting moment and shear to this virtual redundant load by m_x and v_x, respectively, then the redundant reaction when all external loads are applied is found from

$$\sum_{n=1}^{N} \int_{0}^{\ell_n} \left(Mm_x/EI + Vv_x/kA_vG \right) dx = 0 \tag{8}$$

With certain reasonably accurate approximations, the integrals of Equations 7 and 8 can be evaluated and the equations may be replaced by matrix expressions. We have already assumed that the bending moment varies linearly along the length of each structural element and that the shear is constant; i.e.,

$$M = M_i \left(1 - x/\ell \right) + M_o \left(x/\ell \right) \tag{9}$$

$$m = m_i \left(1 - x/\ell \right) + m_o \left(x/\ell \right) \tag{10}$$

$$V, v = \text{constants} \tag{11}$$

within each element. In addition, we shall approximate the reciprocals of the stiffnesses as varying linearly along the length of each element.

$$1/EI = \left(1/E_iI_i \right)\left(1 - x/\ell \right) + \left(1/E_oI_o \right)\left(x/\ell \right) \tag{12}$$

$$1/kA_vG = \left(1/k_iA_{vi}G_i \right)\left(1 - x/\ell \right) + \left(1/k_oA_{vo}G_o \right)\left(x/\ell \right) \tag{13}$$

This form of approximation to the stiffnesses also allows inclusion of variations in material properties, e.g., with temperature. The substitution of Equations 9 to 13 into Equations 7 and 8 permits evaluation of the integrals and leads to the following expressions for the deflection of one of the loaded points.

$$\delta = \sum_{n=1}^{N} \Big[\left(m_{pi} K_{bi} + m_{po} K_{bio} \right) M_i + \left(m_{po} K_{bo} + m_{pi} K_{bio} \right) M_o + v_p K_v V \Big] \tag{14}$$

and for one of the redundants

$$\sum_{n=1}^{N} \Big[(m_{xi} K_{bi} + m_{xo} K_{bio}) M_i + (m_{xo} K_{bo} + m_{xi} K_{bio}) M_o + v_x K_v V \Big] = 0 \tag{15}$$

where the flexibility constants* are

$$K_{bi} = (\ell/12)(3/E_i I_i + 1/E_o I_o) \tag{16}$$

$$K_{bo} = (\ell/12)(3/E_o I_o + 1/E_i I_i) \tag{17}$$

$$K_{bio} = (1/4)(K_{bi} + K_{bo}) \tag{18}$$

$$K_v = (\ell/2)(1/k_i A_{vi} G_i + 1/k_o A_{vo} G_o) \tag{19}$$

Equations 14 and 15 may be written in matrix form as

$$\delta = (\{m_{pi}\}^T \lceil K_{bi} \rfloor + \{m_{po}\}^T \lceil K_{bio} \rfloor)\{M_i\} + (\{m_{po}\}^T \lceil K_{bo} \rfloor + \{m_{pi}\}^T \lceil K_{bio} \rfloor)\{M_o\} + \{v_p\}^T \lceil K_v \rfloor \{V\} \tag{20}$$

and

$$(\{m_{xi}\}^T \lceil K_{bi} \rfloor + \{m_{xo}\}^T \lceil K_{bio} \rfloor)\{M_i\} + (\{m_{xo}\}^T \lceil K_{bo} \rfloor + \{m_{xi}\}^T \lceil K_{bio} \rfloor)\{M_o\} + \{v_x\}^T \lceil K_v \rfloor \{V\} = 0 \tag{21}$$

Because the virtual loads are applied to the statically determinate structure, the virtual reactions may be found from Equations 1 to 3 by identifying corresponding terms

$$\{m_{pi}\} = [M_i/P]\{p\} \tag{22}$$

$$\{m_{po}\} = [M_o/P]\{p\} \tag{23}$$

$$\{v_p\} = [V/P]\{p\} \tag{24}$$

$$\{m_{xi}\} = [M_i/X]\{x\} \tag{25}$$

$$\{m_{xo}\} = [M_o/X]\{x\} \tag{26}$$

$$\{v_x\} = [V/X]\{x\} \tag{27}$$

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*Including torsional and axial flexibilities and assuming linear variations in their reciprocals and constancy of the torque and axial force along the element leads to the torsion and axial flexibility constants $K_t = (\ell/2)(1/G_iJ_i + 1/G_oJ_o)$ and $K_a = (\ell/2)(1/E_iA_{ai} + 1/E_oA_{ao})$, where J and A are the effective polar moment of inertia and area for axial stress for the cross-section, respectively; cf., footnote, p. 4.

where $\{p\}$ and $\{x\}$ are the virtual external load matrix and the virtual redundant load matrix, respectively (N.B., the only nonzero element in $\{p\}$ is taken as unity and corresponds to the point whose deflection is being calculated, and, similarly, the only nonzero element in $\{x\}$ is taken as unity and corresponds to the particular redundant being investigated). By considering a virtual external load to be applied to each point of external loading in turn, one may rewrite Equation 20 a sufficient number of times to determine all of the deflections. The resulting equations may be combined into a single matrix equation*.

$$\begin{aligned}\{\delta\} = &\left([M_i/P]^T \lceil K_{bi} \rfloor + [M_o/P]^T \lceil K_{bio} \rfloor\right) \{M_i\} + \left([M_o/P]^T \lceil K_{bo} \rfloor\right. \\ &\left. + [M_i/P]^T \lceil K_{bio} \rfloor\right) \{M_o\} + [V/P]^T \lceil K_v \rfloor \{V\}\end{aligned} \tag{28}$$

In a similar manner, if we consider a virtual redundant load applied to each cut in turn, Equation 21 may be rewritten a sufficient number of times to determine all of the redundants, and the equations may be summarized in another matrix equation.

$$\begin{aligned}&\left([M_i/X]^T \lceil K_{bi} \rfloor + [M_o/X]^T \lceil K_{bio} \rfloor\right) \{M_i\} + \left([M_o/X]^T \lceil K_{bo} \rfloor\right. \\ &\left. + [M_i/X]^T \lceil K_{bio} \rfloor\right) \{M_o\} + [V/X]^T \lceil K_v \rfloor \{V\} = 0\end{aligned} \tag{29}$$

If Equations 1 to 3 are substituted into Equation 29, the equation for the redundants may be written

$$[A]\{X\} + [B]\{P\} + [C]\{\delta\} = 0 \tag{30}$$

where

$$\begin{aligned}[A] = &\left([M_i/X]^T \lceil K_{bi} \rfloor + [M_o/X]^T \lceil K_{bio} \rfloor\right) [M_i/X] + \left([M_o/X]^T \lceil K_{bo} \rfloor\right. \\ &\left. + [M_i/X]^T \lceil K_{bio} \rfloor\right) [M_o/X] + [V/X]^T \lceil K_v \rfloor [V/X]\end{aligned} \tag{31}$$

$$\begin{aligned}[B] = &\left([M_i/X]^T \lceil K_{bi} \rfloor + [M_o/X]^T \lceil K_{bio} \rfloor\right) [M_i/P] + \left([M_o/X]^T \lceil K_{bo} \rfloor\right. \\ &\left. + [M_i/X]^T \lceil K_{bio} \rfloor\right) [M_o/P] + [V/X]^T \lceil K_v \rfloor [V/P]\end{aligned} \tag{32}$$

$$\begin{aligned}[C] = &\left([M_i/X]^T \lceil K_{bi} \rfloor + [M_o/X]^T \lceil K_{bio} \rfloor\right) [M_i/\delta] + \left([M_o/X]^T \lceil K_{bo} \rfloor\right. \\ &\left. + [M_i/X]^T \lceil K_{bio} \rfloor\right) [M_o/\delta] + [V/X]^T \lceil K_v \rfloor [V/\delta]\end{aligned} \tag{33}$$

Substituting Equation 1 to 3 into Equation 28 yields the following equation for the deflections

$$[B]^T\{X\} + [a_s]\{P\} + [b_s]\{\delta\} = \{\delta\} \tag{34}$$

*This is the result obtained in Reference 5. The erroneous application of Castigliano's First Theorem in Reference 5 achieved the correct result because the deflections were treated as independent variables during the partial differentiation of the strain energy with respect to the external loads.

where

$$[a_s] = ([M_i/P]^T \lceil K_{bi} \rfloor + [M_o/P]^T \lceil K_{bio} \rfloor)\ [M_i/P] + ([M_o/P]^T \lceil K_{bo} \rfloor + [M_i/P]^T \lceil K_{bio} \rfloor)\ [M_o/P] + [V/P]^T \lceil K_v \rfloor\ [V/P] \tag{35}$$

$$[b_s] = ([M_i/P]^T \lceil K_{bi} \rfloor + [M_o/P]^T \lceil K_{bio} \rfloor)\ [M_i/\delta] + ([M_o/P]^T \lceil K_{bo} \rfloor + [M_i/P]^T \lceil K_{bio} \rfloor)\ [M_o/\delta] + [V/P]^T \lceil K_v \rfloor\ [V/\delta] \tag{36}$$

The simultaneous solution of Equations 30 and 34 leads to the deflections and redundants in terms of the external loading. The relationship between the deflections and the external loading defines the structural influence coefficients.

$$\{\delta\} = [a]\ \{P\} \tag{37}$$

where

$$[a] = (\lceil I \rfloor - [b_r])^{-1}\ [a_r] \tag{38}$$

and

$$[a_r] = [a_s] - [B]^T [A]^{-1} [B] \tag{39}$$

$$[b_r] = [b_s] - [B]^T [A]^{-1} [C] \tag{40}$$

We may define a unit final redundant load matrix by writing

$$\{X\} = [X_f/P]\ \{P\} \tag{41}$$

where

$$[X_f/P] = -[A]^{-1} ([B] + [C]\ [a]) \tag{42}$$

Substituting Equations 37 and 41 into Equations 1 to 3 leads to a series of unit final internal reaction matrices.

$$\{M_i\} = [M_{if}/P]\ \{P\} \tag{43}$$

$$\{M_o\} = [M_{of}/P]\ \{P\} \tag{44}$$

$$\{V\} = [V_f/P]\ \{P\} \tag{45}$$

where

$$[M_{if}/P] = [M_i/P] + [M_i/X][X_f/P] + [M_i/\delta][a] \tag{46}$$

$$[M_{of}/P] = [M_o/P] + [M_o/X][X_f/P] + [M_o/\delta][a] \tag{47}$$

$$[V_f/P] = [V/P] + [V/X][X_f/P] + [V/\delta][a] \tag{48}$$

The last consideration of this section is buckling of the structure. The combination of transverse and axial loadings on a beam-column frequently will produce excessive stresses and consequent failure before buckling occurs. Nevertheless, the external loading that makes the matrix $(\lceil I \rfloor - [b_r])$ singular is a useful reference in estimating margins of stability for subcritical loadings and, therefore, is defined as the buckling loading. The matrix $[b_r]$ is a function of the external loading $\{Q\}$. If we assume each of the loads Q to be increased by the same scale factor c, then the buckling loading is found from the lowest value of c for which the matrix $(\lceil I \rfloor - [b_r])$ becomes singular. Letting c_{cr} be this critical value, we define the buckling loading by

$$\{Q_{cr}\} = c_{cr} \{Q\} \tag{49}$$

where c_{cr} is found from the dominant eigenvalue λ of the matrix $[b_r]$; i.e., if we write the eigenvalue problem in canonical form

$$\lambda \{\delta\} = [b_r] \{\delta\} \tag{50}$$

then

$$c_{cr} = 1/\lambda \tag{51}$$

SOLUTION OF THE NONLINEAR CASE

A linear solution for the redundant beam-column system was obtained in the preceding section because it was assumed that the first terms of Equations 4 to 6 were sufficient to account for beam-column effects. In the most general case, beam-column effects cause interaction between the external loading and the redundants through the deflections, and all terms in Equations 4 to 6 are present. We now consider the approximate solution to this general nonlinear problem

For the purpose of an approximate solution, Equations 37, 38, 41, and 42 can be combined to read

$$\{\delta\} = (\lceil I \rfloor - [b_r])^{-1} [a_r] \{P\} \tag{52}$$

$$\{X\} = -[A]^{-1} ([B]\{P\} + [C]\{\delta\}) \tag{53}$$

If we denote the deflections without beam-column effects by

$$\{\delta\}_1 = [a_r] \{P\} \tag{54}$$

and the redundants without beam-column effects by

$$\{X\}_1 = - [A]^{-1} [B] \{P\} \tag{55}$$

then Equations 52 and 53 become

$$\{\delta\} = (\lceil I \rfloor - [b_r])^{-1} \{\delta\}_1 \tag{56}$$

$$\{X\} = \{X\}_1 - [A]^{-1} [C] \{\delta\} \tag{57}$$

The matrices $[b_r]$ and $[C]$, defined in Equations 40 and 33, respectively, are functions of the unknown redundants through Equations 4 to 6, and the possibility of an iterative solution to Equations 56 and 57 becomes evident at this point. Although the choice of an iterative solution is not unique, the following recurrence scheme appears likely to yield a uniform monotonic convergence that will generally be stable.

$$\{\delta\}_{n+1} = ([I] - [b_r]_n)^{-1} \{\delta\}_1 \tag{58}$$

$$\{X\}_{n+1} = \{X\}_1 - [A]^{-1} [C]_n \{\delta\}_n \tag{59}$$

where $[b_r]_n$ and $[C]_n$ are based on $\{X\}_n$.

The rate of convergence of the iterative sequence can be improved by using Aitken's acceleration procedure to obtain a better approximation to each of the deflections and redundants. If convergence is proceeding exponentially, the asymptotic value of each element of the n^{th} iterated deflection mode is

$$\delta_{n+1} = \delta_{n-2} - (\delta_{n-1} - \delta_{n-2})^2 / (\delta_n - 2\delta_{n-1} + \delta_{n-2}) \tag{60}$$

and the asymptotic value of each of the n^{th} iterated redundants is

$$X_{n+1} = X_{n-2} - (X_{n-1} - X_{n-2})^2 / (X_n - 2X_{n-1} + X_{n-2}) \tag{61}$$

The extrapolations are made only if all deflections and all redundants satisfy the conditions

$$\left| (\delta_n - \delta_{n-1}) / (\delta_{n-1} - \delta_{n-2}) \right| < r < 1 \tag{62}$$

$$\left| (X_n - X_{n-1}) / (X_{n-1} - X_{n-2}) \right| < r < 1 \tag{63}$$

in order to maintain uniform convergence. An optimum convergence rate has been observed (Reference 8) with the ratio r about 0.90. Since convergence will not be exactly exponential, the extrapolated asymptotic deflections and redundants will be only approximate and are used as a new basis from which to continue the iterative solution of Equations 58 and 59.

The deflections are sufficient to test convergence. If the maximum deflection agrees on two successive iterations to a given number of significant figures

$$\left| \frac{(\delta_n)_{max}}{(\delta_{n-1})_{max}} - 1 \right| < \epsilon \tag{64}$$

and if each element of the normalized deflection mode agrees on two successive iterations to the same number of decimals

$$\left| \frac{\delta_n}{(\delta_n)_{max}} - \frac{\delta_{n-1}}{(\delta_{n-1})_{max}} \right| < \epsilon \tag{65}$$

then satisfactory convergence has been achieved. For five significant figure convergence, ϵ is equal to 0.5×10^{-5}.

The buckling loading in the nonlinear case can only be defined rigorously in terms of some strength failure criterion; it cannot be defined by the classical stability criterion because of the excessive stresses and deflections at loadings closely approaching the classical critical loading. However, it may also be defined qualitatively (although not without some quantitative meaning) by some arbitrary maximum deflection criteron, such as the loading at which the deflections begin to increase rapidly.

The computer results from Reference 1 have exhibited the following behavior in the nonlinear case as the applied loading is increased toward the buckling loading:

(1) There is a very rapid increase in the number of iterations required for convergence;

(2) There is the aforementioned rapid increase in the deflections;

(3) There is a decrease in the calculated classical buckling loading; and

(4) If the applied loading exceeds the classical buckling loading, the iterative scheme becomes unstable and convergence is not achieved.

These numerical characteristics are illustrated in Example 2 of the next section. The curve shown in Figure 7 indicates the possibility of extrapolation for a "classical buckling load" with some degree of accuracy. However, the arbitrary definition of buckling for this class of problems does not warrant any more refined analysis. Furthermore, the rapidity with which convergence is achieved at subcritical loadings by the particular iterative scheme utilized suggests that the method developed here is adequate for its intended application, viz., for stress and deflection analysis of stable systems.

EXAMPLE PROBLEMS

The following three examples are chosen to illustrate certain aspects of beam-column problems of general interest. The first example is a doubly redundant two member frame subjected to a panel point loading as shown in Figure 2, and is a straightforward redundant beam-column problem since the axial loads in the two members are known at the outset. The second example* is the same frame subjected to a transverse loading at an intermediate point on the vertical member as shown in Figure 3, and leads to a nonlinear beam-column problem that illustrates the general procedure. The third example is a case of non-conservative loading, a clamped column tangentially loaded at its free end, and illustrates the use of unsymmetrical SICs in vibration analyses to determine the buckling load from the condition of frequency coalescence. Only bending flexibility is considered for brevity; additional flexibilities are illustrated in the examples of Reference 1.

Example 1. Consider the uniform doubly redundant frame in Figure 2, and assume that shear and axial deformations may be neglected. Of the various ways the structure can be cut to make it statically determinate, we seek one that makes the deflection mode of the cut structure similar to that of the actual structure in order to maintain accuracy throughout the calculations. We therefore choose to remove the bending restraints at joints B and C and the moments X_1 and X_2 become the two redundants as shown. Since the buckling analysis of Reference 5 was 10 percent in error with the beam divided into three segments of equal length we shall increase the number of segments in each member to 10 here**. Applying the dummy loads at the segment end points as shown will lead to an 18th order matrix of SICs. The axial load Q is carried directly into the horizontal member BC so that six matrices are required to determine the SICs and the buckling load:

$$\left[M_i/P\right],\ \left[M_o/P\right],\ \left[M_i/X\right],\ \left[M_o/X\right],\ \left[M_i/\delta Q\right],\ \text{and}\ \left[M_o/\delta Q\right]$$

(Note: for this configuration the outboard moments can be derived from the inboard values, so only the inboard values will be shown). We denote as outboard the direction around the frame away from the pinned base joint, and choose compression on the outside of the frame as the positive moment convention.

*This is Example 5 of Reference 1

**Something of the order of 10 segments would be recommended normally in dealing with practical configurations having variable stiffness properties.

We begin by deriving the inboard moment matrix for the dummy loads on the statically determinate structure. Consider the dummy load p_1. The reaction at A to p_1 is $9p_1/10$ and the moments below the load become $M_{11} = 0$ and $M_{12} = (9p_1/10)\ (\ell/10)$. Similarly the reaction at B to p_1 is $p_1/10$ and the moments above the load become $M_{13} = (p_1/10)\ (8\ell/10)$, $M_{14} = (p_1/10)\ (7\ell/10)$, etc. The load p_1 causes no moments in member BC. All of the elements of the matrix $[M_i/P]$ are derived in this manner and the matrix of order 20 x 18 finally appears as

$[M_i/P]$ =

Item ↓ / Load →	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}
(1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2)	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0
(3)	8	16	14	12	10	8	6	4	2	0	0	0	0	0	0	0	0	0
(4)	7	14	21	18	15	12	9	6	3	0	0	0	0	0	0	0	0	0
(5)	6	12	18	24	20	16	12	8	4	0	0	0	0	0	0	0	0	0
(6)	5	10	15	20	25	20	15	10	5	0	0	0	0	0	0	0	0	0
(7)	4	8	12	16	20	24	18	12	6	0	0	0	0	0	0	0	0	0
(8)	3	6	9	12	15	18	21	14	7	0	0	0	0	0	0	0	0	0
(9)	2	4	6	8	10	12	14	16	8	0	0	0	0	0	0	0	0	0
(10) $(\ell/100)$	1	2	3	4	5	6	7	8	9	0	0	0	0	0	0	0	0	0
(11)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(12)	0	0	0	0	0	0	0	0	0	9	8	7	6	5	4	3	2	1
(13)	0	0	0	0	0	0	0	0	0	8	16	14	12	10	8	6	4	2
(14)	0	0	0	0	0	0	0	0	0	7	14	21	18	15	12	9	6	3
(15)	0	0	0	0	0	0	0	0	0	6	12	18	24	20	16	12	8	4
(16)	0	0	0	0	0	0	0	0	0	5	10	15	20	25	20	15	10	5
(17)	0	0	0	0	0	0	0	0	0	4	8	12	16	20	24	18	12	6
(18)	0	0	0	0	0	0	0	0	0	3	6	9	12	15	18	21	14	7
(19)	0	0	0	0	0	0	0	0	0	2	4	6	8	10	12	14	16	8
(20)	0	0	0	0	0	0	0	0	0	1	2	3	4	5	6	7	8	9

The outboard matrix is obtained from the above by deleting the top row, shifting each remaining row upward one row, and adding a row of zeros at the bottom.

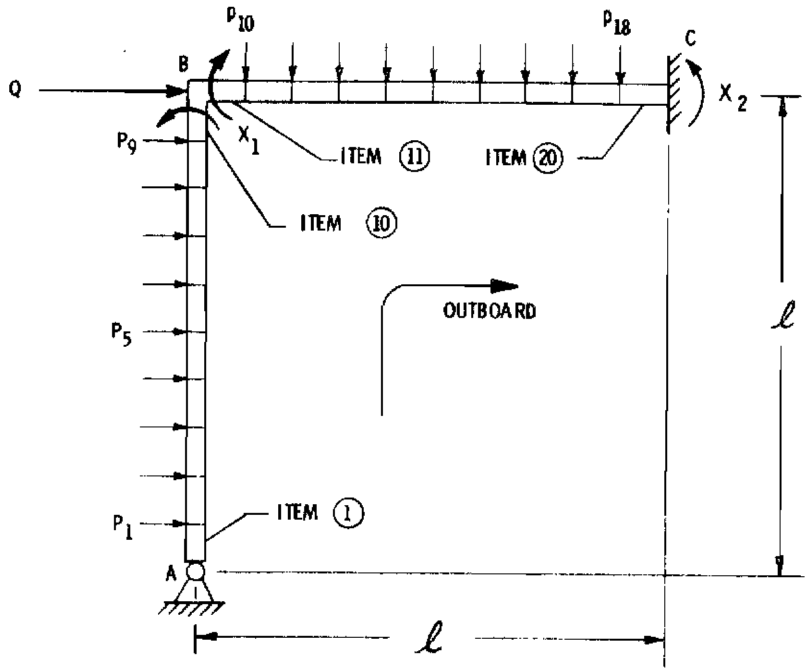

Figure 2. Corner Loaded Redundant Frame

We next derive the inboard moment matrix for the redundants. Consider the dummy redundant x_1. The horizontal reaction at A to x_1 is x_1/ℓ and the moments in member AB are $M_{i1} = 0$, $M_{i2} = (x_1/\ell)\ (\ell/10)$, $M_{i3} = (x_1/\ell)\ (2\ell/10)$, etc. The reaction at C to x_1 is also x_1/ℓ and the moments in BC are $M_{i11} = x_1$, $M_{i12} = (x_1/\ell)\ (9\ell/10)$, $M_{i13} = (x_1/\ell)\ (8\ell/10)$, etc. The moments caused by x_2 are found in the same manner and by noting that it does not affect AB.

The matrix $[M_i/X]$ of order 20 x 2 finally appears as

$$
[M_i/X] = \begin{array}{cc}
 & \begin{matrix} \text{Load} \rightarrow \\ \begin{matrix} x_1 & x_2 \end{matrix} \end{matrix} \\
\begin{matrix} \text{Item} \downarrow \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20 \end{matrix} &
\begin{matrix} \\ \begin{bmatrix}
0 & 0 \\
0.1 & 0 \\
0.2 & 0 \\
0.3 & 0 \\
0.4 & 0 \\
0.5 & 0 \\
0.6 & 0 \\
0.7 & 0 \\
0.8 & 0 \\
0.9 & 0 \\
1.0 & 0 \\
0.9 & 0.1 \\
0.8 & 0.2 \\
0.7 & 0.3 \\
0.6 & 0.4 \\
0.5 & 0.5 \\
0.4 & 0.6 \\
0.3 & 0.7 \\
0.2 & 0.8 \\
0.1 & 0.9
\end{bmatrix} \end{matrix}
\end{array}
$$

The outboard matrix is obtained from the above by deleting the top row, shifting the remaining rows upward, and adding the row $\lfloor 0 \quad 1.0 \rfloor$ at the bottom.

The inboard moment matrix for beam-column effects may be written by inspection of Figure 3 merely by observing the moments caused by the horizontal axial load* Q and the deflections δ. Noting that the force Q does not induce moments in member AB, we immediately deduce the following moments in BC: $M_{i_{11}} = 0$, $M_{i_{12}} = Q\delta_{10}$, $M_{i_{13}} = Q\delta_{11}$, etc.

*This is a typical case of conservative loading. Example 3 illustrates a case of nonconservative loading.

The inboard moment matrix for unit beam-column loading then appears as

$$\left[M_i/\delta Q\right] =$$

Deflection →

Item ↓

$$\begin{array}{cc}
 & \begin{array}{cccccccccccccccccc} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 & \delta_7 & \delta_8 & \delta_9 & \delta_{10} & \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} & \delta_{15} & \delta_{16} & \delta_{17} & \delta_{18} \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}$$

The outboard matrix is obtained by shifting the rows upward and adding a row of zeros at the bottom.

For the numerical work, we assume the length $\ell = 200$ in. and the uniform stiffness $EI = 10^8$ lb sq in. The axial load Q is assumed to be 40,000 lb. Utilization of the computer program of Reference 1 with the foregoing geometric and stiffness data and the moment matrices results in an 18th order matrix of SICs and the buckling load. The SICs will not be shown but the 5-5 element is found to be

$$a_{5\text{-}5} = 0.00139245 \text{ in./lb}$$

and the lowest buckling load is

$$Q_{cr} = 68{,}640 \text{ lb}$$

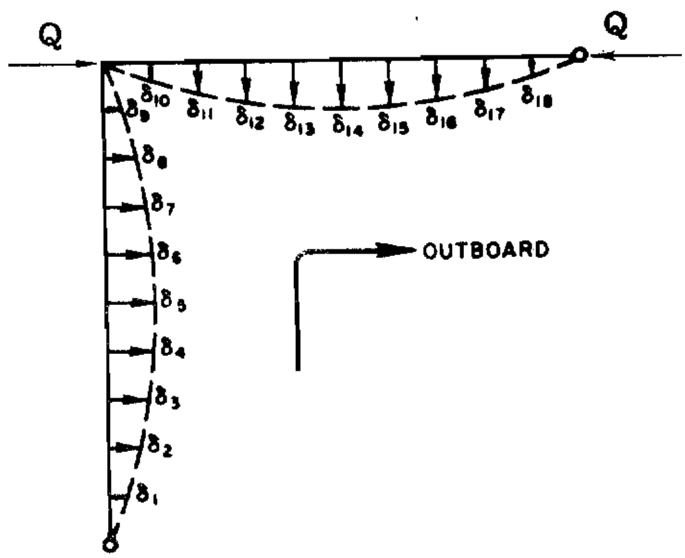

Figure 3. Interaction of Loading and Deflections

The exact solution to this problem may be found from well known uniform beam deflection formulae (Reference 9) and the requirement for continuity of rotation at joint B. The rotation of joint B in terms of the loading P_5 on member AB is given by

$$EIy' = k_1 P_5 \ell^2 + k_2 X_1 \ell$$

where $k_1 = 1/16$ and $k_2 = 1/3$. The rotation of joint B in terms of the loading on the member BC is (see Roark Reference 9, Table VI, Case 8 adjusted to have zero slope at one end).

$$EIy' = -k_3 X_1 \ell$$

where

$$k_3 = (2 - 2\cos\mu_h\ell - \mu_h\ell\sin\mu_h\ell)/\mu_h\ell(\sin\mu_h\ell - \mu_h\ell\cos\mu_h\ell)$$

and

$$\mu_h = \sqrt{Q/EI}$$

Equating the rotations yields

$$X_1 = k_1 P_5 \ell/(k_2 + k_3)$$

The deflection under P_5 is

$$y_5 = P_5 \ell^3/48EI + X_1 \ell^2/16EI$$

from which the SIC of control point 5 is

$$a_{5\text{-}5} = y_5/P_5$$
$$= (\ell^3/48EI)\left[1 - 3k_1/(k_2 + k_3)\right]$$

The condition for buckling is

$$k_2 + k_3 = 0$$

For the load Q = 40,000 lb. the exact solutions are found to be

$$a_{5-5} = 0.00140312 \text{ in./lb}$$

and

$$Q_{cr} = 26.9582 \; EI/\ell^2$$
$$= 67{,}396 \text{ lb}$$

(The buckling load is found from a numerical solution for the lowest root of the transcendental equation, $k_2 + k_3 = 0$). We note that the original assumption of a linear variation in moment along the length of each element implies a linear deflection curve between control points when calculating the effect of axial load. This restraint in treating beam-column effects results in an effective increase in stiffness. Hence, the calculated SICs are lower and the buckling loads are higher than the corresponding theoretical values. In the present example, the division of each member into 10 segments results in errors in a_{5-5} and Q_{cr} of 0.77 and 1.85 percents respectively.

Example 2. Reconsider the doubly redundant frame of Figure 2, but with the horizontal load Q applied at the midpoint of the vertical member AB as shown in Figure 4. This example illustrates the interaction between external loading and internal redundants when the loading is sufficiently large that beam-column effects must be accounted for. For this general nonlinear beam-column problem, ten matrices are required to determine the SICs and the buckling load: $[M_i/P]$, $[M_i/X]$, $[M_i/\delta Q]$, $[M_i/\delta X_1]$, $[M_i/\delta X_2]$, and the corresponding outboard values (which are again derived from the inboard values).

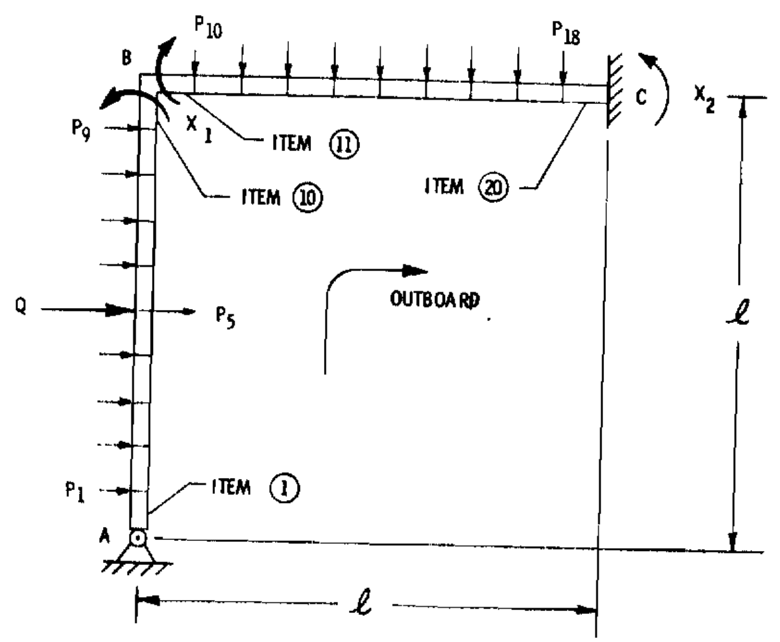

Figure 4. Redundant Frame With Intermediate Member Loading

By making the same choice of redundants and considering the same 18 virtual loads as in Example 1, the matrices $[M_i/P]$, $[M_o/P]$, $[M_i/X]$, and $[M_o/X]$ remain the same. The inboard moment matrices for beam-column effects may be written by considering the reactions to the load Q and the redundants X_1 and X_2, and observing Figures 5a, 5b and 5c. A comparison of Figure 5a with Figure 3 shows that the elements of the matrices $[M_i/\delta Q]$ and $[M_o/\delta Q]$ have 1/2 of the values of the corresponding matrices in Example 1, because only half of the load Q is reacted by the member BC in the present case. These two matrices, therefore, need not be shown. From Figure 5b it is seen that the interaction between the first redundant and the deflections induces moments in both members. We observe the following moments in AB: $M_{i1} = 0$, $M_{i2} = -X_1\delta_1/\ell$, $M_{i3} = -X_1\delta_2/\ell$, etc., and the moments in BC are: $M_{i11} = 0$, $M_{i12} = -X_1\delta_{10}/\ell$, $M_{i13} = -X_1\delta_{11}/\ell$, etc. The inboard moment matrix for the first redundant appears as

$$[M_i/\delta X_1] =$$

$$\begin{array}{cc}
\text{Item} \downarrow \;\; \text{Deflection} \rightarrow & \begin{array}{cccccccccccccccccc} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 & \delta_7 & \delta_8 & \delta_9 & \delta_{10} & \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} & \delta_{15} & \delta_{16} & \delta_{17} & \delta_{18} \end{array} \\
\begin{array}{c} (1)\\(2)\\(3)\\(4)\\(5)\\(6)\\(7)\\(8)\\(9)\\(10)\\(11)\\(12)\\(13)\\(14)\\(15)\\(16)\\(17)\\(18)\\(19)\\(20) \end{array} & (1/\ell)\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1
\end{bmatrix}
\end{array}$$

From Figure 5c, it is seen that interaction between the second redundant and the deflections only induces moments in member AB. The moments are: $M_{i1} = 0$, $M_{i2} = X_2\delta_1/\ell$, $M_{i3} = X_2\delta_2/\ell$, etc.,

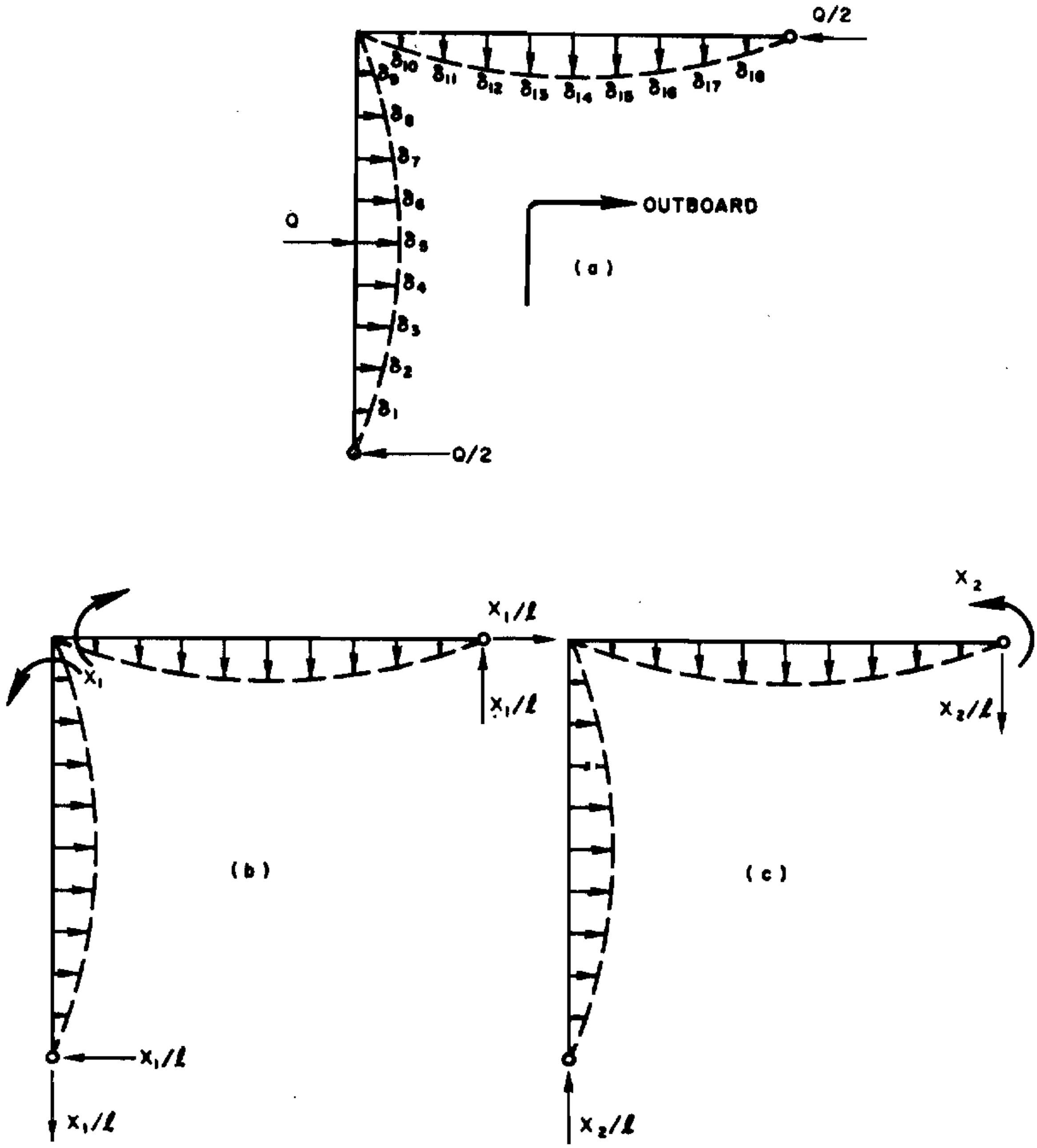

Figure 5. Interaction of External Loading, Redundants, and Deflections

and the inboard moment matrix for the second redundant appears as

$$\left[M_i / \delta X_2\right] =$$

Item \ Deflection	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{16}	δ_{17}	δ_{18}
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
10 $(1/\ell)$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The outboard moment matrices for redundant beam-column effects are derived from the inboard matrices by shifting the rows upward and adding a row of zeros at the bottom.

The computer results for a_{5-5} and the buckling load Q_{cr} are shown in Figures 6 and 7, respectively, for a number of loading conditions. For loading Q = 45,000 lb, the following values were found,

$$a_{5-5} = 0.0020036 \text{ in./lb}$$

$$Q_{cr} = 85{,}223 \text{ lb}$$

Since $Q < Q_{cr}$ the buckling load must exceed 45,000 lb. The extrapolation for the buckling load is discussed at the end of this section.

The exact solution to this problem begins with the requirement for continuity of rotation at the frame corner.

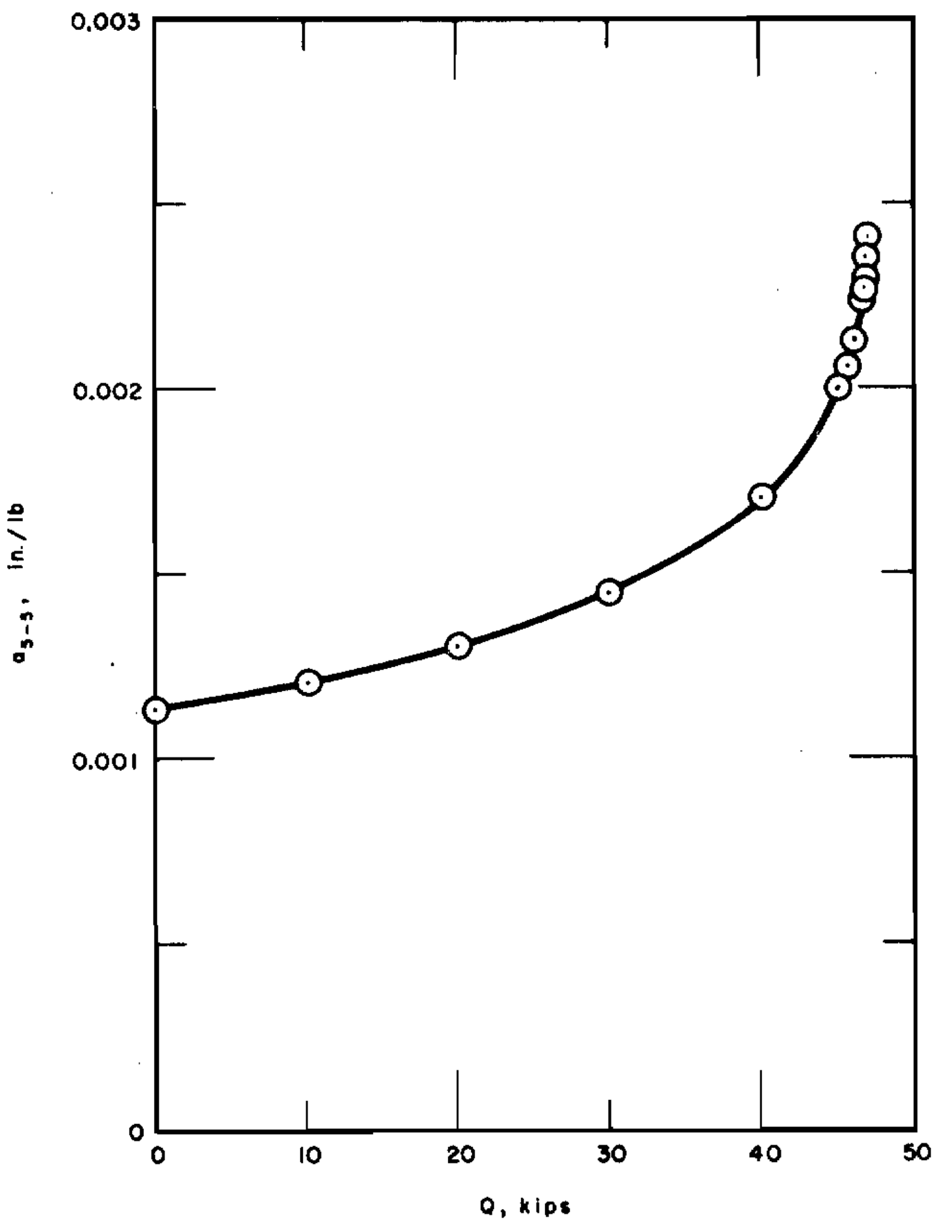

Figure 6. Variation of Largest Influence Coefficient With Applied Load

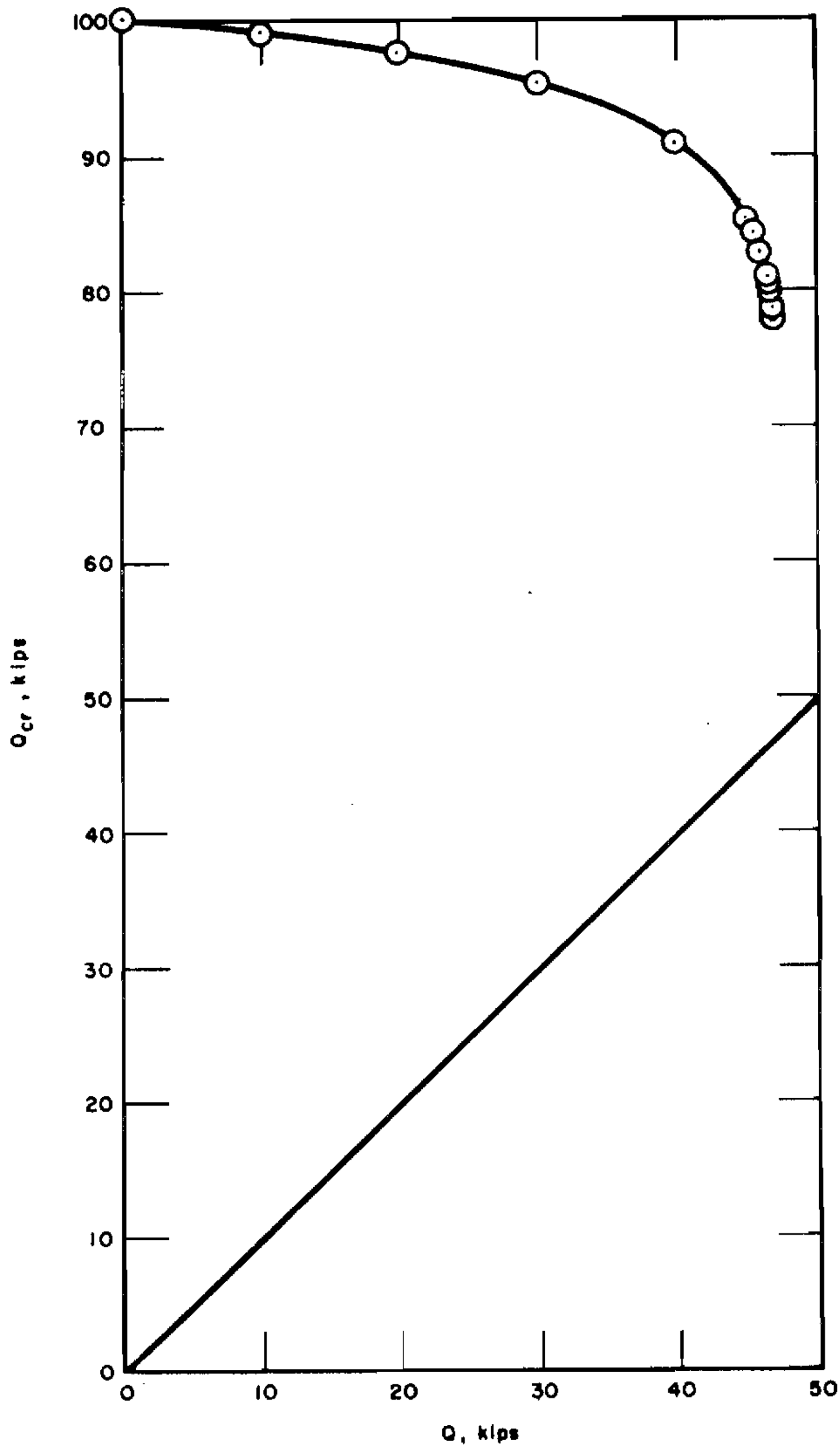

Figure 7. Variation of Classical Buckling Load with Applied Load

The first redundant is proportional to the external load

$$X_1 = -KQ\ell$$

where the coefficient K will be found from the condition of rotational continuity. The second redundant is related to the first through the carry-over factor C

$$X_2 = -CX_1$$

All of the reactions on the frame can be expressed in terms of the coefficients K and C as shown in Figure 8. Having the frame reactions permits utilizing the single span beam-column slope formulae to determine the coefficient K. The corner joint rotation in terms of the loading on the vertical member may be written (see Roark Reference 9, Table VI, Cases 3 and 8)

$$EIy' = k_1 Q\ell^2 + k_2 X_1 \ell$$

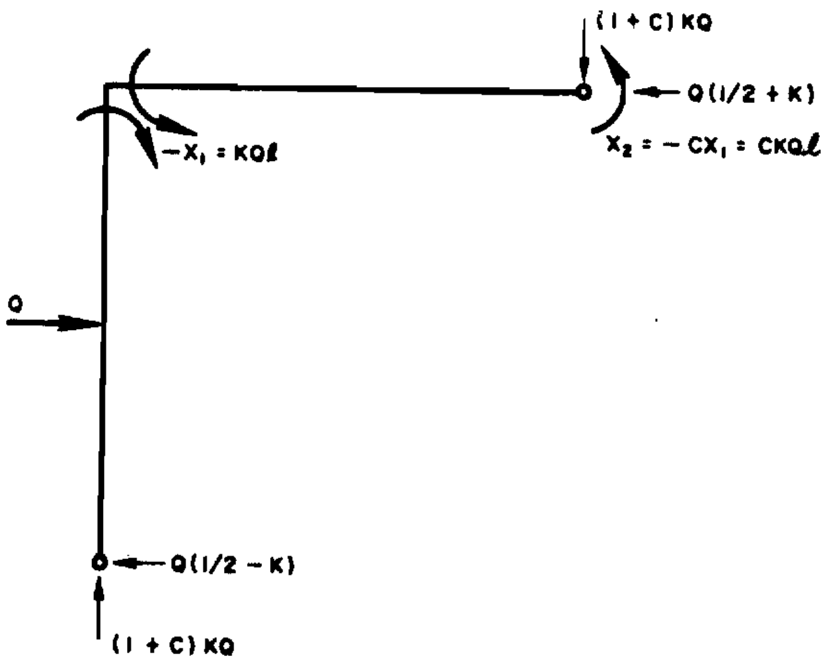

Figure 8. Frame Reactions

where

$$k_1 = [1 - \cos(\mu_v \ell/2)/2(\mu_v \ell)^2 \cos(\mu_v \ell/2)]$$

$$k_2 = (\sin\mu_v \ell - \mu_v \ell \cos\mu_v \ell)/(\mu_v \ell)^2 \sin\mu_v \ell$$

and

$$\mu_v = \sqrt{(1+C)\,KQ/EI}$$

The rotation in terms of loading on the horizontal member is (see Roark Reference 9, Table VI, Case 8 adjusted to have zero slope at one end)

$$EIy' = -k_3 X_1 \ell$$

where

$$k_3 = (2 - 2\cos\mu_h\ell - \mu_h\ell\sin\mu_h\ell) / \mu_h\ell(\sin\mu_h\ell - \mu_h\ell\cos\mu_h\ell)$$

and

$$\mu_h = \sqrt{(1/2 + K)Q/EI}$$

The carry-over factor is

$$C = (\mu_h\ell - \sin\mu_h\ell) / (\sin\mu_h\ell - \mu_h\ell\cos\mu_h\ell)$$

The continuity of rotation leads to the coefficient K

$$K = k_1 / (k_2 + k_3)$$

The solution of the foregoing equations for a given value of Q requires a trial-and-error technique. The following sequence was found to achieve a reasonably rapid convergence until the buckling load was approached:

(1) A value of K was estimated beginning with the value $K = 3/28$ for negligible beam-column effects.

(2) The following quantities were calculated in order:

(a) μ_h
(b) k_3 and C
(c) μ_v
(d) k_1 and k_2

(3) The original estimate of K was checked by calculating $K = k_1/(k_2 + k_3)$. If it was not sufficiently close, the sequence was repeated using the check value as a new estimate until satisfactory accuracy was obtained.

A converged value of K permits calculation of the frame deflections. In particular, the structural influence coefficient under the external load is given by

$$a_{5\text{-}5} = (\ell^3/EI)\left\{\left[\tan(\mu_v\ell/2) - \mu_v\ell/2(\mu_v\ell)^3\right] - K\left[\sin(\mu_v\ell/2)/\sin(\mu_v\ell) - 1/2\right]/(\mu_v\ell)^2\right\}$$

For the load $Q = 45{,}000$ lb the converged iterative "exact" solution results in

$$K = 0.12935$$
$$C = 1.17725$$

and the influence coefficient of the loaded point is

$$a_{5\text{-}5} = 0.0020523 \text{ in./lb}$$

The computer calculation of a_{5-5} has an error of 2.4 percent for this particular loading. Calculations were also carried out for a number of additional loadings in Reference 1. The inability to obtain convergence in the region of $Q = 46,500$ lb suggests that this load can reasonably be called the buckling load, inasmuch as the definition of the buckling load in this case is somewhat arbitrary, as has been discussed in the preceding section.

The computer solutions, as shown in Figures 6 and 7, lead to an estimate of buckling load of approximately 46,900 lb. In Figure 7, extrapolation of the (Q, Q_{cr}) curve to obtain an intersection with the true buckling line $Q = Q_{cr}$ does not appear to be reliable with a sufficient degree of accuracy. However, the value estimated from the curves is consistent with the arbitrary definition of buckling in the general case, and provides an adequate measure of the elastic stability of the frame.

Example 3. A uniform cantilevered column loaded transversely by 10 virtual loads and axially by a tangential load at the free end is shown in Figure 9 in its deflected position. We consider the determination of the buckling load for this non-conservative loading. It is first necessary to find the static SICs and then to carry out a vibration analysis to determine the axial load that causes coalescence of the first two natural frequencies. The dynamic analysis for the non-conservative buckling load was first given by Beck (Reference 10).

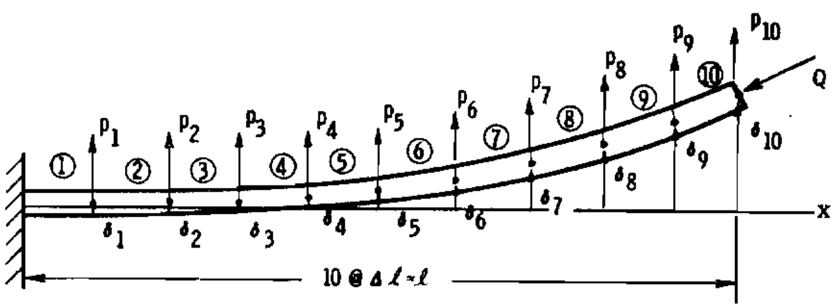

Figure 9. Column Subjected to Non-Conservative Loading

The matrix of inboard moments caused by the 10 virtual loads is easily written by inspection of Figure 9. Defining a positive moment as one that causes compression on the upper side, the matrix is

$$\left[M_i / P\right] =$$

Load →

Item ↓

$$\begin{array}{cc}
 & \begin{matrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & P_{10} \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{matrix} & (\ell/10)\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}$$

The outboard moment matrix follows by deleting the top row, shifting the remaining rows upward and adding a row of zeros at the bottom.

The inboard moment matrix for beam-column effects may also be derived from Figure 9 but not quite so easily. We note that in general, the inboard moment from the end load is

$$M_{i(n+1)} = -Q\left[\delta_n - \delta_{10} + (\ell - x_n)\, y'(\ell)\right]$$

The slope at the free end, $y'(\ell)$, can be expressed in terms of the deflections by numerical differentiation formulas. Taking the quadratic formula of Milne (Reference 11) for equally spaced points we have

$$y'(\ell) = (2/\Delta\ell)(\delta_8 - 4\delta_9 + 3\delta_{10})$$

and the elements of the matrix $[M_i/\delta Q]$ can be calculated to the accuracy of parabolic differentiation. The matrix becomes

$$\left[M_i/\delta Q\right] =$$

Item ↓ / Deflection →	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{10}
1	0	0	0	0	0	0	0	-5.0	20.0	-14.0
2	-1.0	0	0	0	0	0	0	-4.5	18.0	-12.5
3	0	-1.0	0	0	0	0	0	-4.0	16.0	-11.0
4	0	0	-1.0	0	0	0	0	-3.5	14.0	-9.5
5	0	0	0	-1.0	0	0	0	-3.0	12.0	-8.0
6	0	0	0	0	-1.0	0	0	-2.5	10.0	-6.5
7	0	0	0	0	0	-1.0	0	-2.0	8.0	-5.0
8	0	0	0	0	0	0	-1.0	-1.5	6.0	-3.5
9	0	0	0	0	0	0	0	-2.0	4.0	-2.0
10	0	0	0	0	0	0	0	-0.5	1.0	-0.5

Again the outboard moment matrix follows by shifting the rows upward and adding a row of zeros at the bottom.

The foregoing matrices are sufficient to obtain the SICs from the computer program. Thirteen sets of SICs were obtained for the values of $Q\ell^2/\pi^2 EI$ = 0, 0.5, 1.0, 1.5, 2.0, 2.02, 2.04, 2.06, 2.08, 2.09, 2.10, 2.11, and 2.12. A typical result for $Q\ell^2/\pi^2 EI$ = 1.0 illustrates the nonsymmetry of the matrices.

$$[a] = (\ell^3/EI) \times 10^{-3} \times$$

$$\begin{bmatrix}
0.328 & 0.788 & 1.172 & 1.441 & 1.571 & 1.548 & 1.375 & 1.069 & 0.657 & 0.178 \\
0.820 & 0.254 & 4.185 & 5.420 & 6.128 & 6.241 & 5.749 & 4.698 & 3.180 & 1.332 \\
1.312 & 0.446 & 8.166 & 11.240 & 13.222 & 13.920 & 13.267 & 11.325 & 8.249 & 4.303 \\
1.804 & 0.638 & 12.311 & 18.027 & 22.157 & 24.135 & 23.770 & 21.097 & 16.296 & 9.755 \\
2.296 & 0.830 & 16.457 & 24.979 & 32.060 & 36.193 & 36.810 & 33.854 & 27.464 & 18.114 \\
2.787 & 1.022 & 20.602 & 31.931 & 42.128 & 49.218 & 51.692 & 49.148 & 41.592 & 29.521 \\
3.279 & 1.215 & 24.747 & 38.883 & 52.195 & 62.407 & 67.542 & 66.284 & 58.235 & 43.822 \\
3.771 & 1.406 & 28.892 & 45.835 & 62.262 & 75.596 & 83.556 & 84.388 & 76.699 & 60.582 \\
4.263 & 1.598 & 33.037 & 52.787 & 72.330 & 88.785 & 99.570 & 102.655 & 96.116 & 79.128 \\
4.755 & 1.790 & 37.183 & 59.739 & 82.397 & 101.974 & 115.584 & 120.923 & 115.690 & 98.612
\end{bmatrix}$$

The vibration analysis is carried out by using the computer program for flutter analysis of Reference 12 because the natural frequencies become complex conjugates as the frequency coalescence condition is passed with increasing end load. The eigenvalue subprogram is based on a variation of the power method capable of finding complex conjugate roots and is described in Appendix A of Reference 12. A coupled mass matrix may be used to increase the accuracy of the vibration analysis of the 10-degree-of-freedom system. The derivation of a coupled mass matrix is discussed in Appendix B of Reference 12. If the distributed mass is lumped at 20 points, 10 at the control points for the SICs, and 10 half way between the control points, and if linear interpolation between control points is assumed, then the mass matrix appears as

$$[M] = (m\ell/20) \begin{bmatrix}
1.50 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.25 & 1.50 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.25 & 1.50 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.25 & 1.50 & 0.25 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.25 & 1.50 & 0.25 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.25 & 1.50 & 0.25 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.25 & 1.50 & 0.25 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 1.50 & 0.25 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 1.50 & 0.25 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.75
\end{bmatrix}$$

where M now denotes the elements of the mass matrix and m denotes the distribution mass per unit beam length.

The results of the vibration analysis are summarized in Table 1. By extrapolating the imaginary parts of the post-buckling frequencies (the square of the imaginary part is approximately linear) back to the coalescence point, the buckling load is estimated to be

$Q\ell/\pi^2 EI = 2.082$ and is 3.7 percent higher than the value 2.008 given by Timoshenko and Gere (Reference 13).*

Table 1. Comparison of Computed and Exact** Frequencies

$Q\ell^2/\pi^2 EI$	$\omega_1 \ell^2 \sqrt{m/EI}$		$\omega_2 \ell^2 \sqrt{m/EI}$	
	Computed	Exact	Computed	Exact
0	3.510	3.49	22.124	21.8
0.5	4.221	5.03	20.636	20.2
1.0	5.173	5.41	18.921	17.9
1.5	6.566	6.69	16.786	15.9
2.0	9.369	9.67	13.268	9.97
2.02	9.608		13.002	
2.04	9.892		12.689	
2.06	10.262		12.293	
2.08	10.941		11.585	
2.09	11.305 + i3.425		11.305 − i3.425	
2.10	11.360 + i5.201		11.360 − i5.201	
2.11	11.416 + i6.533		11.416 − i6.533	
2.12	11.473 + i7.654		11.473 − i7.654	

CONCLUDING REMARKS

A matrix solution by a Force Method for structural analysis of an elastic redundant beam-column system having variable bending and shear stiffnesses has been presented. The development leads to the SICs, the internal load distribution, and critical buckling loads for a system subjected to either conservative or nonconservative beam-column loading. The nonlinear problem that arises when the external loading, internal redundants, and deflections all interact because of beam-column effects has been solved by a reasonably rapidly convergent iterative procedure. Three example problems have been solved to illustrate the new features of the method.

The present development has only considered the effects of temperature to the extent that the material properties of each structural element are determined by the temperature. No thermally induced stresses have been considered. However, the basis for an extension of the Ogness method for the general solution of the thermoelastic deflection, internal load, and stability problems is provided in the present extension and the earlier extension of Reference 4. The general thermoelastic problem will be the subject of a later investigation.

The SICs derived here can be used directly in vibration and flutter analyses of systems restrained in space as has been illustrated in the third example. It is not apparent, however, that the SICs, if beam-column effects are present, can be used directly in the analysis of systems free in space by collocation methods such as Reference 12 without some modification

*The buckling load $Q\ell^2/\pi^2 EI = 2.031$ is given by Beck Reference 10, but the later result of Reference 13 appears to have been based on a more refined calculation.

**Derived from Table 2-15 of Reference 13.

of the free-free boundary conditions to include the generalized forces from the beam-column loading. An example is the problem of the transverse vibration frequencies of an axially accelerated booster; exact solutions for the idealized case of an axially accelerated uniform beam have been obtained by Silverberg (Reference 14) and Seide (Reference 15). This aspect of free-free vibration analysis is currently under investigation.

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