SECTION 6

ESTIMATES FOR TRUNCATION ERRORS OF INFINITE DIMENSIONAL SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

by

M. L. Bandy
and
P. K. C. Wang

IBM San Jose Research Laboratory
San Jose, California

ABSTRACT: The problem of determining the motion of distributed parameter dynamical systems governed by partial differential equations can often be reduced to that of solving a denumerably infinite system of ordinary differential equations. Although approximate solutions can be obtained by using various closure techniques, their usefulness depends to a large extent on whether error estimates can be obtained. In this section, explicit estimates for the truncation errors associated with particular classes of infinite systems of first and second order linear ordinary differential equations are obtained. The results are applied to a system arising in a distributed parameter control process.

6.1 INTRODUCTION

In many distributed parameter dynamical systems whose motion is describable by a set of partial differential equations, it is often possible to separate the equations into spatial-dependent and time-dependent parts. The latter takes on the form of a denumerably infinite system of ordinary differential equations. Although approximate solutions to these equations can be obtained by using appropriate closure techniques, their usefulness depends to a large extent on whether estimates on the error can be obtained.

This section is concerned with establishing explicit estimates for the truncation errors associated with particular classes of denumerably infinite systems of linear ordinary differential equations. In the subsequent development, the process of separating the time-dependent part from a given partial differential equation into the form of an infinite system of ordinary differential equations will be illustrated by specific examples. This will be followed by a formal mathematical derivation of the error estimates for the particular classes of equations under consideration. The application of the main results will be illustrated by a specific example.

6.2 EXAMPLES OF INFINITE SYSTEMS

For many partial differential equations arising from physical problems, the solutions may be expressed in the form of an infinite series of products of separable functions of time and spatial coordinates. The sequence of spatially dependent functions is generally taken to be any convenient complete set of orthogonal functions which satisfy the given boundary conditions. For example, in aeroelastic systems one may take these functions to be the eigenfunctions corresponding to the elastic system without aerodynamic loading.

To illustrate the mathematical details involved in deriving an infinite system of ordinary differential equations from a given partial differential equation, we shall consider two specific systems which arise in problems of automatic control and elasticity. One of these examples will also be used to illustrate the applications of the main results.

Example 1:

Consider a simple linear diffusion equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} \qquad (6.2\text{-}1)$$

defined on a spatial domain $(0,1)$, with initial and boundary conditions given by:

$$u(0,x) = u_0(x)$$

$$u(t,0) = 0 \quad , \quad u(t,1) = \int_0^1 g(\xi)u(t,\xi)\, d\xi \; . \tag{6.2-2}$$

In the physical situation, the above equations describe the dynamic behavior of a temperature regulator for a thick slab, in which the temperature at the surface $x = 1$ is made to be proportional to a spatially weighted average of the instantaneous temperature distribution of the slab.

First, let us introduce a transformation:

$$w(t,x) = u(t,x) - x \int_0^1 g(\xi)u(t,\xi)\, d\xi \tag{6.2-3}$$

which transforms (6.2-1) and (6.2-2) into the form:

$$\frac{\partial w(t,x)}{\partial t} = \frac{\partial^2 w(t,x)}{\partial x^2} - x \int_0^1 g(\xi) \frac{\partial^2 w(t,\xi)}{\partial \xi^2}\, d\xi \tag{6.2-1'}$$

and

$$\left.\begin{aligned} w(0,x) &= u_0(x) - x \int_0^1 g(\xi)\, u_0(\xi)\, d\xi \\ u(t,0) &= w(t,1) = 0 \; . \end{aligned}\right\} \tag{6.2-2'}$$

Assume the solutions to Eqs. (6.2-1') and (6.2-2') can be expressed in the form:

$$w(t,x) = \sum_{n=1}^{\infty} a_n(t) \sin n\pi x. \tag{6.2-4}$$

Substituting (6.2-4) into (6.2-1') and making use of the orthogonality property of $\{\sin n\pi x\}$ lead to the following infinite system of ordinary differential equations for $a_n(t)$:

$$\frac{d a_n(t)}{dt} + (n^2\pi^2 + 2n\pi(-1)^n g_n)\, a_n(t) = -2\pi \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{(-1)^n m^2}{n} g_m a_m(t)$$

$$n = 1, 2, \ldots \tag{6.2-5}$$

with initial conditions

$$a_n(0) = u_{o(n)} + \frac{(-1)^n}{n\pi} \hat{u}_o \qquad ; \; n = 1, 2, \ldots \tag{6.2-6}$$

where

$$g_m = \int_0^1 g(\xi) \sin m\pi\xi \, d\xi$$

$$u_{o(n)} = \int_0^1 u_o(\xi) \sin n\pi\xi \, d\xi \qquad (6.2\text{-}7)$$

$$\hat{u}_o = \int_0^1 g(\xi)\, u_o(\xi)\, d\xi \ .$$

In order to obtain the solution to the original equations, (6.2-1) and (6.2-2), it is necessary to solve (6.2-3) for u in terms of w. This can be accomplished by first differentiating (6.2-3) with respect to x and substituting the results into (6.2-3) to obtain a differential equation relating u and w:

$$\frac{d}{dx}\,(x^{-1}u(t,x) = \frac{d}{dx}\,(x^{-1}u(t,x)) \qquad (6.2\text{-}8)$$

It follows that if $\int_0^1 \xi\, g(\xi)\, d\xi \neq 1$, then u can be expressed in the form:

$$u(t,x) = w(t,x) + (1-K)^{-1}\, x\, v(t) \qquad (6.2\text{-}9)$$

where

$$v(t) = \int_0^1 g(\xi)\, w(t,\xi)\, d\xi$$

$$(6.2\text{-}10)$$

$$K = \int_0^1 \xi\, g(\xi)\, d\xi$$

Example 2:

A simply supported elastic panel with finite chord, infinite span, and uniform thickness will be considered here. It is assumed that the equation of motion for this panel has the form:

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^4 u(t,x)}{\partial x^4} + f\left[t, u(t,x)\,,\ \frac{\partial u(t,x)}{\partial t}\,,\ \frac{\partial u(t,x)}{\partial x}\right]$$

$$(6.2\text{-}11)$$

defined on a spatial domain (0,1), with initial and boundary conditions given by

$$u(0,x) = u_o(x) \quad , \quad \left.\frac{\partial u(t,x)}{\partial t}\right|_{t=0} = \dot{u}_o(x) \quad ,$$

$$u(t,0) = u(t,1) = \left.\frac{\partial^2 u(t,x)}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 u(t,x)}{\partial x^2}\right|_{x=1} = 0. \tag{6.2-12}$$

The function f represents a distributed external load. For the present discussion, f is taken to be of the form:

$$f = \int_o^1 \sum_{j=1}^{M} q_j \sin j\pi\xi \;\; u(t,\xi)d\xi + \int_o^1 \sum_{j=1}^{M} p_j \sin j\pi\xi \;\; \frac{\partial u(t,\xi)}{\partial t} \, d\xi \tag{6.2-13}$$

Similar to Example 1, we assume a solution of the form:

$$u(t,x) = \sum_{n=1}^{\infty} b_n(t) \sin n\pi x \tag{6.2-14}$$

A straightforward computation leads to the following infinite system of ordinary differential equations for $b_n(t)$:

$$\frac{d^2 b_n(t)}{dt^2} + 2\gamma_n \frac{db_n(t)}{dt} + \alpha_n b_n(t) = \sum_{\substack{m=1 \\ m \neq n}}^{M} \left[\gamma_{nm} \frac{db_n(t)}{dt} + \alpha_{nm} b_n(t) \right] \tag{6.2-15}$$

with initial conditions:

$$b_n(0) = b_{o(n)} = \int_o^1 u_o(\xi) \sin n\pi\xi \; d\xi \quad ,$$

$$\dot{b}_n(0) = \dot{b}_{o(n)} = \int_o^1 \dot{u}_o(\xi) \sin n\pi\xi \; d\xi \quad , \tag{6.2-16}$$

where

$$\gamma_n = \begin{cases} -p_n \delta_n / n\pi & \text{for } n \le M \\ 0 & \text{for } n > M \end{cases} \quad , \quad \alpha_n = \begin{cases} n^4\pi^4 - \dfrac{2q_n \delta_n}{n\pi} & \text{for } n \le M \\ n^4\pi^4 & \text{for } n > M \end{cases}$$

$$\gamma_{nm} = \frac{2p_m \delta_n}{n\pi} \quad , \quad \alpha_{nm} = \frac{2q_m \delta_n}{n\pi} \tag{6.2-17}$$

$$\delta_n = \begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

In order to ensure physical realism for the initial and boundary value problem, we shall restrict the initial data to that class of functions corresponding to finite total (kinetic plus strain) energy, i.e.:

$$\int_0^1 \left[\dot{u}_0(x)^2 + \left(\frac{\partial^2 u_o^2(x)}{\partial x^2} \right)^2 \right] dx < +\infty \qquad (6.2\text{-}18)$$

The above condition implies that $b_{o(n)}$ and $\dot{b}_{o(n)}$, $n = 1, 2, \ldots$ must satisfy the following conditions:

$$\sum_{n=1}^{\infty} n^4 \pi^4 b_{o(n)}^2 < +\infty \quad , \quad \sum_{n=1}^{\infty} \dot{b}_{o(n)}^2 < +\infty \, . \qquad (6.2\text{-}19)$$

6.3 ERROR ESTIMATES

In this section, estimates for truncation errors for both first and second order systems will be derived.

6.3.1 First Order System

For convenience, the following notation will be adopted: Let $V(t) = (v_1(t), v_2(t), \ldots)$ be any sequence of functions defined for $0 \le t \le T$. We set

$$\|V(t)\|^2 = \sum_{i=1}^{\infty} v_i^2(t) \;\; ; \;\; \|V(t)\|^2_N = \sum_{i=1}^{N} v_i^2(t) \;\; ; \;\; \|V(t)\|^2_{-N} = \sum_{i=N+1}^{\infty} v_i^2(t)$$

(6.3-1)

Further notation will be introduced as required, and will be signalled by (*) in the equation number.

Let there be given a denumerably infinite system of first order ordinary differential equations of the form:

$$\frac{dx_n(t)}{dt} + a_n x_n(t) = \sum_{\substack{m=1 \\ m \ne n}}^{\infty} a_{nm} x_m(t) \quad ; \quad n = 1, 2, \ldots \qquad (6.3\text{-}2)$$

where a_n and a_{nm} are constants. Let the solution to (6.3-2) corresponding to initial condition $X(0) = (x_{o(1)}, x_{o(2)}, \ldots)$ be denoted by $X(t) = (x_1(t), x_2(t), \ldots)$. Also, let $Y_N(t) = (y_{N(1)}(t), \ldots, y_{N(N)}(t))$ be a solution to the truncated system corresponding to (6.3-2) :

$$\frac{dy_{N(n)}(t)}{dt} + a_n y_{N(n)}(t) = \sum_{\substack{m=1 \\ m \neq n}}^{N} a_{nm} y_{N(m)}(t) \quad ; \quad n = 1,2,\ldots,N \qquad (6.3\text{-}2')$$

with initial condition $Y_N(0) = (x_{o(1)}, \ldots, x_{o(N)})$. The problem is to derive an estimate for the error corresponding to the difference between the solutions of (6.3-2) and (6.3-3); i.e.,

$$E(t) = (e_1(t), e_2(t), \ldots) \text{ where } e_n(t) = \begin{cases} x_n(t) - y_{N(n)}(t) & \text{for } 1 \leq n \leq N \\ x_n(t) & \text{for } n > N \end{cases} \qquad (6.3\text{-}3)$$

For the case where the coefficient matrix is a bounded linear operator, the above problem is trivial since the fundamental matrix has the usual exponential form. For the case where the coefficient matrix is unbounded, Lewis[1] and Bellman[2] have obtained results for certain special forms of the coefficient matrix. Here, a more general class of equations having unbounded coefficient matrices will be considered.

Theorem 1: If the initial condition $X(0)$ and the coefficients $\{a_{nm}\}$ and $\{a_n\}$ of (6.3-2) satisfy:

(i) $\|X(0)\| < \infty$,

(ii) $\underset{n}{\text{Min}} \{a_n\} = k_\infty > -\infty$, (6.3-4*)

(iii) $\left[\sum_{\substack{n,m=1 \\ n \neq m}}^{\infty} a_{nm}^2 \right]^{1/2} = A < \infty$,

then the error satisfies the estimate:

$$\|E(t)\| \leq K(N,t) \exp(A - k_\infty) t \qquad (6.3\text{-}5)$$

where

$$K(N,t) = \|X(0)\|_{-N} + A_{-N} \|X(0)\| t \qquad (6.3\text{-}6)$$

and

$$A_{-N} = \left[\sum_{n=N+1}^{\infty} \sum_{m=1}^{N} a_{nm}^2 \right]^{1/2} \qquad (6.3\text{-}7*)$$

Proof: We first obtain an estimate for $\|Y_N(t)\|_N$. In view of (6.3-2'), each component of $Y_N(t)$ satisfies an integral equation

$$y_{N(n)}(t) = \exp(-a_n t)\, x_{o(n)} + \int_o^t \exp(-a_n(t-\tau)) \sum_{\substack{m=1\\m\neq n}}^{N} a_{nm} y_{N(m)}(\tau)\,d\tau \tag{6.3-8}$$

A straightforward computation using Minkowski's inequalities for sums and integrals leads to

$$\|Y_N(t)\|_N \leq \left[\sum_{n=1}^{N} \exp(-2a_n t) x_{o(n)}^2\right]^{1/2} + \int_o^t \left\{\sum_{n=1}^{N} \exp(-2a_n(t-\tau)) \left(\sum_{\substack{m=1\\m\neq n}}^{N} a_{nm} y_{N(m)}(\tau)\right)^2\right\}^{1/2} d\tau \tag{6.3-9}$$

Extracting an exponential term from the right-hand side of (6.3-9) and applying Schwarz' inequality, we have

$$\|Y_N(t)\|_N \leq \exp(-k_N t)\left(\|X(0)\|_N + \int_o^t A_N \|Y_N(\tau)\|_N \exp(k_N \tau)\,d\tau\right) \tag{6.3-10}$$

where

$$k_N = . \min_{n\leq N} \{a_n\}\ , \quad A_N = \left[\sum_{\substack{n,m=1\\n\neq m}}^{N} a_{nm}^2\right]^{1/2} \tag{6.3-11*}$$

Applying Gronwall's lemma[3] to (6.3-10) gives the desired estimate for $Y_N(t)$:

$$\|Y_N(t)\|_N \leq \|X(0)\|_N \exp(A_N - k_N)\, t \tag{6.3-12}$$

We shall now proceed to prove the theorem. Assuming that $X(t)$ is a solution to (6.3-2), the components of $E(t)$ satisfy the following integral equation:

$$e_n(t) = \begin{cases} \displaystyle\int_o^t \exp(-a_n(t-\tau)) \left(\sum_{\substack{m=1\\m\neq n}}^{\infty} a_{nm} e_m(\tau)\right) d\tau & ;\ \text{for } n \leq N \\[2ex] \displaystyle\exp(-a_n t) x_{o(n)} + \int_o^t \exp(-a_n(t-\tau)) \left(\sum_{\substack{m=1\\m\neq n}}^{\infty} a_{nm} e_m(\tau) + \sum_{m=1}^{N} a_{nm} y_{N(m)}(\tau)\right) d\tau & \text{for } n > N \end{cases} \tag{6.3-13}$$

By a series of manipulations identical to those used in deriving (6.3-10), we have

$$\|E(t)\| \le \exp(-k_{-N}t)\ \|X(0)\|_{-N} + \int_0^t A_{-N}\|Y_N(\tau)\|_N \exp(-k_N(t-\tau))\, d\tau$$

$$+ \int_0^t A\|E(\tau)\| \exp(-k_\infty(t-\tau))d\tau \qquad (6.3\text{-}14)$$

where

$$k_{-N} = \underset{n > N}{\mathrm{Min}} \{a_n\} \qquad (6.3\text{-}15^*)$$

Using (6.3-12) to eliminate $\|Y_N(t)\|_N$ from (6.3-14) and applying a slightly modified form of Gronwall's lemma (see Appendix 1) lead to the desired estimate (6.3-5). The convergence of $K(N,t)$ to zero as $N \to \infty$ is an immediate consequence of conditions (i) and (iii) in the theorem. Hence, the proof is complete.

Remark 1: Conditions (i)-(iii) in Theorem 1 are sufficient to ensure existence and uniqueness of solutions to (6.3-2), since we may replace k_N by k_∞, A_N by A, and $\|X(0)\|_N$ by $\|X(0)\|$ in inequality (6.3-10) so that (6.3-12) becomes

$$\|Y_N(t)\|_N \le \|X(0)\| \ \exp(A-k_\infty)\ t \qquad (6.3\text{-}16)$$

and the right-hand side of (6.3-16) is independent of N. It is now a straightforward but tedious argument to show that $\{Y_N\}$ forms a convergent sequence, and that the limit function satisfies (6.3-2). Also, note that the solution X(t) satisfies the estimate (6.3-16).

Remark 2: Error estimates for a non-homogeneous system may be derived in a similar manner. Here, it is necessary to assume that the non-homogeneous terms satisfy a boundedness condition similar to condition (iii) of Theorem 1. For example, if a term $f_n(t)$ is introduced in each equation of (6.3-2), then a condition of the form

$$\left[\sum_{n=1}^{N} f_n^2(t)\right]^{1/2} = F_N \le F < \infty \quad \text{for all } N \qquad (6.3\text{-}17)$$

is needed.

In this case, (6.3-12) becomes

$$\|Y_N(t)\|_N \le \left\{ \|X(0)\|_N + \int_0^t F_N \exp(-(A_N-k_N)\tau)d\tau \right\} \exp(A_N-k_N)t .$$

The coefficient K(N,t) in the error estimate will contain two additional terms, and will tend to zero as $N \to \infty$ for $t \in [0,T]$.

Remark 3: The form of the coefficient corresponding to K(N,t) obtained from applying Gronwall's modified lemma to (6.3-14) is considerably more complex than K(N,t) as given by (6.3-6). If we set $k_N = k_\infty$ for $N \ge N_o$, then the error estimate for $N \ge N_o$ takes on the form:

$$\|E(t)\| \le \left\{ \|X(0)\|_{-N}(AR_1 + \exp(-C_1 t)) + A_N A_{-N} \|X(0)\|_N \left(\frac{R_2-R_1}{C_1-C_2}\right) + A_{-N} \|X(0)\|_N R_1 C_1 \right\} \exp(A-k_\infty)\, t \qquad (6.3\text{-}5')$$

where

$$C_1 = A + k_{-N} - k_\infty \quad , \quad C_2 = A - A_N \quad , \quad C_3 = A - k_\infty \; ,$$

$$R_1 = C_1^{-1}(1-\exp(-C_1 t)) \, , \quad R_2 = C_2^{-1}(1-\exp(-C_2 t)) \quad .$$

It can be shown that the $\{\ldots\}$ term in (6.3-5') is bounded above by K(N,t).

Remark 4: If (6.3-2) is derived from a partial differential equation using the approach discussed in Section 6.2, then it can be readily shown by using Parseval's formula that (6.3-5) corresponds to a L_2 type error estimate for the approximate solution to the original partial differential equation.

6.3.2 Second Order System

The approach used in the previous section can be used to derive error estimates for an infinite system of second order ordinary differential equations.

Consider the system

$$\frac{d^2 x_n(t)}{dt^2} + 2c_n \frac{dx_n(t)}{dt} + a_n x_n(t) = \sum_{m=1}^{\infty} \left(c_{nm} \frac{dx_m(t)}{dt} + a_{nm} x_m(t) \right)$$

$$; \; n = 1,2,\ldots \qquad (6.3\text{-}18)$$

with $c_{nn} = a_{nn} = 0$ and initial conditions

$$x_n(0) = x_{o(n)} \quad , \quad \left.\frac{dx_n(t)}{dt}\right|_{t=0} = \dot{x}_{o(n)} \quad ; \quad n = 1, 2, \ldots \qquad (6.3\text{-}19)$$

and the corresponding truncated system:

$$\frac{d^2 y_{N(n)}(t)}{dt^2} + 2c_n \frac{dy_{N(n)}(t)}{dt} + a_n y_{N(n)}(t)$$
$$\sum_{m=1}^{N} \left(c_{nm} \frac{dy_{N(m)}(t)}{dt} + a_{nm} y_{N(m)}(t) \right)$$
$$; \quad n = 1, \ldots, N \qquad (6.3\text{-}18')$$

with $c_{nn} = a_{nn} = 0$ and initial conditions

$$y_{N(n)}(0) = x_{o(n)} \quad , \quad \left.\frac{dy_{N(n)}(t)}{dt}\right|_{t=0} = \dot{x}_{o(n)}$$
$$; \quad n = 1, \ldots, N \qquad (6.3\text{-}19')$$

Each component $x_n(t)$ of the solution $X(t)$ of (6.3-18) satisfies an integral equation of the form:

$$x_n(t) = \exp(-c_n t) \left(q_n(t) x_{o(n)} + p_n(t) \dot{x}_{o(n)} \right)$$
$$= \int_o^t \exp(-c_n(t-\tau))\, p_n(t-\tau) \left[\sum_{m=1}^{\infty} \left(c_{nm} \frac{dx_m(\tau)}{d\tau} + a_{nm} x_m(\tau) \right) \right] d\tau$$
$$(6.3\text{-}20)$$

where

$$p_n(t) = \lambda_n^{-1} \operatorname{Sinh} \lambda_n t \quad , \quad q_n(t) = c_n \lambda_n^{-1} \operatorname{Sinh} \lambda_n t + \operatorname{Cosh} \lambda_n t$$
$$\text{for } \lambda_n^2 = c_n^2 - a_n > 0$$

$$p_n(t) = \lambda_n^{-1} \operatorname{Sin} \lambda_n t \quad , \quad q_n(t) = c_n \lambda_n^{-1} \operatorname{Sin} \lambda_n t + \operatorname{Cos} \lambda_n t$$
$$\text{for } \lambda_n^2 = a_n - c_n^2 > 0$$

$$p_n(t) = t \quad , \quad q_n(t) = c_n t + 1 \quad \text{for } a_n = c_n^2 .$$

The solutions to (6.3-18') satisfy a similar integral equation with the upper summation limits replaced by N.

Using the notations given in (6.3-1) and (6.3-4) and Appendix 2, an error estimate for the solutions of (6.3-18') can be summarized in the following theorem:

<u>Theorem 2:</u> Let $X(t) = (x_1(t), x_2(t), \ldots)$ and $Y_N(t) = (y_{N(1)}(t),$ $y_{N(2)}(t), \ldots, y_{N(N)}(t))$ be solutions of (6.3-18) and (6.3-18') respectively. If the initial conditions and coefficients of (6.3-18) satisfy:

(i) $\|X(0)\| < \infty$, $\|\dot{X}(0)\| < \infty$

(ii) $a_n \geq -r_a > -\infty$, $|c_n| \leq -r_c < \infty$,

(iii) $A < \infty$, $C < \infty$, $M(T) < \infty$,

where r_a and r_c are positive constants; then the error $E(t) = (e_1(t), e_2(t), \ldots)$ defined by (6.3-3) satisfies an estimate of the form:

$$\|E(t)\|_{AC} \leq \hat{K}(N,T) \exp \hat{k}(N,T)t \quad \text{for } t \in [0,T] \tag{6.3-21}$$

where $\|E(t)\|_{AC}$ is defined by

$$\|E(t)\|_{AC} = A\|E(t)\| + C\left\|\frac{dE(t)}{dt}\right\| , \tag{6.3-22*}$$

and

$$\hat{k}(N,T) = (AP_\infty(T) + C\dot{P}_\infty(T)) - k_\infty$$

$$\hat{K}(N,T) = AQ_{-N}(T)\|X(0)\|_{-N} + CM_{-N}(T) + (AP_{-N}(T) + C\dot{P}_{-N}(T)) \cdot$$
$$\{\|\dot{X}(0)\|_{-N} + (A_{-N}A_N^{-1} + C_{-N}C_N^{-1})(t + \tfrac{1}{4}(A_N P_N(T)$$
$$+ C_N\dot{P}_N(T))^{-1})\ \hat{K}_o(N,T)\}$$

$$\hat{K}_o(N,T) = A_N Q_N(T)\|X(0)\|_N + C_N M_N(T) + (A_N P_N(T)$$
$$+ C_N\dot{P}_N(T))\|X(0)\|_N \tag{6.3-23}$$

<u>Proof:</u> Each component of E(t) satisfies an integral equation of the form:

$$e_n(t) = \begin{cases} \int_o^t \exp(-c_n(t-\tau))p_n(t-\tau)\left[\sum_{m+1}^{\infty}\left(c_{nm}\frac{de_n(\tau)}{d\tau} + a_{nm}e_m(\tau)\right)\right]d\tau \quad \text{for } n \le N \\ \exp(-c_nt)(q_n(t)x_{o(n)} + p_n(t)\dot{x}_{o(n)}) + \int_o^t \exp(-c_n(t-\tau))p_n(t-\tau)\cdot \end{cases}$$

$$\left\{\sum_{m=1}^{\infty}\left(c_{nm}\frac{de_m(\tau)}{d\tau} + a_{nm}e_m(\tau)\right) + \sum_{m=1}^{N}\left(c_{nm}\frac{dy_{N(n)}(\tau)}{d\tau} + a_{nm}y_{N(n)}(\tau)\right)\right\}d\tau \quad \text{for } n > N \tag{6.3-24}$$

It follows that

$$\|E(t)\| \le \exp(-k_{-N}t)\Big\{Q_{-N}(T)\|X(0)\|_{-N} + P_{-N}\|\dot{X}(0)\|_{-N} + \int_o^t \exp(k_{-N}\tau)\Big[P_{-N}(T)\|Y_N(\tau)\|_{NA_{-N}C_{-N}} + \|E(\tau)\|_{AC}P_\infty(T)\exp(k_{-N}-k_\infty)(t-\tau))\Big]d\tau\Big\} \tag{6.3-25}$$

Similarly, by differentiating (6.3-24), we obtain an estimate for dE(t)/dt:

$$\left\|\frac{dE(t)}{dt}\right\| \le \exp(-k_{-N}t)\Big\{M_{-N}(T) + \dot{P}_{-N}(T)\|\dot{X}(0)\|_{-N} + \int_o^t \exp(k_{-N}\tau)\Big[\dot{P}_{-N}(T)\|Y_N(\tau)\|_{NA_{-N}C_{-N}} + \|E(\tau)\|_{AC}\dot{P}_\infty(T)\exp((k_{-N}-k_\infty)(t-\tau))\Big]d\tau\Big\} \tag{6.3-26}$$

Combining (6.3-25) and (6.3-26), we have

$$\|E(t)\|_{AC}\exp(k_{-N}t) \le AQ_{-N}(T)\|X(0)\|_{-N} + CM_{-N}(T) + (AP_{-N}(T) + C\dot{P}_{-N}(T))\Big\{\|\dot{X}(0)\|_{-N} + 2(A_{-N}A_N^{-1} + C_{-N}C_N^{-1})\cdot \int_o^t \exp(k_{-N}\tau)\|Y_N(\tau)\|_{NA_NC_N}d\tau + \int_o^t \exp(k_{-N}\tau)\|E(\tau)\|_{AC}(AP_\infty(T) + C\dot{P}_\infty(T))\exp((k_{-N}-k_\infty)(t-\tau))d\tau\Big\} \tag{6.3-27}$$

where

$$\|Y_N(t)\|_{N^A N^C N} = A_N \|Y_N(t)\|_N + C_N \left\| \frac{dY_N(t)}{dt} \right\|_N \qquad (6.3\text{-}28^*)$$

Following the same approach, it can be shown that $\|Y_N(t)\|_{N^A N^C N}$ satisfies the following inequality:

$$\|Y_N(t)\|_{N^A N^C N} \leq \tilde{K}_0(N,T) \exp \tilde{k}_1(N,T)\, t \qquad (6.3\text{-}29^*)$$

where $K_0(N,T)$ is defined in (6.3-23), and

$$\tilde{k}_1(N,T) = A_N P_N(T) + C_N \dot{P}_N(T) - k_N \; . \qquad (6.3\text{-}30^*)$$

The desired estimate (6.3-21) can be obtained by substituting (6.3-29) into (6.3-27) and applying Gronwall's lemma. This completes the proof.

Remark 5: For the second order system considered here, statements analogous to those of Remarks 1-4 can be made. Here, the expression corresponding to (6.3-5') is considerably more complex than K(N,T) as given by (6.3-23), and will not be given here.

6.4 NUMERICAL EXAMPLE

Here, the system given by Example 1 of Section 6.2 will be considered. Let

$$w_N(t,x) = \sum_{n=1}^{N} a_{N(n)}(t) \sin n\pi x \qquad (6.4\text{-}1)$$

where $a_{N(n)}(t)$ is the nth component of the solution of the truncated system corresponding to (6.2-5). Also, we set

$$u_N(t,x) \equiv w_N(t,x) + (1-K)^{-1} x \int_0^1 g(\xi)\, w_N(t,\xi)\, d\xi \; . \qquad (6.4\text{-}2)$$

Thus,

$$\|u(t,x)-u_N(t,x)\| = \left[\int_0^1 |u(t,x)-u_N(t,x)|^2\, dx \right]^{1/2} \leq \|w(t,x)-w_N(t,x)\| + 3^{-1/2}\, |1-k|^{-1}\, |v_N(t)| \qquad (6.4\text{-}3)$$

where

$$v_N(t) = \int_0^1 g(\xi)(w(t,\xi)-w_N(t,\xi))d\xi \leq \|g(x)\| \; \|w(t,x)-w_N(t,x)\| \qquad (6.4\text{-}4)$$

It follows from Parseval's formula that

$$\|u(t,x)-u_N(t,x)\| \le \|\underline{a}(t)-\underline{a}_N(t)\| \quad (1 + \sqrt{3}\ |1-K|)^{-1} \|g(x)\|) \tag{6.4-5}$$

where $\underline{a}(t) = (a_1(t), a_2(t), \ldots)$ and $\underline{a}_N(t) = (a_1(t), \ldots, a_N(t))$.

Applying Theorem 1, we have the following estimate for the error due to approximation:

$$\|u(t,x)-u_N(t,x)\| \le \{ \|\underline{a}(0)\|_{-N} + A_{-N}\|\underline{a}(0)\|\ t \} \ .$$

$$(1 + (\sqrt{3}\ |1-K|)^{-1} \|g(x)\|)\exp(A-k_\infty)\ t \tag{6.4-6}$$

To check the sharpness of the error estimate, we consider the special case where

$$g(\chi) = \frac{3\pi}{4} \sin \pi\chi \quad , \quad u_0(x) = \sin(\pi x/2) \tag{6.4-7}$$

The system (6.2-5) then becomes:

$$\frac{da_1(t)}{dt} + \frac{\pi^2}{4} a_1(t) = 0 \ , \tag{6.4-8a}$$

$$\frac{da_n(t)}{dt} + n^2\pi^2 a_n(t) = - \frac{3\pi^2(-1)^n}{4n} a_1(t) \quad ; \quad n = 2,3,\ldots \tag{6.4-8b}$$

$$a_n(0) = \frac{(-1)^{n+1}}{n\pi(4n^2-1)} \quad , \quad n = 1,2,\ldots \tag{6.4-8c}$$

and

$$K = \int_0^1 \xi g(\xi)\, d\xi = 3/4 \neq 1$$

The above condition permits the solution representation in the form of (6.2-9).

Now, the terms in the error estimate (6.4-6) can be easily computed; viz.,

$$\|\underline{a}(0)\| = \frac{1}{\pi} \left[\sum_{n=1}^{\infty} n^{-2}(4n^2-1)^{-2} \right]^{1/2} = \left[\frac{5}{12} - \frac{4}{\pi^2} \right]^{1/2} \ ,$$

$$\|\underline{a}(0)\|_{-N} = \frac{1}{\pi}\left[\sum_{n=N+1}^{\infty} n^{-2}(4n^2-1)^{-2}\right]^{1/2} < \left[\frac{1}{8\pi^2 N^5}\right]^{1/2} ,$$

$$A = \frac{3\pi^2}{4}\left[\sum_{n=2}^{\infty} n^{-2}\right]^{1/2} = \frac{3\pi^2}{4}\left[\frac{\pi^2}{6} - 1\right]^{1/2} ,$$

$$A_{-N} = \frac{3\pi^2}{4}\left[\sum_{n=N+1}^{\infty} n^{-2}\right]^{1/2} \leq \frac{3\pi^2}{4\sqrt{N}} ,$$

$$k_\infty = \text{Min}\left\{\frac{\pi^2}{4}, \quad \text{Min}\{i^2\pi^2\}\right\} = \frac{\pi^2}{4} ,$$

$$\|g(x)\| = \left[\int_0^1 |g(\xi)|^2 d\xi\right]^{1/2} = \frac{3\pi}{4}\left[\int_0^1 \sin^2\pi\xi \, d\xi\right]^{1/2} = \frac{3\pi}{4\sqrt{2}} .$$

Thus,

$$\|u(t,x)-u_N(t,x)\| \leq \frac{1}{\sqrt{N}}\left\{\frac{1}{2\pi N^2\sqrt{2}} + \frac{3\pi^2}{4}\left(\frac{5}{12} - \frac{4}{\pi^2}\right)^{1/2} t\right\} .$$
$$\left\{1 + \frac{3\pi}{4\sqrt{6}}\right\} \exp\left[\frac{3\pi^2}{4}\left(\frac{\pi^2}{6} - 1\right)^{1/2} \frac{\pi^2}{4}\right] t$$

(6.4-9)

On the other hand, (6.4-8) can be solved explicitly be first solving for $a_1(t)$:

$$a_1(t) = \frac{1}{3\pi} \exp\left(-\frac{\pi^2}{4} t\right) . \qquad (6.4\text{-}10)$$

Substituting (6.4-10) into (6.4-8b), we have

$$\frac{da_n(t)}{dt} + n^2\pi^2 a_n(t) = \frac{\pi(-1)^{n+1}}{4n} \exp(-\pi^2 t/4) \quad ; \quad n = 2,3,\ldots$$

(6.4-11)

The solution to (6.4-11) is

$$a_n(t) = \frac{(-1)^{n+1}}{n(4n^2-1)\pi} \exp(-\pi^2 t/4) \quad ; \quad n = 2,3,\ldots \qquad (6.4\text{-}12)$$

By direct computation,

$$\|\underline{a}(t)-\underline{a}_N(t)\| = \|\underline{a}(0)\|_{-N} \cdot \exp(-\pi^2 t/4) \qquad (6.4\text{-}13)$$

Now, the estimate for the error obtained by applying Theorem 1 is given by:

$$\|\underline{a}(t) - \underline{a}_N(t)\| \le \left\{ \|\underline{a}(0)\|_{-N} - \frac{3\pi^2}{4}\left(\sum_{n=N+1}^{\infty} n^{-2}\right)^{1/2}\left(\frac{5}{12} - \frac{4}{\pi^2}\right)^{1/2} t \right\}$$

$$\exp\left[\frac{3\pi}{4}\left(\frac{\pi^2}{6} - 1\right)^{1/2} - \frac{\pi^2}{4}\right] t \qquad (6.4\text{-}14)$$

The difference between the estimate (6.4-14) and the exact result (6.4-13) can be interpretated as follows: The $\{\ldots\}$ term in (6.4-14) is an estimate of the effect of the first N components of the solution on the remaining components through the coupling of the general system; the $[\ldots]$ term in the exponential coefficient of (6.4-14) is an estimate of the over-all coupling effect in the general system. In this special case of the example, our choice of g as given by (6.4-7) effectively decouples the system equations as apparent from (6.4-11).

6.5 CONCLUDING REMARKS

The straightforward truncation of an infinite dimensional system of linear ordinary differential equations considered here is certainly the simplest possible form of approximation. However, this approach has been commonly used in the engineering literature without consideration of the errors involved. From the mathematical standpoint, we have demonstrated in effect that the first and second order infinite system of linear ordinary differential equations under consideration satisfy the "principe des réduites."

6.6 APPENDIX 1

Gronwall's Lemma (slightly modified): Let u(t) satisfy the following integral inequality:

$$u(t) \le \delta + \int_0^t (u(\tau)\, A \exp k(t-\tau) + v(\tau))\, d\tau \qquad (6.6\text{-}1)$$

where A and k are constants with $A > 0$, and v is a specified function of τ. Then,

$$u(t) \le (\delta + \hat{v}(t)) \exp(A + k)\, t \qquad (6.6\text{-}2)$$

where

$$\hat{v}(t) = \int_0^t \left[v(\tau) - k(\delta + \int_0^\tau v(s)\, ds) \right] \exp(-A+k)\, \tau\, d\tau\ . \qquad (6.6\text{-}3)$$

Proof: Let w(t) equal to the right-hand side of (6.6-1). Then,

$$u(t) \le w(t) \quad , \qquad w(0) = \delta \qquad (6.6\text{-}4)$$

and

$$\frac{dw(t)}{dt} = A\, u(t) + v(t) + \int_0^t k u(\tau)\, A \exp k(t-\tau)\, d\tau \qquad (6.6\text{-}5)$$

In view of (6.6-1) and (6.6-4), we have

$$\frac{dw(t)}{dt} - (A+k)\, w(t) \le v(t) - k\left(\delta + \int_0^t v(\tau)\, d\tau\right). \qquad (6.6\text{-}6)$$

It follows that

$$\frac{d}{dt}\left[w(t) \exp(-(A+k)t)\right] \le \exp(-(A+k)\, t) \left[v(t) - k\left(\delta + \int_0^t v(\tau)\, d\tau\right)\right]$$

$$(6.6\text{-}7)$$

Direct integration leads to the desired estimate (6.6-2). Hence, the proof is complete.

6.7 APPENDIX 2

Notation for the second order system:

(i) $$A = \left[\sum_{n,m=1}^{\infty} a_{nm}^2 \right]^{1/2}, \quad A_N = \left[\sum_{n,m=1}^{N} a_{nm}^2 \right]^{1/2}, \quad A_{-N} \left[\sum_{n,m=1}^{\infty} a_{nm}^2 \right]^{1/2} \qquad (6.7\text{-}1)$$

(ii) $$C = \left[\sum_{n,m=1}^{\infty} c_{nm}^2 \right]^{1/2}, \quad C_N = \left[\sum_{n,m=1}^{N} c_{nm}^2 \right]^{1/2}, \quad C_{-N} = \left[\sum_{n,m=1}^{\infty} c_{nm}^2 \right]^{1/2} \qquad (6.7\text{-}2)$$

(iii) $$k_N = \underset{n \le N}{\text{Min}} \{c_n\}, \quad k_{-N} = \underset{n > N}{\text{Min}} \{c_n\}, \quad k_\infty = \text{Min}\{k_N, k_{-N}\} \qquad (6.7\text{-}3)$$

(iv) $$P_N(T) = \text{Max}\{|p_n(t)| : n \le N, 0 \le t \le T\},$$

$$P_{-N}(T) = \sup\{|p_n(t)| : n > N, 0 \le t \le T\},$$

$$P_\infty(T) = \text{Max}\{P_N(T), P_{-N}(T)\} \qquad (6.7\text{-}4)$$

(v) $$Q_N(T) = \text{Max}\{|q_n(t)| : n \le N, 0 \le t \le T\},$$

$$Q_{-N}(T) = \sup\{|q_n(t)| : n > N, 0 \le t \le T\}, \qquad (6.7\text{-}5)$$

(vi) $$\dot{P}_{N(T)} = \left\{ \text{Max} \left| \frac{dp_n(t)}{dt} - c_n p_n(t) \right| : n \le N, 0 \le t \le T \right\},$$

$$\dot{P}_{-N}(T) = \sup\left\{ \left| \frac{dp_n'(t)}{dt} - c_n p_n(t) \right| : n > N, 0 \le t \le T \right\},$$

$$\dot{P}_\infty(T) = \text{Max}\{\dot{P}_N(T), \dot{P}_{-N}(T)\} \qquad (6.7\text{-}6)$$

(vii) $$M_N(T) = \underset{t \in [0,T]}{\text{Max}} \left[\sum_{n=1}^{N} (a_n p_n(t)\, x_{o(n)})^2 \right]^{1/2},$$

$$M_{-N}(T) = \underset{t \in [0,T]}{\sup} \left[\sum_{n>N}^{\infty} (a_n p_n(t) x_{o(n)})^2 \right]^{1/2}$$

$$M(T) = \text{Max}\{M_N(T), M_{-N}(T)\} \qquad (6.7\text{-}7)$$

6.8 REFERENCES

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