DYNAMIC STIFFNESS MATRIX FORMULATION BY MEANS OF HERMITIAN POLYNOMIALS

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Hermitian polynomials are used for the approximation of the deformation of structural components, such as beam-and plate - elements. In the case of one-dimensional problems (bars, beam, arches, frames) the establishment of the elastic stiffness matrix and of the mass matrix is straightforward, if the order of the Hermitian polynomials is equal to the order of the differential equation for the structural component. However, higher-order Hermitian polynomials can be employed in order to obtain the dynamic stiffness matrix for such structural element. Due to the matrix condensation, necessary in this case, the dynamic stiffness matrix can no longer be split into separate mass- and elastic matrices. It has been found that the use of higher-order polynomials yields more accurate results at less computational expense compared with the procedure where the structure is divided into more individual segments whose stiffness and mass matrices are established by means of polynomials of order equal to that of the differential equation. The application to plates is somewhat more intricate, since the condensation technique is somewhat problematic. Two ways of solving this dilemma are indicated.

INTRODUCTION

Matrix Methods require, by their very nature, the discretization of the structure under consideration into a finite number of components. The inertial and elastic properties of each component can be fully described by means of its transfer matrix, or by its dynamic stiffness matrix, or by its dynamic flexibility matrix. From the treatment of beam problems we know that the elements of these matrices consist of transcendental functions whose arguments reflect the inertial and elastic properties of the structural component. Thus, whenever the exact matrix representation is possible, we find that we cannot split such matrix into separate mass and elastic matrices.

Separation becomes possible only when in the approximation of the deformation of the structural component the number of free constants assembled in the vector $\mathbf{a}$ is equal to the number of forces $\mathbf{p}$ (and, likewise, to the number of the associated displacements $\mathbf{u}$) applied to the boundaries of the structural component. Then we have the linear relationship

$$\mathbf{u} = \mathbf{L}\,\mathbf{a}$$

where $\mathbf{L}$ is a square matrix and it is possible to establish the matrix equation

$$\mathbf{p} = \mathbf{S}\,\mathbf{u} = (-\omega^2 \mathbf{M} + \mathbf{K})\,\mathbf{u}$$

where $\mathbf{S}$ is called the dynamic stiffness matrix, while $\mathbf{M}$ and $\mathbf{K}$ are mass and elastic stiffness

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matrices respectively. However, when the number of the free constants $\mathbf{a}$ is larger than the number of forces $\mathbf{p}$ applied to the boundaries of the structural component, then the separation of the dynamic stiffness matrix $\mathbf{S}$ into separate mass and stiffness matrices is no longer possible. Let the vector $\mathbf{u}$ denote the displacements associated with the forces $\mathbf{p}$ and let the vector $\mathbf{v}$ denote the boundary displacements (higher derivatives) to which there are no associate boundary forces. Then we can find the following relationship

$$\begin{bmatrix} \mathbf{p} \\ \hline \mathbf{o} \end{bmatrix} \quad \begin{bmatrix} \mathbf{D}_1 & \mathbf{D}_2 \\ \hline \mathbf{D}_3 & \mathbf{D}_4 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u} \\ \hline \mathbf{v} \end{bmatrix}$$

which upon elimination of $\mathbf{v}$ leads to the matrix equation

$$\mathbf{p} = \mathbf{S}\,\mathbf{u}$$

where

$$\mathbf{S} = \mathbf{D}_1 - \mathbf{D}_2 \mathbf{D}_4^{-1} \mathbf{D}_3$$

is the dynamic stiffness matrix which now no longer can be split into separate mass and stiffness matrices. These dynamic stiffness matrices of the individual structural components are then assembled in the usual manner to obtain the dynamic stiffness matrix of the whole structure. It has been found that the approximate formulation of the individual dynamic stiffness matrix for bar-, beam-, rectangular and parallelogram disc- and plate-elements can be effectively accomplished by means of Hermitian polynomials.

HERMITIAN POLYNOMIALS

For the benefit of those unacquainted with the nature of Hermitian polynomials (Reference 1) a short summary of their properties may be injected. We define a polynomial $w(\xi)$ $(o \le \xi \le 1)$ as follows

$$w(\xi) = q_1\, p_1(\xi) + q_2\, p_2(\xi) + \dots + q_m\, p_m(\xi) + + q_{m+1}\, p_{m+1}(\xi) + \dots + q_{2m}\, p_{2m}(\xi) \tag{1}$$

If we choose

$$q_1 = w(o) = w_o\,,\quad q_2 = w'(o) = w'_o \;\dots;\quad q_{m+1} = w(1) = w_1\,,\; q_{m+2} = w'(1) = w'_1 \tag{2}$$

where the primes denote derivatives with respect to ξ, the polynomials $p_k(\xi)$ (k=1,2,. . .,2m) become Hermitian polynomials, and we can find their structure in the following manner. For example, for $\xi = o$ we deduce from Equations 1 and 2 immediately that

$$p_1(o) = 1\;;\; p_2(o) = o\;;\; p_3(o) = o \dots;\; p_{2m}(o) = o$$

Furthermore, differentiating Equation 1 with respect to ξ, we find in view of Equations 2 for $\xi = o$

$$w'(o) = w_o\, p'_1(o) + w'_o\, p'_2(o) + \dots + w_1^{(m-1)}\, p'_{2m}(o)$$

and therefore we conclude that

$$p'_1(o) = o\;;\; p'_2(o) = 1\;;\; p'_3(o) = o\;;\; \dots,\; p'_{2m}(o) = o$$

In this way we obtain $(2m)^2$ equations, and if we define the Hermitian polynomials $p_k(\xi)$ (k=1,2, . . . ,2m) as

$$p_k(\xi) = h_{k1} + h_{k2}\xi + h_{k3}\xi^2 + \ldots + h_{k,2m}\xi^{2m-1}$$

we realize that the above $(2m)^2$ equations suffice to compute the $(2m)^2$ coefficients $h_{k\ell}$ (k,ℓ = 1,2, . . . , 2m) in the 2m Hermitian polynomials $p_k(\xi)$. Since the polynomials $p_k(\xi)$ contain 2m terms each, they are said to be of order 2m. Consider the two cases that 2m = 2 and 2m = 4. Now it is obvious that in the first case

$$p_1(\xi) = h_{11} + h_{12}\xi \qquad (2m = 2)$$

and in the second

$$p_1(\xi) = h_{11} + h_{12}\xi + h_{13}\xi^2 + h_{14}\xi^3 \qquad (2m = 4)$$

Hence we find that $p_k(\xi)$ is different for different orders of the polynomial, and thus we shall from now on also denote the order of the Hermitian polynomial as follows,

$$p_k(\xi) \quad \overset{2m}{H}_k(\xi) \tag{3}$$

For example, in the case 2m = 4 the four Hermitian polynomials (k=1,2,3,4) are written

$$\begin{aligned} \overset{4}{H}_1(\xi) &= h_{11} + h_{12}\xi + h_{13}\xi^2 + h_{14}\xi^3 \\ \overset{4}{H}_2(\xi) &= h_{21} + h_{22}\xi + h_{23}\xi^2 + h_{24}\xi^3 \\ \overset{4}{H}_3(\xi) &= h_{31} + h_{32}\xi + h_{33}\xi^2 + h_{34}\xi^3 \\ \overset{4}{H}_4(\xi) &= h_{41} + h_{42}\xi + h_{43}\xi^2 + h_{44}\xi^3 \end{aligned} \tag{4}$$

This set of equations may then be formulated in matrix notation:

$$\overset{4}{h}(\xi) = \begin{bmatrix} \overset{4}{H}_1 \\ \overset{4}{H}_2 \\ \overset{4}{H}_3 \\ \overset{4}{H}_4 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \xi \\ \xi^2 \\ \xi^3 \end{bmatrix} \tag{5}$$

As shown above, we have $(2m)^2$ = 16 equations to determine the 16 coefficients $h_{k\ell}$. Thereby we obtain the Hermitian coefficient matrix

$$\overset{4}{H} = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{6}$$

Likewise we may compute Reference 1 the Hermitian coefficient matrices for the cases 2m = 2, 2m = 6, and 2m = 8

$$\overset{2}{H} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{7}$$

$$\overset{6}{H} = \begin{bmatrix} 1 & 0 & 0 & -10 & 15 & -6 \\ 0 & 1 & 0 & -6 & 8 & -3 \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 10 & -15 & 6 \\ 0 & 0 & 0 & -4 & 7 & -3 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{2}{2} & \frac{1}{2} \end{bmatrix} \tag{8}$$

$$\overset{8}{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & -35 & 84 & -70 & 20 \\ 0 & 1 & 0 & 0 & -20 & 45 & -36 & 10 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{10}{2} & \frac{20}{2} & -\frac{15}{2} & \frac{4}{2} \\ 0 & 0 & 0 & \frac{1}{6} & -\frac{4}{6} & \frac{6}{6} & -\frac{4}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 35 & -84 & 70 & -20 \\ 0 & 0 & 0 & 0 & -15 & 39 & -34 & 10 \\ 0 & 0 & 0 & 0 & \frac{5}{2} & -\frac{14}{2} & \frac{13}{2} & -\frac{4}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{6} & \frac{3}{6} & -\frac{3}{6} & \frac{1}{6} \end{bmatrix} \tag{9}$$

From these matrices we read, for example, that

$$\overset{8}{H}_3 = \frac{1}{2}\left(\xi^2 - 10\,\xi^4 + 20\,\xi^5 - 15\,\xi^6 + 4\,\xi^7\right)$$

Figures 1,2,3 and 4 depict the Hermitian polynomials of second, fourth, sixth, and eighth order, respectively. In view of Equations 1,2,3 and 5 the Hermitian polynomials are then assembled to form $w(\xi)$ as follows:

$$w(\xi) = w^T \overset{2m}{h}(\xi) \tag{10}$$

where

$$\mathbf{w}^T = \left[w_0 \; w_0' \cdots w_0^{(m-1)} \; ; \; w_1 \; w_1' \cdots w_1^{(m-1)} \right] \tag{11}$$

Formulation of the Individual Dynamic Stiffness Matrix.

The formulation of the individual dynamic stiffness matrix is most easily accomplished by means of Lagrange's Equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k \tag{12}$$

where L = T - U is the so-called Lagrangean with T as the kinetic energy and U as the potential energy, while q_k is the k-th generalized coordinate, and Q_k the k-th generalized force. For example, let us compute the dynamic stiffness matrix relation for a straight beam element i (Figure 5) in harmonic vibration of circular frequency ω. Then we have

$$T = \frac{1}{2}\omega^2 \int_0^{\ell_i} \mu \; w^2 dx \; ; \qquad U = \frac{1}{2} \int_0^{\ell_i} EI \left(\frac{d^2 w}{dx^2}\right)^2 dx$$

or with $\xi = \frac{x}{\ell_i}$

$$T = \frac{1}{2}\omega^2 \ell_i \int_0^1 \mu \; w^2 d\xi \quad \text{and} \quad U = \frac{1}{2}\frac{1}{\ell_i^3}\int_0^1 EI \, (w'')^2 d\xi \tag{13}$$

Approximating the deflection shape $w(\xi)$ by a Hermitian polynomial of fourth order (cf. Equations 4, 6 and 10)

$$\begin{aligned} w(\xi) = \mathbf{w}^T \mathbf{h}^4(\xi) = w_i(1-3\xi^2+2\xi^3) + w_i'(\xi-2\xi^2+\xi^3) \\ + w_{i+1}(3\xi^2-2\xi^3) + w_{i+1}'(-\xi^2+\xi^3) \end{aligned} \tag{14}$$

and using w_i, w_i', w_{i+1} and w_{i+1}' as the four generalized coordinates q_1, q_2, q_3, and q_4 respectively, we obtain the following four Lagrangean equations

$$\begin{aligned}
&\frac{1}{\ell_i^3}\int_0^1 EI\, w'' \frac{\partial w''}{\partial w_i} d\xi - \ell_i \omega^2 \int_0^1 \mu \, w \frac{\partial w}{\partial w_i} d\xi = Q_1 = V_i + \ell_i \int_0^1 f_i \frac{\partial w}{\partial w_i} d\xi \\
&\frac{1}{\ell_i^2}\int_0^1 EI\, w'' \frac{\partial w''}{\partial w_i'} d\xi - \ell_i^2 \omega^2 \int_0^1 \mu \, w \frac{\partial w}{\partial w_i'} d\xi = Q_2 = M_i + \ell_i^2 \int_0^1 f_i \frac{\partial w}{\partial w_i'} d\xi \\
&\frac{1}{\ell_i^3}\int_0^1 EI\, w'' \frac{\partial w''}{\partial w_{j+1}} d\xi - \ell_i \omega^2 \int_0^1 \mu \, w \frac{\partial w}{\partial w_{i+1}} d\xi = Q_3 = V_{i+1} + \ell_i \int_0^1 f_i \frac{\partial w}{\partial w_{i+1}} d\xi \\
&\frac{1}{\ell_i^2}\int_0^1 EI\, w'' \frac{\partial w''}{\partial w_{i+1}'} d\xi - \ell_i^2 \omega^2 \int_0^1 \mu \, w \frac{\partial w}{\partial w_{i+1}'} d\xi = Q_4 = M_{i+1} + \ell_i^2 \int_0^1 f_i \frac{\partial w}{\partial w_{i+1}'} d\xi
\end{aligned} \tag{15}$$

In these equations V_i, M_i, V_{i+1}, M_{i+1} are the boundary forces, and $f_i(\xi)$ describes the harmonic force/unit length of frequency ω applied to the beam element i. Let us consider the special case that only boundary forces V_i, M_i, V_{i+1} and M_{i+1} are applied, and evaluate the integrals in the above equations for EI = const. and μ = const. Then we obtain readily with $\omega^2 \mu \ell_i^4/EI = \beta^4$, $V\ell^3/EI = \overline{V}$ and $M\ell^2/EI = \overline{M}$ the four Lagrangean Equations (15) in matrix notation (cf. Equations (14) and (4))

$$w^T \int_0^1 \overset{4}{h}'' \overset{4}{H}_1'' \, d\xi - \beta^4 w^T \int_0^1 \overset{4}{h} \overset{4}{H}_1 \, d\xi = \overline{V}_i$$

$$w^T \int_0^1 \overset{4}{h}'' \overset{4}{H}_2'' \, d\xi - \beta^4 w^T \int_0^1 \overset{4}{h} \overset{4}{H}_2 \, d\xi = \overline{M}_i \tag{16}$$

$$w^T \int_0^1 \overset{4}{h}'' \overset{4}{H}_3'' \, d\xi - \beta^4 w^T \int_0^1 \overset{4}{h} \overset{4}{H}_3 \, d\xi = \overline{V}_{i+1}$$

$$w^T \int_0^1 \overset{4}{h}'' \overset{4}{H}_4'' \, d\xi - \beta^4 w^T \int_0^1 \overset{4}{h} \overset{4}{H}_4 \, d\xi = \overline{M}_{i+1}$$

or with $p = \{ \overline{V}_i \; \overline{M}_i \; \overline{V}_{i+1} \; \overline{M}_{i+1} \}$ in one matrix equation

$$\left(\int_0^1 \overset{4}{h}'' (\overset{4}{h}'')^T \, d\xi - \beta^4 \int_0^1 \overset{4}{h} (\overset{4}{h})^T \, d\xi \right) w = p \tag{17}$$

Putting

$$\int_0^1 \overset{4}{h}'' (\overset{4}{h}'')^T \, d\xi = K_i \quad \text{(dimensionless stiffness matrix)} \tag{18}$$

and

$$\int_0^1 \overset{4}{h} (\overset{4}{h})^T \, d\xi = M_i \quad \text{(dimensionless mass matrix)} \tag{19}$$

we may also write

$$(K_i - \beta^4 M_i) w = p \tag{20}$$

The evaluation of Equation 17* yields the well known result (Reference 2)

$$K_i = \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

*The integrals $\int_0^1 h(h)^T d\xi$ etc. have been evaluated by S. Falk for polynomials up to order eight (Reference 1).

and

$$\mathbf{M}_i = \frac{1}{420}\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}.$$

If we employ Hermitian polynomials of sixth order, we obtain naturally matrices **K** and **M** of sixth order:

$$\mathbf{K} = \frac{1}{70}\begin{bmatrix} 1200 & 600 & 30 & -1200 & 600 & -30 \\ 600 & 384 & 22 & -600 & 216 & -8 \\ 30 & 22 & 6 & -30 & 8 & 1 \\ -1200 & -600 & -30 & 1200 & -600 & 30 \\ 600 & 216 & 8 & -600 & 384 & -22 \\ -30 & -8 & 1 & 30 & -22 & 6 \end{bmatrix}$$

$$\mathbf{M} = \frac{1}{55440}\begin{bmatrix} 21720 & 3732 & 281 & 6000 & -1812 & 181 \\ 3732 & 832 & 69 & 1812 & -532 & 52 \\ 281 & 69 & 6 & 181 & -52 & 5 \\ 6000 & 1812 & 181 & 21720 & -3732 & 281 \\ -1812 & -532 & -52 & -3732 & 832 & -69 \\ 181 & 52 & 5 & 281 & -69 & 6 \end{bmatrix}$$

with the displacement vector

$$\mathbf{w} = \left\{ w_i \quad w_i' \quad w_i'' \; ; \; w_{i+1} \quad w_{i+1}' \quad w_{i+1}'' \right\}$$

and the force vector

$$\mathbf{p} = \left\{ \overline{V}_i \quad \overline{M}_i \quad 0 \; ; \; \overline{V}_{i+1} \quad \overline{M}_{i+1} \quad 0 \right\}$$

The third and sixth elements in the force vector are zero, because the virtual work of $\overline{V}_i$, $\overline{M}_i$, $\overline{V}_{i+1}$ and $\overline{M}_{i+1}$ is zero for the virtual displacements $q_3 = \delta w_i''$ and $q_6 = \delta w_{i+1}''$. However this is not the case, if there is a load $f(\xi)$ between points i and i+1. Then, for example, the third and sixth component in the force vector would be

$$\overline{Q}_3 = \frac{\ell_i^4}{EI}\int_0^1 f(\xi)\,\frac{\partial w}{\partial w_i''}\,d\xi \quad \text{and} \quad \overline{Q}_6 = \frac{\ell_i^4}{EI}\int_0^1 f(\xi)\,\frac{\partial w}{\partial w_{i+1}''}\,d\xi \text{, respectively.}$$

Likewise the use of eighth - order Hermitian polynomials leads of **K** - and **M** - matrices of order eight:

$$
\mathbf{K} = \frac{1}{13860}\begin{bmatrix}
352800 & 176400 & 16800 & 630 & -352800 & 176400 & -16800 & 630 \\
176400 & 108000 & 11370 & 480 & -176400 & 68400 & -5430 & 150 \\
16800 & 11370 & 3000 & 140 & -16800 & 5430 & -30 & -25 \\
630 & 480 & 140 & 8 & -630 & 150 & 25 & -3 \\
-352800 & -176400 & -16800 & -630 & 352800 & -176400 & 16800 & -630 \\
176400 & 68400 & 5430 & 150 & -176400 & 108000 & -11370 & 480 \\
-16800 & -5430 & -30 & 25 & 16800 & -11370 & 3000 & -140 \\
630 & 150 & -25 & -3 & -630 & 480 & -140 & 8
\end{bmatrix}
$$

$$
\mathbf{M} = \frac{1}{12972960}\begin{bmatrix}
5251680 & 978480 & 98640 & 4596 & 1234800 & -411480 & 55800 & -3126 \\
978480 & 237600 & 26460 & 1296 & 411480 & -134280 & 17910 & -990 \\
98640 & 26460 & 3096 & 156 & 55800 & -17910 & 2358 & -129 \\
4596 & 1296 & 156 & 8 & 3126 & -990 & 129 & -7 \\
1234800 & 411480 & 55800 & 3126 & 5251680 & -978480 & 98640 & -4596 \\
-411480 & -134280 & -17910 & -990 & -978480 & 237600 & -26460 & 1296 \\
55800 & 17910 & 2358 & 129 & 98640 & -26460 & 3096 & -156 \\
-3126 & -990 & -129 & -7 & -4596 & 1296 & -156 & 8
\end{bmatrix}
$$

with the displacement vector

$$\mathbf{w} = \left\{ w_i \;\; w_i' \;\; w_i'' \;\; w_i''' \;;\;\; w_{i+1} \;\; w_{i+1}' \;\; w_{i+1}'' \;\; w_{i+1}''' \right\}$$

and the force vector (no load between points i and i+ 1)

$$\mathbf{p} = \left\{ \bar{V}_i \;\; \bar{M}_i \;\; 0 \;\; 0 \;;\;\; \bar{V}_{i+1} \;\; \bar{M}_{i+1} \;\; 0 \;\; 0 \right\}$$

In case we use Hermitian polynomials of order 2m larger than the number of non-zero components of the force vector **p** there are two ways to proceed further, in order to determine the natural frequencies of the whole structure:

(a) Assemble the individual **K** - and **M**-matrices in the usual manner. Consider the boundary conditions of the whole structure and evaluate the frequency determinant. This is, in effect, equivalent to a matricized Ritz-procedure, considering derivatives up to the order m-1.

(b) Condense the matrix relation for each individual component after splitting the displacement vector

$$\mathbf{w} = \left[\mathbf{u}^T \;,\; \mathbf{v}^T \right]$$

as pointed out in the Introduction, and rearranging $K_i - \beta^4 M_i = D_i$ accordingly:

$$\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}_i \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix} \qquad (21)$$

Since

$$D_3 \, \underline{u} + D_4 \, v = 0$$

we have

$$v = -D_4^{-1} \, D_3 \, u$$

and thus obtain the dynamic stiffness matrix relationship

$$p = S \, u$$

where

$$S = D_1 - D_2 \, D_4^{-1} \, D_3 \qquad (22)$$

is the dynamic stiffness matrix, which no longer may be split into separate mass- and stiffness matrices, as was the case when the order 2m of the Hermitian polynomial was equal to the number of nonzero components of the force vector p. Thereafter the dynamic stiffness matrices S are assembled to yield a smaller overall dynamic stiffness matrix compared with procedure (a).

The comparison of procedures (a) and (b) as applied to beam vibrations has shown that (b) yields better results in shorter time. Furthermore it was found that the use of Hermitian polynomials of order higher than four resulted in more accurate eigenfrequencies at less computational expense. In Figures 6 to 13, graphs are presented that permit the comparison of the accuracy achieved in using Hermitian polynomials of fourth, sixth, and eighth order with the number w of segments (individual beam elements) as parameter. In the treatment of forced vibration Equation 21 is replaced by

$$\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \qquad (23)$$

With

$$D_3 \, u + D_4 \, v = q$$

or

$$v = D_4^{-1} \, q - D_4^{-1} \, D_3 \, u$$

we obtain the relationship

$$(D_1 - D_2 \, D_4^{-1} \, D_3) \, u = S \, u = p - D_2 \, D_4^{-1} \, q = f \qquad (24)$$

For the given forcing frequency ω the dynamic stiffness matrix S and the new force vector f

can be computed for each structural component. The shearing forces and the bending moments drop out (action = reaction), when the individual matrix relationships are assembled so that only the given external load is considered in the computation of **f**.

USE OF HERMITIAN POLYNOMIALS IN DISC AND PLATE PROBLEMS

In two-dimensional problems, such as discs, plates, and folding structures the application of Hermitian polynomials is straightforward, when the problem can be expressed in rectangular, polar, or skew-angular coordinates. Let us discuss solely the case of free vibration of rectangular plates. Here the Lagrangean is (Kirchhoff's theory)

$$L = \frac{1}{2} D \int_A (\Delta w)^2 \, dA - D(1-\nu) \int_A \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \, \partial y} \right)^2 \right] dA - \omega^2 \int_A \mu w^2 \, dA \tag{25}$$

where

$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

w is the deflection,

A the area of the plate, and

$D = \dfrac{Eh^3}{12(1-\nu^2)}$ the bending stiffness

Dividing the plate into a finite number of rectangular plate elements, Hermitian polynomials can be used to describe the deflection $w(\xi, \eta)$ of each element (Figure 14):

$$w(\xi, \eta) = \overset{2m}{\mathbf{h}}{}^T(\xi) \, \overset{2m}{\mathbf{W}} \, \overset{2m}{\mathbf{h}}(\eta) \tag{26}$$

The square matrix $\overset{2m}{\mathbf{W}}$ contains the deflection, slope, etc. at the four corners 00, 01, 10, 11 of the plate element. For example, if we use sixth-order polynomials (2m = 6), then we have

$$\mathbf{W} = \begin{array}{c} 00 \\ \\ \\ 10 \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} w & w_\eta & w_{\eta\eta} & w & w_\eta & w_{\eta\eta} \\ w_\xi & w_{\xi\eta} & w_{\xi\eta\eta} & w_\xi & w_{\xi\eta} & w_{\xi\eta\eta} \\ w_{\xi\xi} & w_{\xi\xi\eta} & w_{\xi\xi\eta\eta} & w_{\xi\xi} & w_{\xi\xi\eta} & w_{\xi\xi\eta\eta} \\ \hline w & w_\eta & w_{\eta\eta} & w & w_\eta & w_{\eta\eta} \\ w_\xi & w_{\xi\eta} & w_{\xi\eta\eta} & w_\xi & w_{\xi\eta} & w_{\xi\eta\eta} \\ w_{\xi\xi} & w_{\xi\xi\eta} & w_{\xi\xi\eta\eta} & w_{\xi\xi} & w_{\xi\xi\eta} & w_{\xi\xi\eta\eta} \end{array} \right] \begin{array}{c} 01 \\ \\ \\ 11 \\ \\ \\ \end{array} \tag{27}$$

where the subscripts $\xi, \eta, \xi\eta, \ldots$ denote partial derivatives with respect to these independent variables.

Inserting Equation 26 in Equation 25 and using the $(2m)^2$ elements of the matrix **W** as generalized coordinates, we establish $(2m)^2$ Lagrangean equations 12. In contrast to problems with only one independent variable (bars, beams, frames and arches), now the right sides of all $(2m)^2$ Lagrangean equations are nonzero, because the forces and moments distributed along the four edges of the plate elements do virtual work due to the variation of every element in the matrix **W**. Consider, for example, the variation of $w_{\eta\eta}$ at corner 00, assuming that we have used 6th-order Hermitian polynomials. There will be no virtual displacement and deformation of the faces $\eta = 0$, $\eta = 1$, and $\xi = 1$, solely the face $\xi = 0$ will be deflected such that its neutral axis assumes a virtual displacement like the curve H_3 in Fig. 3. Then (cf. Fig.15) work is done only by the shear $v_{\xi\zeta}$ $(\xi = o, \eta)$ and the torsional moment $m_{\xi\xi}$ $(\xi = o, \eta)$. Hence the generalized force (right side of the Lagrangean equation) corresponding to the variation of $w_{\eta\eta}$ is:

$$\int_0^1 v_{\xi\zeta} \frac{\partial w}{\partial w_{\eta\eta,00}} d\eta + \int_0^1 m_{\xi\xi} \frac{\partial(\partial w/\partial\eta)}{\partial w_{\eta\eta,00}} d\eta \qquad (28)$$

Thus the obvious condensation technique, as used for the beam problem, can no longer be applied. There are several, as yet untried, recourses possible. For example, we could lump the distributed forces at each corner into the statically equivalent shear V_ζ and the moments M_ξ and M_η. Then only 12 of the $(2m)^2$ Lagrangean equations have non-zero right sides, namely those associated with w, w_ξ, w_η at the four corners. If we arrange the sequence of the Lagrangean equation such that these twelve equations are on top, we have obtained a matrix relationship corresponding to Equation 21

$$\begin{array}{c c} & \begin{array}{cc} 12 & 2m-12 \end{array} \\ \begin{array}{c} 12 \\ 2m-12 \end{array} & \left[\begin{array}{c|c} D_1 & D_2 \\ \hline D_3 & D_4 \end{array}\right] \end{array} \left[\begin{array}{c} u \\ \hline v \end{array}\right] = \left[\begin{array}{c} p \\ \hline 0 \end{array}\right] \qquad (29)$$

where

$$u = \{ w_{00}\ w_{\xi 00}\ w_{\eta 00};\ w_{01}\ w_{\xi 01}\ w_{\eta 01};\ w_{10}\ w_{\xi 10}\ w_{\eta 10};\ w_{11}\ w_{\xi 11}\ w_{\eta 11} \}$$

$$v = \{ w_{\xi\xi 00}\ w_{\xi\eta 00} \cdots;\ w_{\xi\xi 01}\ w_{\xi\eta 01} \cdots;\ w_{\xi\xi 10}\ w_{\xi\eta 10} \cdots;\ w_{\xi\xi 11}\ w_{\xi\eta 11} \cdots \}$$

and

$$p = \{ V_{\zeta 00}\ M_{\xi 00}\ M_{\eta 00};\ V_{\zeta 01}\ M_{\xi 01}\ M_{\eta 01};\ V_{\zeta 10}\ M_{\xi 10}\ M_{\eta 10};\ V_{\zeta 11}\ M_{\xi 11}\ M_{\eta 11} \}$$

From here on the condensation technique leads, as shown above, to the stiffness matrix relation (Equation 22)

$$p = S\ u$$

where **S** is now a 12 x 12 dynamic stiffness matrix. Thereafter the assembly of these individual stiffness matrices, the consideration of the boundary conditions, etc. is standard procedure.

Another approach is indicated by the well-known relationships between the moments $m_{\xi\xi}$, $m_{\eta\eta}$, $m_{\xi\eta}$, $m_{\eta\xi}$ and shear $v_{\xi\zeta}$ and $v_{\eta\zeta}$ distributed along the four edges (Figure 15) and certain derivatives of the deflection w:

$$m_{\xi\eta} = -\frac{Eh^3}{12(1-\nu^2)}\left(\frac{w_{\xi\xi}}{k^2} + \nu\frac{w_{\eta\eta}}{\ell^2}\right)$$

$$m_{\eta\xi} = \frac{Eh^3}{12(1-\nu^2)}\left(\frac{w_{\eta\eta}}{\ell^2} + \nu\frac{w_{\xi\xi}}{k^2}\right)$$

$$m_{\xi\xi} = -m_{\eta\eta} = \frac{Eh^3}{12(1-\nu^2)}\;\frac{w_{\xi\eta}}{k\ell} \tag{30}$$

$$v_{\xi\zeta} = \left(\frac{\partial m_{\xi\eta}}{k\,\partial\xi} + \frac{\partial m_{\eta\eta}}{\ell\,\partial\eta}\right)$$

$$v_{\eta\zeta} = -\left(\frac{\partial m_{\xi\xi}}{k\,\partial\xi} + \frac{\partial m_{\eta\xi}}{\ell\,\partial\eta}\right)$$

Then we could insert the relationships (30) in the right sides of the $(2m)^2$ Lagrangean equations, except for those corresponding to variations of, say, w_{oo}, $w_{\xi oo}$, $w_{\eta oo}$, $w_{\xi\eta oo}$, ... , $w_{\eta 11}$, and $w_{\xi\eta 11}$. Now $(2m)^2$ 16 right sides are expressed in terms of $w(\xi,\eta)$ (Equation 26) and its partial derivatives, and hence the condensation of the original 2m x 2m matrix into a 16 x 16 "dynamic stiffness matrix" can be carried out. The evaluation of the vector $\mathbf{p}$, containing the generalized forces corresponding to the variations of the not eliminated 16 displacements, assembled in the vector $\mathbf{u} = \{w_{oo}\ w_{\xi oo} \ldots\ w_{\xi\eta 11}\}$, is not necessary, because these generalized forces cancel each other, when the individual dynamic stiffness matrices are complied into the corresponding matrix for the whole plate.

Since the application of Hermitian polynomials of sixth and eighth order to beam problems has yielded considerable improvement in accuracy of the results at less computational expense compared with the use of fourth order polynomials, there is hope that the two condensation approaches outlined above may also prove advantageous over the straight-forward Ritz-procedure with Hermitian polynomials. In conclusion it should be mentioned that the latter was successfully applied to the static investigation of rectangular plates (Reference 3) and parallelogram plates (Reference 4). It was especially gratifying to note the good accuracy achieved in the computation of stresses, when Hermitian polynomials of order 6 and 8 were employed.

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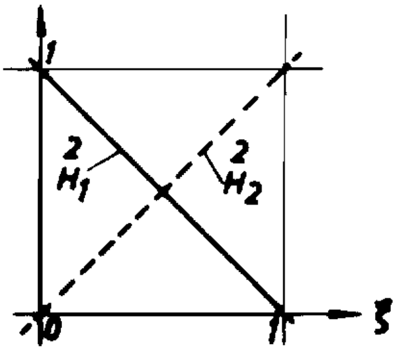

Figure 1. Hermitian Polynomials Of Order 2

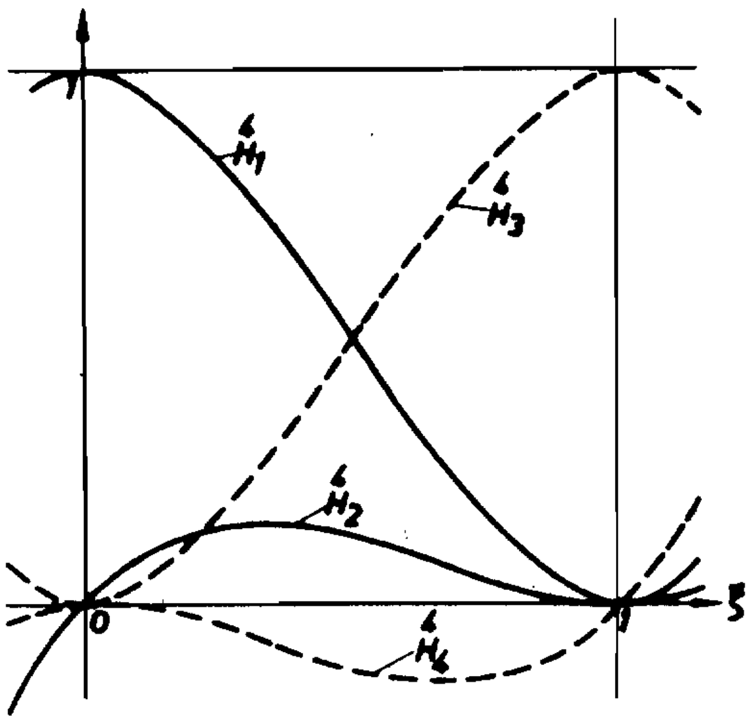

Figure 2. Hermitian Polynomials Of Order 4

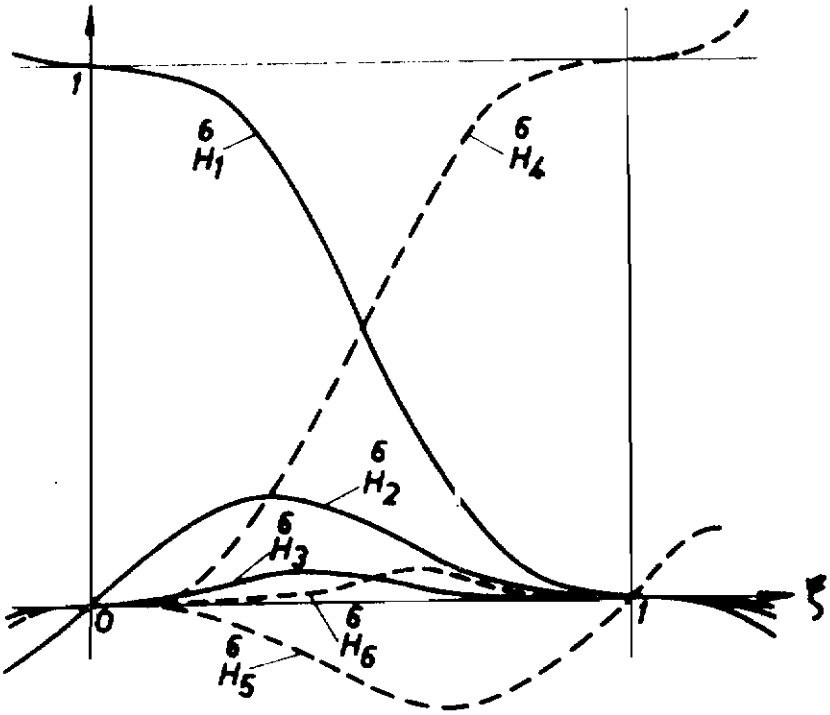

Figure 3. Hermitian Polynomials Of Order 6

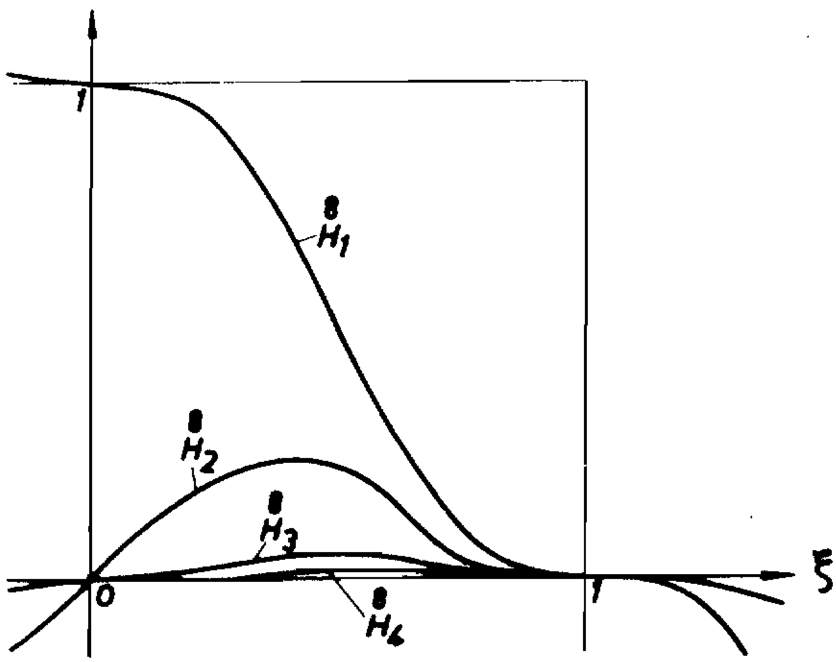

Figure 4. Hermitian Polynomials of Order 8

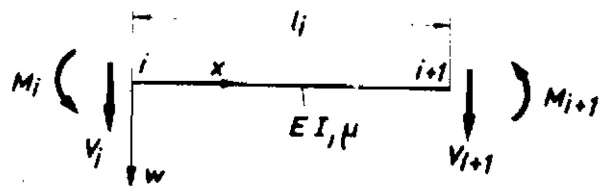

Figure 5. Beam Element i

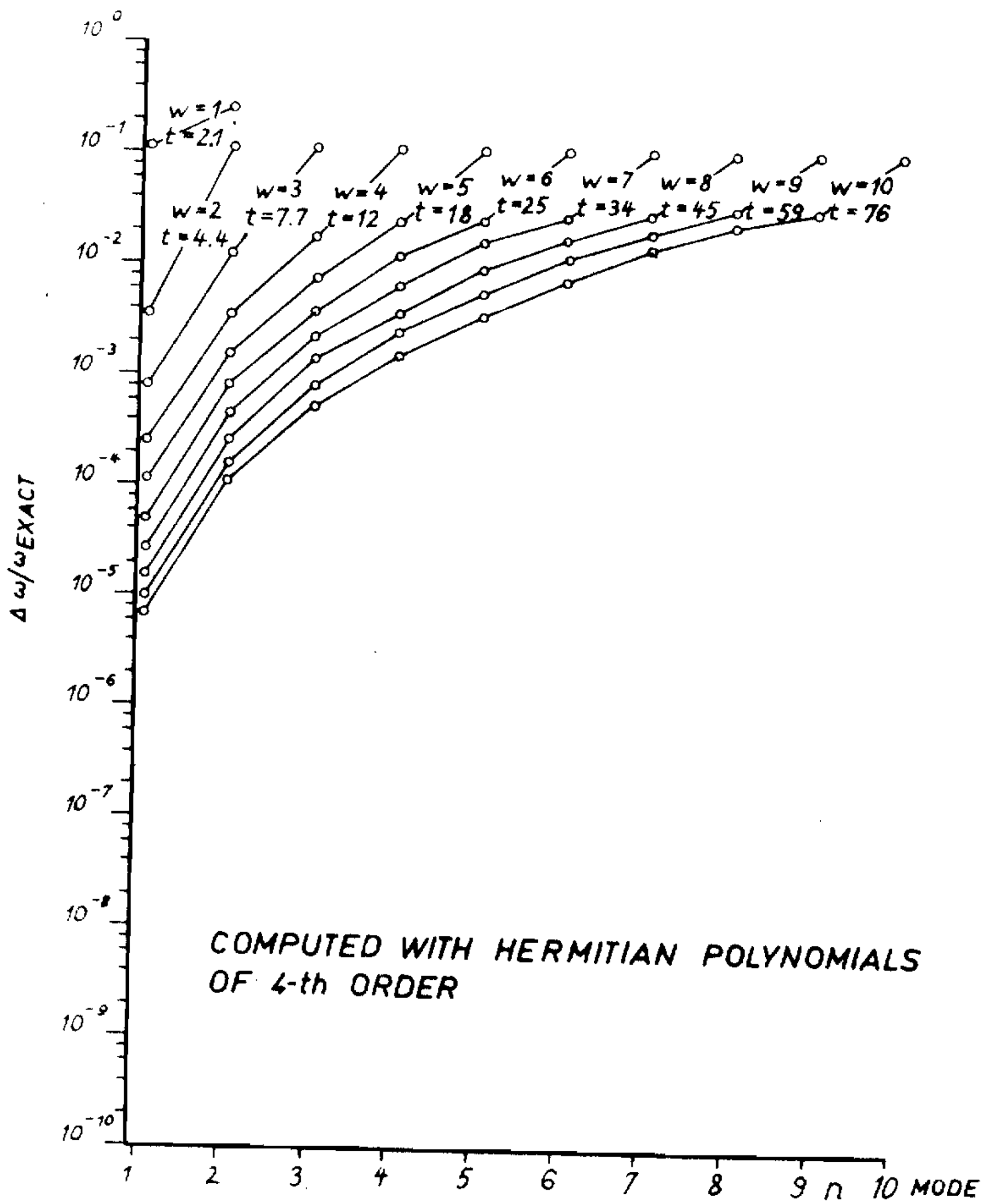

Figure 6. Error of Eigenfrequencies For The Simply Supported Beam

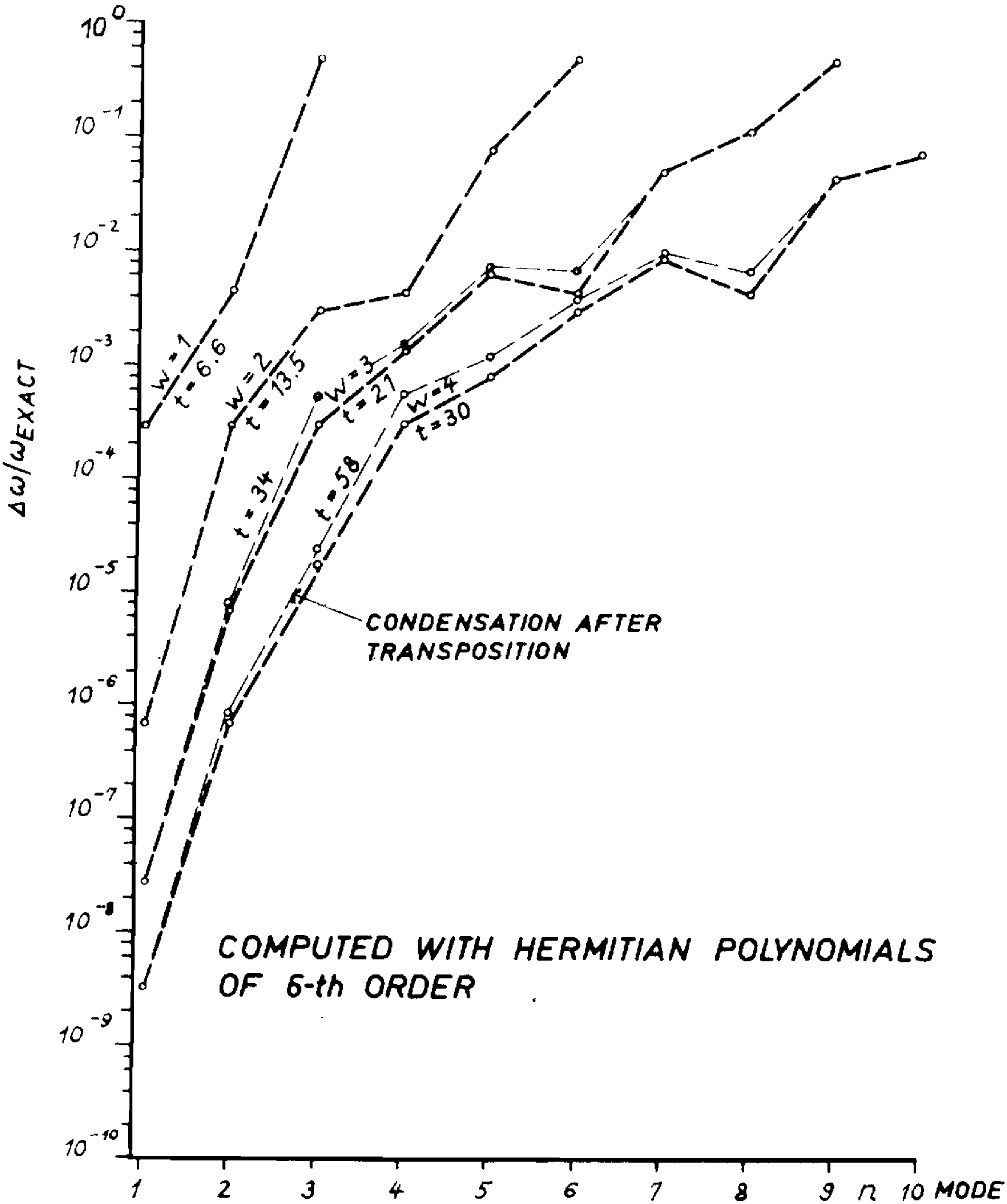

Figure 7. Error Of Eigenfrequencies For The Simply Supported Beam

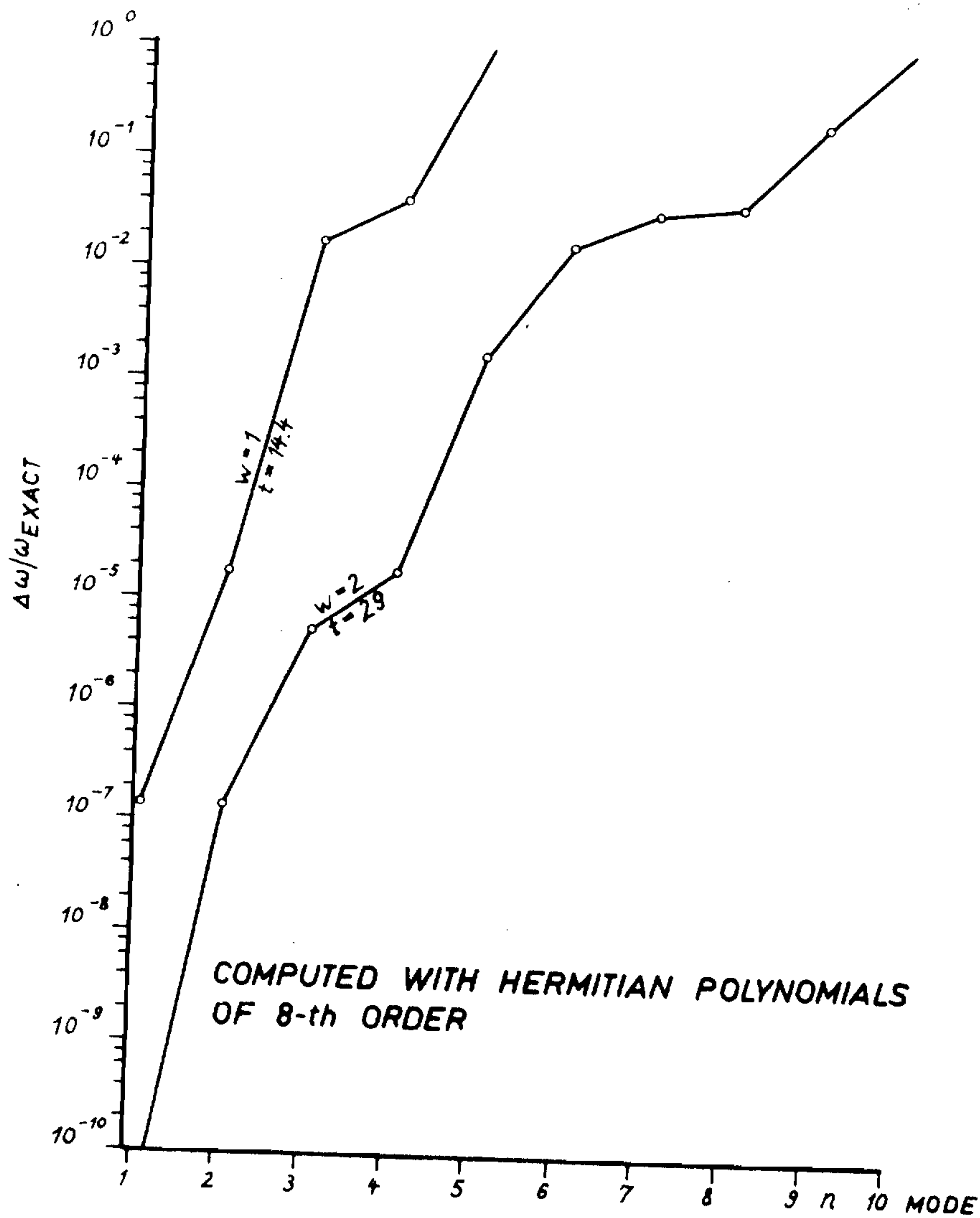

Figure 8. Error Of Eigenfrequencies For The Simply Supported Beam

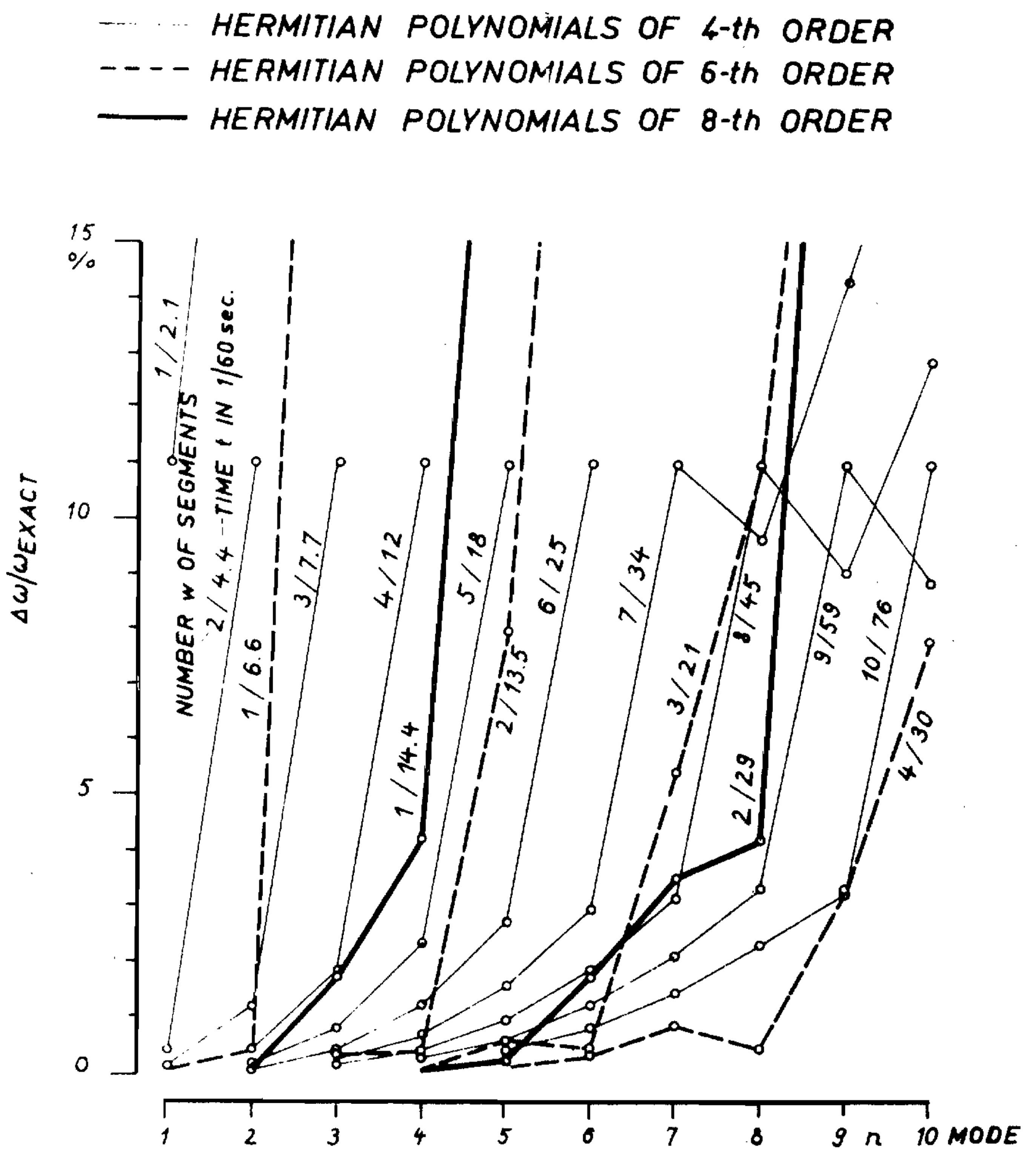

Figure 9. Error Of Eigenfrequencies For The Simply Supported Beam

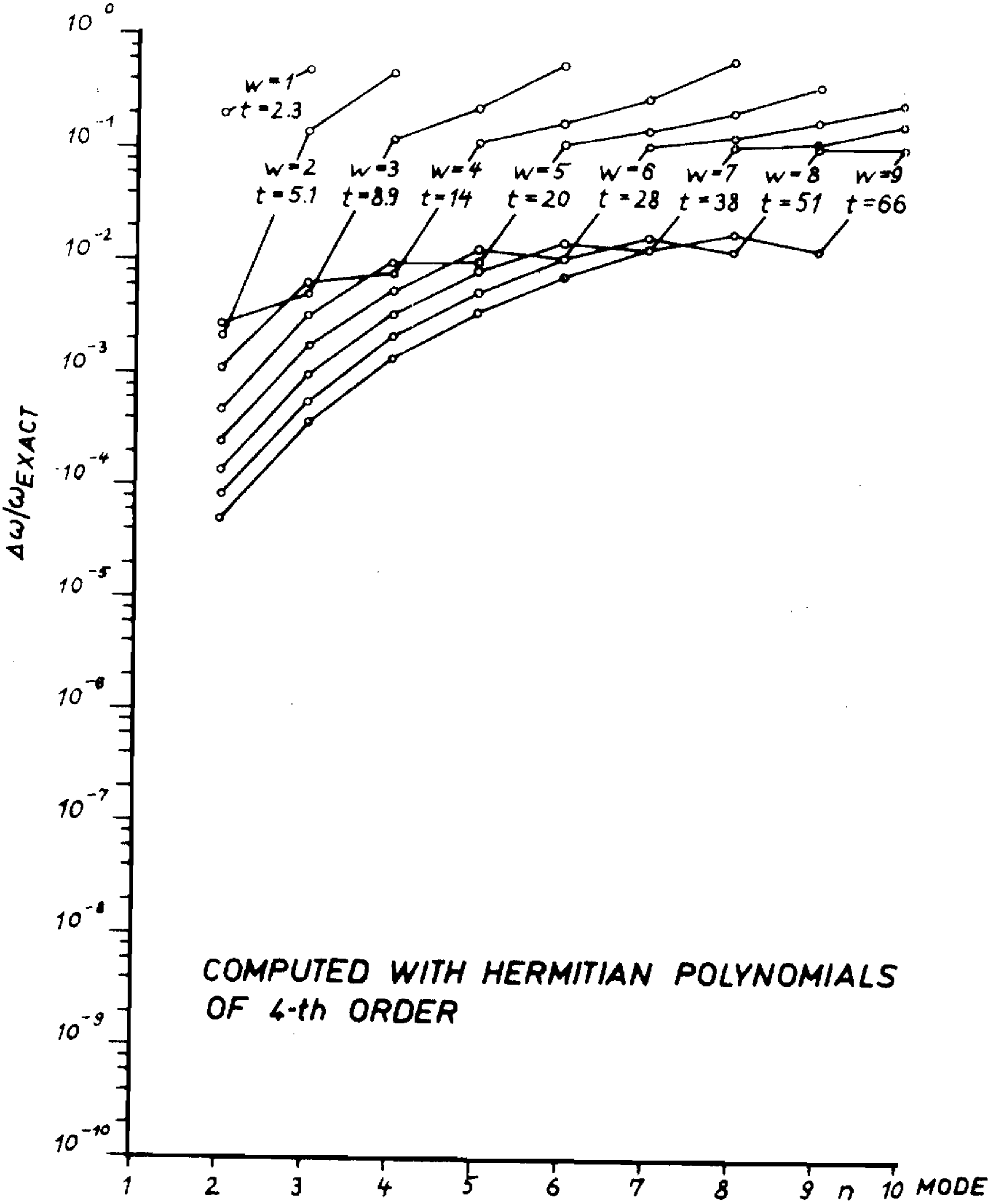

Figure 10. Error Of Eigenfrequencies For The Free-Free Beam

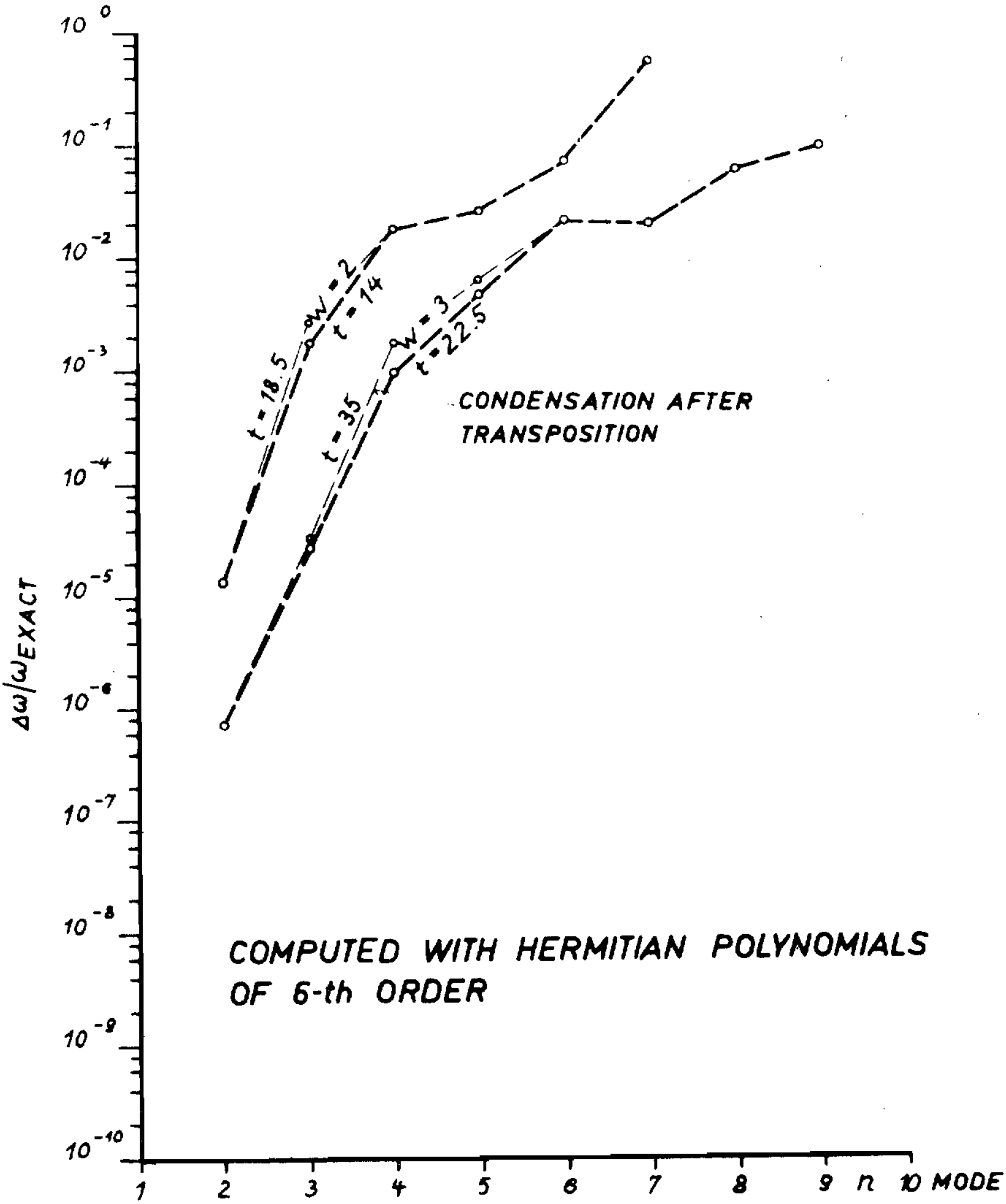

Figure 11. Error Of Eigenfrequencies For The Free-Free Beam.

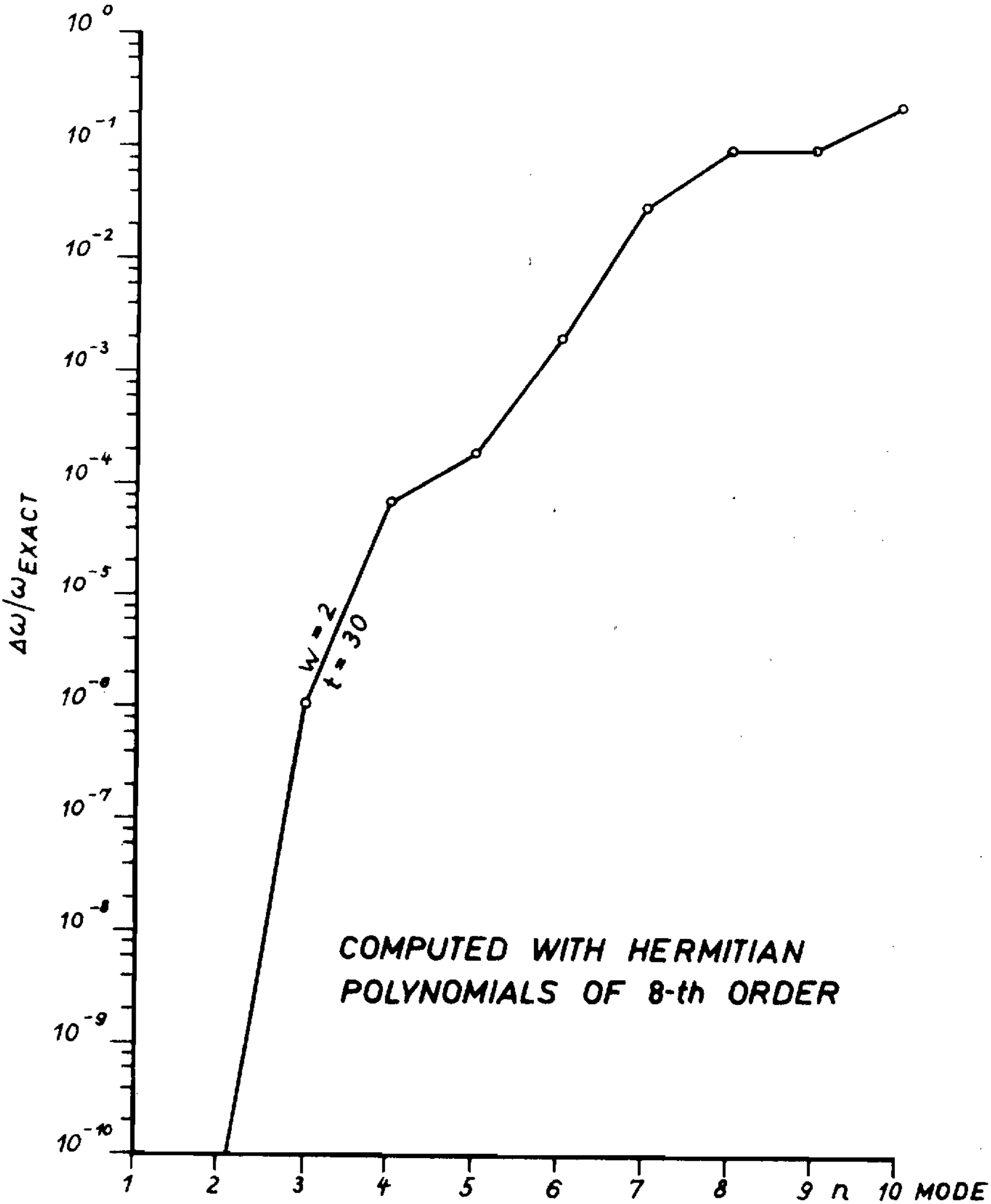

Figure 12. Error Of Eigenfrequencies For The Free-Free Beam

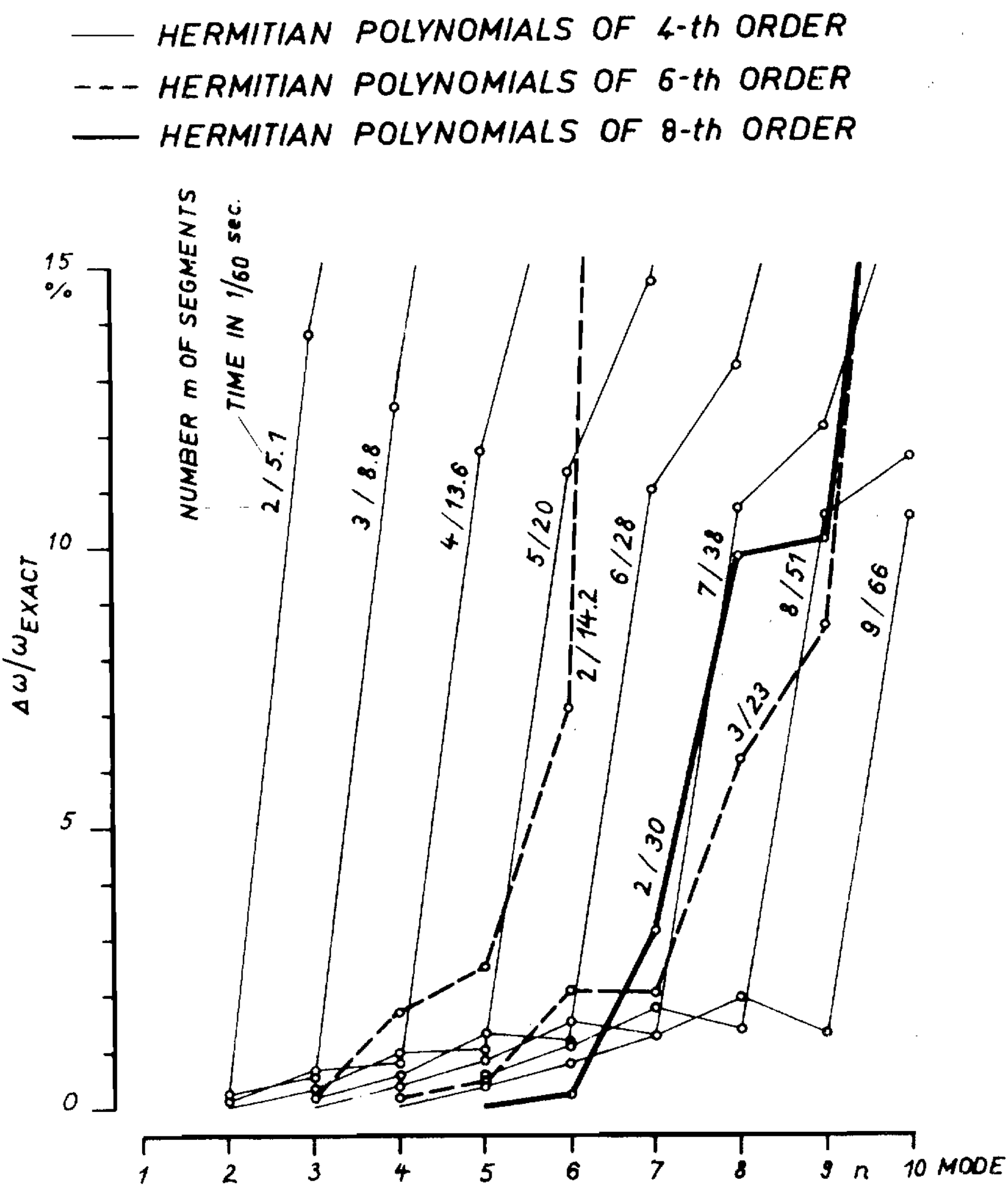

Figure 13. Error Of Eigenfrequencies For The Free-Free Beam

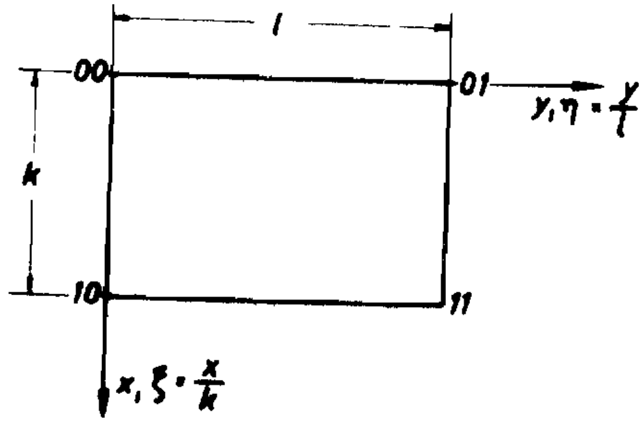

Figure 14. Rectangular Plate Element

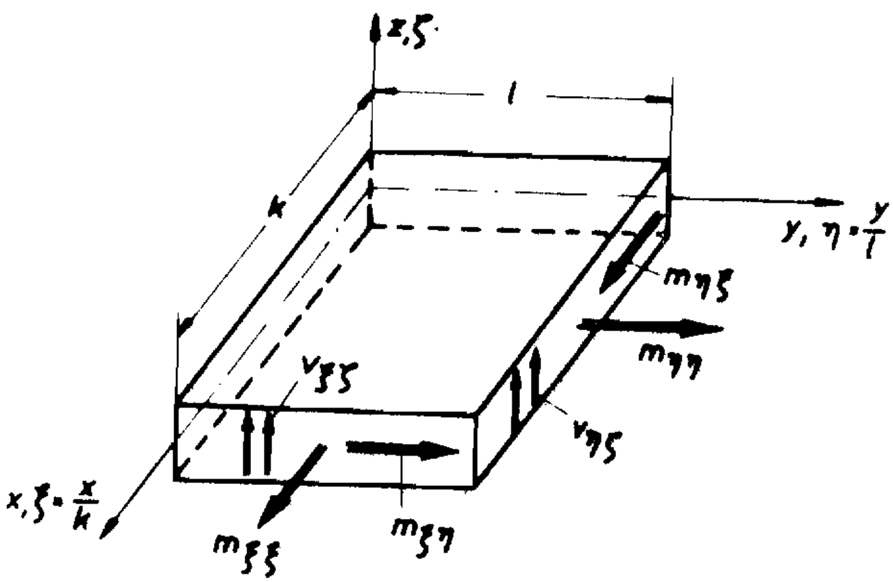

Figure 15. Forces And Moments Distributed Along Faces Of Plate Element