

RTD-TDR-63-4139  
Vol II

## FOREWORD

The research work in this report was performed by the IIT Research Institute, Chicago, Illinois, for the Aerospace Dynamics Branch, Vehicle Dynamics Division, AF Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio, under Contract AF33-(657)-8231. This research is part of a continuing effort to provide rational and reliable dynamic load design criteria for flight vehicles and is part of the Research and Technology Division, Air Force Systems Command's exploratory development program. The Department of Defense Program Element Number is 6.24.05.33.4, "Aircraft Flight Dynamics". This work was performed under Project No. 1367 "Structural Design Criteria", Task No. 136701, "Design Criteria for Ground-Induced Dynamic Loads". Walter P. Dunn and later Bernard H. Groomes, of the Vehicle Dynamics Division, AF Flight Dynamics Laboratory, were the Project Engineers. The research was conducted from March 1962 to October 1963.

IIT Research Institute personnel who contributed to the program are C. E. Gebhart, Dr. E. E. Hahn, Dr. E. Saleme, R. E. Wheeler, and W. J. Wheeler.

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ABSTRACT

The purpose of this research program was to collect all available measured prepared surface profile and power spectral data, to ascertain the reliability of these data, to establish from these data design criteria for vehicles operating on prepared surfaces, and to demonstrate the applicability of these design criteria.

Both deterministic and statistical analyses are used to determine the responses of a five-degree-of-freedom vehicle to some of the criteria developed in Volume I of this report.

Volume I is concerned with the collection of profile and power spectra data for runways, taxiways and ramps, ascertaining the reliability of these data, and establishing from the data sets of design criteria for vehicles operating on prepared surfaces.

PUBLICATION REVIEW

This report has been reviewed and is approved.

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## LIST OF SYMBOLS

$A_1$	lb-sec/ft	System Constant
$A_2$	lb/ft	System Constant
$A_3$	lb-sec	System Constant
$A_4$	lb	System Constant
$A_5$	lb-sec	System Constant
$A_6$	lb	System Constant
$A_7$	lb-sec	System Constant
$A_8$	lb	System Constant
$A_9$	lb-sec-ft	System Constant
$A_{10}$	lb-ft	System Constant
$A_{11}$	lb-sec-ft	System Constant
$A_{12}$	lb-ft	System Constant
$a$	ft	Amplitude of Forcing Function
$a, b, d, e, f$	ft	Dimensions of Vehicle
$B_1$	lb	Forcing Function
$B_2, B_3$	ft-lb	Forcing Function
$C$	lb-sec/ft	Damping Constant
$C_1$	lb-sec/ft	Damping Constant for Front Vehicle Support
$C_2$	lb-sec/ft	Damping Constant for Auxiliary Mass Supports
$C_3$	lb-sec/ft	Damping Constant for Rear Vehicle Supports
$C_{critical}$	lb-sec/ft	Critical Damping Constant

## LIST OF SYMBOLS (Cont.)

$D, E$	ft	Coefficients
$E[ \quad ]$	dimensionless	Expectation of [ $\quad$ ]
$F$	ft	Amplitude of Steady State Vibration
$F_j(\omega) q$	ft/(rad/sec)	Fourier Transform of Excitation at Support j
$f_j(t)$	ft	Excitation at Support j
$\bar{f}_j^2$	ft <sup>2</sup>	Mean Square of Excitation at Support j
$G_j$	ft	Amplitude of Input Displacement at Support j
$g_j(t)$	ft	Input Displacement at Support j
$\dot{g}_j(t)$	fps	Input Velocity at Support j
$H(\omega)$	dimensionless	Complex Frequency Response
$H_j(\omega)$	dimensionless	Complex Frequency Response Related to Support j
$h_j(t)$	dimensionless	Response of Vehicle to Unit Impulse at Support j
$I_{xx}$	slug-ft <sup>2</sup>	Moment of Inertia of Main Mass about x Axis
$I_{yy}$	slug-ft <sup>2</sup>	Moment of Inertia of Main Mass about y Axis
$j$	dimensionless	Index Denoting Support
$K$	lb/ft	Spring Constant
$K_1$	lb/ft	Spring Constant for Front Vehicle Support
$K_2$	lb/ft	Spring Constant for Auxiliary Vehicle Support
$K_3$	lb/ft	Spring Constant for Rear Vehicle Support



# Contrails

## LIST OF SYMBOLS (Cont.)

$k$	dimensionless	Index Denoting Support
$M$	slugs	Mass of Vehicle
$M_1$	slugs	Main Vehicle Mass
$M_2$	slugs	Auxiliary Vehicle Mass
$n_Y$	times/sec	Number of Times $y$ exceeds $Y$ , on the Average, per Unit Time
$n_{\dot{Y}}$	times/sec	Number of Times $\dot{y}$ exceeds $\dot{Y}$ , on the Average, per Unit Time
$n_{\ddot{Y}}$	times/sec	Number of Times $\ddot{y}$ exceeds $\ddot{Y}$ , on the Average, per Unit Time
$n_o^y$	times/sec	Number of Times $y$ Passes through Zero, on the Average, per Unit Time
$n_o^{\dot{y}}$	times/sec	Number of Times $\dot{y}$ Passes through Zero, on the Average, per Unit Time
$n_o^{\ddot{y}}$	times/sec	Number of Times $\ddot{y}$ Passes through Zero, on the Average, per Unit Time
$P_y(y)$	$ft^{-1}$	Probability Density for $y$
$P_{\dot{y}}(\dot{y})$	sec/ft	Probability Density for $\dot{y}$
$P_{\ddot{y}}(\ddot{y})$	$sec^2/ft$	Probability Density for $\ddot{y}$
$R$	ft	Amplitude of Response $z$
$R_i$	ft	Amplitude of $z_i$ , $i = 1, 2, \text{ and } 3$
$R_4$	rad	Amplitude of $\theta$
$R_5$	rad	Amplitude of $\phi$
$R_{jb}(\tau)$	$ft^2$	Auto Correlation Function of $f_j(t)$
$R_{jj}(\tau)$	$ft^2$	Cross Correlation Function of $f_j(t)$ and $f_b(t)$
$R_y(\tau)$	$ft^2$	Auto Correlation Function, Mass Displacement

## LIST OF SYMBOLS (Cont.)

$R_{\dot{y}}(\tau)$	$ft^2/sec^2$	Auto Correlation Function, Mass Velocity
$R_{\ddot{y}}(\tau)$	$ft^2/sec^4$	Auto Correlation Function, Mass Acceleration
$R_{y_o}(\tau)$	$ft^2$	Auto Correlation Function, Input to Vehicle
$R'_{y_o}(\gamma)$	$ft^2$	Auto Correlation Function, Track
$R_z(\tau)$	$ft^2$	Auto Correlation Function of the Response $z$
$S_{j_b}(\omega)$	$ft^2/(rad/sec)$	Cross Power Spectral Density of $f_j(t)$ and $f_b(t)$
$S_{j_j}(\omega)$	$ft^2/(rad/sec)$	Power Spectral Density of $f_j(t)$
$S_y(\omega)$	$ft^2/(rad/sec)$	Power Spectral Density, Mass Displacement
$S_{y_o}(\omega)$	$ft^2/(rad/sec)$	Power Spectral Density, Input to Vehicle
$S_{\dot{y}}(\omega)$	$ft^2/sec^2(rad/sec)$	Power Spectral Density, Mass Velocity
$S_{\ddot{y}}(\omega)$	$ft^2/sec^4(rad/sec)$	Power Spectral Density, Mass Acceleration
$S'_{y_o}(\Omega)$	$ft^2/(rad/sec)$	Power Spectral Density, Track
$T$	sec	Period: Dummy Constant
$t$	sec	Time
$V_o$	fps	Velocity of Vehicle
$X$	ft	Dummy Constant
$Y$	ft	A Particular Mass Displacement
$\dot{Y}$	fps	A Particular Mass Velocity
$\ddot{Y}$	$ft/sec^2$	A Particular Mass Acceleration

## LIST OF SYMBOLS (Cont.)

$Y'_0$	ft	Track Elevation as a Function of Time
$x, y, z$	ft	Coordinate Axes
$z_1$	ft	Vertical Displacement of Main Mass
$\dot{z}_1$	fps	Vertical Velocity of Main Mass
$\ddot{z}_1$	ft/sec <sup>2</sup>	Vertical Acceleration of Main Mass
$z_2$	ft	Vertical Displacement of Starboard Auxiliary Mass
$\dot{z}_2$	fps	Vertical Velocity of Starboard Auxiliary Mass
$\ddot{z}_2$	ft/sec <sup>2</sup>	Vertical Acceleration of Starboard Auxiliary Mass
$z_3$	ft	Vertical Displacement of Port Auxiliary Mass
$\dot{z}_3$	fps	Vertical Velocity of Port Auxiliary Mass
$\ddot{z}_3$	ft/sec <sup>2</sup>	Vertical Acceleration of Port Auxiliary Mass
$\gamma$	dimensionless	Damping Ratio
$\gamma$	ft	Dummy Variable
$\Delta$	lb <sup>5</sup> /ft	Determinant
$\theta$	rad	Main Mass Rotation about x Axis
$\dot{\theta}$	rad/sec	Angular Velocity of Main Mass about x Axis
$\ddot{\theta}$	rad/sec <sup>2</sup>	Angular Acceleration of Main Mass about x Axis
$\lambda_1, \lambda_2$	rad/sec	Exponent, Transient Solution

## LIST OF SYMBOLS (Cont.)

$\sigma_y$	ft	Standard Deviation of $y$
$\sigma_{\dot{y}}$	fps	Standard Deviation of $\dot{y}$
$\sigma_{\ddot{y}}$	ft/sec <sup>2</sup>	Standard Deviation of $\ddot{y}$
$\tau, \tau_1, \tau_2, \tau_3$	sec	Dummy Variables
$\phi$	rad	Main Mass Rotation about $y$ Axis
$\dot{\phi}$	rad/sec	Angular Velocity of Main Mass about $y$ Axis
$\ddot{\phi}$	rad/sec <sup>2</sup>	Angular Acceleration of Main Mass about $y$ Axis
$\Omega$	rad/ft	Spatial Frequency
$\omega$	rad/sec	Frequency
$\omega_n$	rad/sec	Natural Frequency of Vehicle

SECTION I  
DETERMINISTIC ANALYSIS

Analyses of a vehicle to be operated on prepared surfaces are given here to illustrate the application of the design criteria developed in Volume One of this report. Both deterministic and statistical analyses are given. For the deterministic analysis, the vehicle was considered a linear five-degree-of-freedom system, a very simplified model of an airplane. The model vehicle is shown in Fig. 2-1.

The total weight of the model is

$$MG = M_1G + 2M_2G = 150,000 + 2(75,000) = 300,000 \text{ lb.}$$

The roll moment of inertia about the center of gravity of the vehicle acting as a rigid body is

$$\begin{aligned} I_{\text{roll}} &= I_{xx} + 2 M_2 a^2 \\ &= 932,488 + 2(75,000/32.172)(40.0)^2 = 8,392,391 \text{ lb-sec}^2\text{-ft.} \end{aligned}$$

In terms of the total mass  $M$

$$I_{\text{roll}} = M \bar{r}_{\text{roll}}^2$$

where  $\bar{r}_{\text{roll}} = \sqrt{I_{\text{roll}}/M} = \sqrt{8,392,391/(300,000/32.172)} = 30 \text{ ft.}$

The pitch moment of inertia about the center of gravity of the vehicle acting as a rigid body is

$$I_{\text{pitch}} = I_{yy} + M_1 \bar{x}^2 + 2 M_2 (f - \bar{x})^2$$

where  $\bar{x}$  is the distance from  $M_1$  to the center of gravity of the entire vehicle.

$$M_1 \bar{x} - 2 M_2 (f - \bar{x}) = 0,$$

therefore,

$$\begin{aligned} \bar{x} &= 2 M_2 f / (M_1 + 2 M_2) = 2 M_2 Gf / (M_1 G + 2 M_2 G) \\ &= 2 (75,000) (20) / (300,000) = 10.0 \text{ ft.} \end{aligned}$$

Manuscript released by author in August 1963 for publication as an RTD Technical Documentary Report.

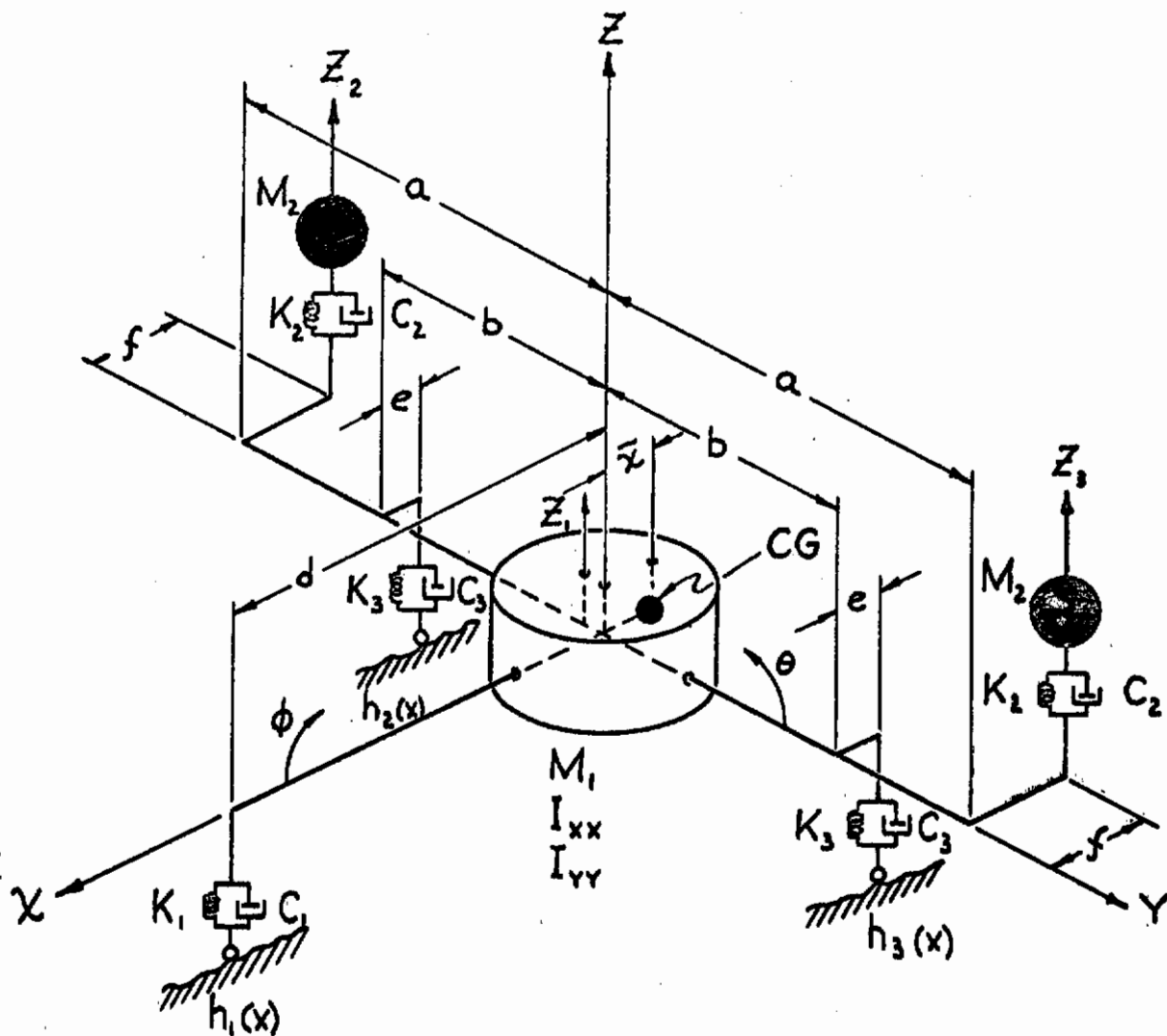


Fig. 2-1 Five-Degree-of-Freedom Model Vehicle

<u>Dimensions</u>	a = 40.0 ft	$M_1 G = 150,000 \text{ lb}$
	b = 10.0 ft	$M_2 G = 75,000 \text{ lb}$
	d = 35.0 ft	$I_{xx} = 932,488 \text{ lb-sec}^2\text{-ft}$
	e = 15.0 ft	$I_{yy} = 2,797,164 \text{ lb-sec}^2\text{-ft}$
	f = 20.0 ft	
	$K_1 = 10,000 \text{ lb/in.}$	$C_1 = 185 \text{ lb-sec/in.}$
	$K_2 = 15,000 \text{ lb/in.}$	$C_2 = 105 \text{ lb-sec/in.}$
	$K_3 = 25,000 \text{ lb/in.}$	$C_3 = 185 \text{ lb-sec/in.}$

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Using this result we obtain

$$\begin{aligned} I_{\text{pitch}} &= 2,797,464 + (150,000/32.172) (10.0)^2 \\ &\quad + 2 (75,000/32.172) (20.0 - 10.0)^2 \\ &= 3,729,952 \text{ lb-sec}^2\text{-ft.} \end{aligned}$$

In terms of the total mass  $M$ ,

$$I_{\text{pitch}} = M \bar{r}_{\text{pitch}}^2$$

where

$$\bar{r}_{\text{pitch}} = \sqrt{I_{\text{pitch}}/M} = \sqrt{3,729,952/(300,000/32.172)} = 20.0 \text{ ft}$$

The ratio of the distance from the front support to the center of gravity of the entire vehicle to the distance from the front support to the rear supports is

$$\frac{d + \bar{x}}{d + e} = \frac{35 + 10}{35 + 15} = 0.90.$$

The natural frequency of the vehicle in a rigid plunge (vertical motion with  $K_2 = \infty$ ) is given by

$$\omega_p = \sqrt{K/M} = \sqrt{G/\delta_{ST}}$$

where  $\delta_{ST}$  is the static deflection of the CG of the vehicle. Since the CG is located 0.9 of the distance between the vehicle supports to the rear of the front support 0.9 of the weight of the vehicle will be carried by the rear supports and 0.1 of the weight by the front support. Also, the deflection of the CG will be

$$\delta_{ST} = \delta_1 + 0.9 (\delta_3 - \delta_1) = 0.1 \delta_1 + 0.9 \delta_3$$

where  $\delta_1$  is the static deflection of support 1 and  $\delta_3$  is the static deflection of support 3. Using these results in the formula for the rigid plunge natural frequency

$$\begin{aligned} p &= \sqrt{G/(0.1 \delta_1 + 0.9 \delta_3)} \\ &= \sqrt{G/[0.1 (0.1 MG/K_1) + 0.9 (0.9 MG/2K_3)]} \\ &= \sqrt{G/MG (0.01/K_1 + 0.405/K_3)} \\ &= \sqrt{32.172/300,000 (0.01/10,000 \cdot 12 + 0.405/25,000 \cdot 12)} \\ &= 8.650 \text{ rad/sec} = 1.377 \text{ cps.} \end{aligned}$$

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The natural frequency of the mass-spring combination  $M_2, K_2$  is

$$\begin{aligned}\omega_n &= \sqrt{K_2/M_2} = \sqrt{(15,000 \cdot 12)/(75,000/32.172)} \\ &= 8.786 \text{ rad/sec} \\ &= 1.398 \text{ cps}\end{aligned}$$

The ratio of the damping to critical damping in the  $(M_2, K_2, C_2)$  system is

$$\begin{aligned}z_2 &= C_2 / 2\sqrt{K_2 M_2} = 105 \cdot 12 / 2 \sqrt{15,000 \cdot 12 (75,000/32.172)} \\ &= 0.03075 \\ &= 3.075 \text{ percent.}\end{aligned}$$

## 1. Derivation of Equations of Motion

Suppose support 1 is displaced according to the function  $g_1(t)$ , support 2 according to  $g_2(t)$  and support 3 according to  $g_3(t)$ . Then summing the forces acting on the vehicle, we find

$$\begin{aligned}M_1 \ddot{Z}_1 &= - [Z_1 + d\phi - g_1(t)] K_1 - [\dot{Z}_1 + d\dot{\phi} - \dot{g}_1(t)] C_1 \\ &\quad - [Z_1 - e\phi - b\theta - g_2(t)] K_3 - [\dot{Z}_1 - e\dot{\phi} - b\dot{\theta} - \dot{g}_2(t)] C_3 \\ &\quad - [Z_1 - e\phi + b\theta - g_3(t)] K_3 - [\dot{Z}_1 - e\dot{\phi} + b\dot{\theta} - \dot{g}_3(t)] C_3 \\ &\quad - [Z_1 - f\phi - a\theta - Z_2] K_2 - [\dot{Z}_1 - f\dot{\phi} - a\dot{\theta} - \dot{Z}_2] C_2 \\ &\quad - [Z_1 - f\phi + a\theta - Z_3] K_2 - [\dot{Z}_1 - f\dot{\phi} + a\dot{\theta} - \dot{Z}_3] C_2\end{aligned}$$

$$M_2 \ddot{Z}_2 = [Z_1 - f\phi - a\theta - Z_2] K_2 + [\dot{Z}_1 - f\dot{\phi} - a\dot{\theta} - \dot{Z}_2] C_2$$

$$M_2 \ddot{Z}_3 = [Z_1 - f\phi - a\theta - Z_3] K_2 + [\dot{Z}_1 - f\dot{\phi} + a\dot{\theta} - \dot{Z}_3] C_2$$

$$\begin{aligned}I_{xx} \ddot{\theta} &= [Z_1 - e\phi - b\theta - g_2(t)] K_3 b + [\dot{Z}_1 - e\dot{\phi} + b\dot{\theta} - \dot{g}_2(t)] C_3 b \\ &\quad - [Z_1 - e\phi + b\theta - g_3(t)] K_3 b - [\dot{Z}_1 - e\dot{\phi} + b\dot{\theta} - \dot{g}_3(t)] C_3 b \\ &\quad + [Z_1 - f\phi - a\theta - Z_2] K_2 a + [\dot{Z}_1 - f\dot{\phi} - a\dot{\theta} - \dot{Z}_2] C_2 a \\ &\quad - [Z_1 - f\phi + a\theta - Z_3] K_2 a - [\dot{Z}_1 - f\dot{\phi} + a\dot{\theta} - \dot{Z}_3] C_2 a\end{aligned}$$



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$$\begin{aligned}
 I_{yy}\ddot{\phi} = & - [Z_1 + d\dot{\phi} - g_1(t)] K_1 d - [\dot{Z}_1 + d\dot{\phi} - \dot{g}_1(t)] C_1 d \\
 & + [Z_1 - e\dot{\phi} - b\dot{\theta} - g_2(t)] K_3 e + [\dot{Z}_1 - e\dot{\phi} - b\dot{\theta} - \dot{g}_2(t)] C_3 e \\
 & + [Z_1 - e\dot{\phi} + b\dot{\theta} - g_3(t)] K_3 e + [\dot{Z}_1 - e\dot{\phi} + b\dot{\theta} - \dot{g}_3(t)] C_3 e \\
 & + [Z_1 - f\dot{\phi} - a\dot{\theta} - Z_2] K_2 f + [\dot{Z}_1 - f\dot{\phi} - a\dot{\theta} - \dot{Z}_2] C_2 f \\
 & + [Z_1 - f\dot{\phi} + a\dot{\theta} - Z_3] K_2 f + [\dot{Z}_1 - f\dot{\phi} + a\dot{\theta} - \dot{Z}_3] C_2 f.
 \end{aligned}$$

These equations can be rewritten

$$\begin{aligned}
 M_1 \ddot{Z}_1 + (C_1 + 2C_2 + 2C_3) \dot{Z}_1 + (K_1 + 2K_2 + 2K_3) Z_1 - C_2 \dot{Z}_2 - K_2 Z_2 \\
 - C_2 \dot{Z}_3 - K_2 Z_3 + (dC_1 - 2fC_2 - 2eC_3) \dot{\phi} + (dK_1 - 2fK_2 - 2eK_3) \phi \\
 = C_1 \dot{g}_1(t) + K_1 g_1(t) + C_3 \dot{g}_2(t) + K_3 g_2(t) + C_3 \dot{g}_3(t) + K_3 g_3(t) \\
 - C_2 \dot{Z}_1 - K_2 Z_1 + M_2 \ddot{Z}_2 + C_2 \dot{Z}_2 + K_2 Z_2 + aC_2 \dot{\theta} + aK_2 \theta + fC_2 \dot{\phi} + fK_2 \phi = 0 \\
 - C_2 \dot{Z}_1 - K_2 Z_1 + M_2 \ddot{Z}_3 + C_2 \dot{Z}_3 + K_2 Z_3 - aC_2 \dot{\theta} - aK_2 \theta + fC_2 \dot{\phi} + fK_2 \phi = 0 \\
 aC_2 \dot{Z}_2 + aK_2 Z_2 - aC_2 \dot{Z}_3 - aK_2 Z_3 + I_{xx} \ddot{\theta} + (2a^2 C_2 + 2b^2 C_3) \dot{\theta} \\
 + (2a^2 K_2 + 2b^2 K_3) \theta = -bC_3 \dot{g}_2(t) - bK_3 g_2(t) + bC_3 \dot{g}_3(t) + bK_3 g_3(t) \\
 (dC_1 - 2fC_2 - 2eC_3) \dot{Z}_1 + (dK_1 - 2fK_2 - 2eK_3) Z_1 + fC_2 \dot{Z}_2 + fK_2 Z_2 \\
 + fC_2 \dot{Z}_3 + fK_2 Z_3 + I_{yy} \ddot{\phi} + (d^2 C_1 + 2f^2 C_2 + 2e^2 C_3) \dot{\phi} \\
 + (d^2 K_1 + 2f^2 K_2 + 2e^2 K_3) \phi = dC_1 \dot{g}_1(t) + dK_1 g_1(t) \\
 - eC_3 \dot{g}_2(t) - eK_3 g_2(t) - eC_3 \dot{g}_3(t) - eK_3 g_3(t).
 \end{aligned}$$

If the vehicle moves at a constant horizontal velocity  $V_0$  the functions  $g_1(t)$ ,  $g_2(t)$ , and  $g_3(t)$  are related to  $h_1(x)$ ,  $h_2(x)$ , and  $h_3(x)$  by the following relationships provided  $t = 0$  at  $x = 0$ .

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$$g_1(t) = h_1(v_0 t + d + e)$$

$$g_2(t) = h_2(v_0 t)$$

$$g_3(t) = h_3(v_0 t).$$

Let

$$B_1 = C_1 \ddot{g}_1(t) + K_1 g_1(t) + C_3 \dot{g}_2(t) + K_3 g_2(t) + C_3 \dot{g}_3(t) + K_3 g_3(t)$$

$$B_2 = -bC_3 \dot{g}_2(t) - bK_3 g_2(t) + bC_3 \dot{g}_3(t) + bK_3 g_3(t)$$

$$B_3 = dC_1 \dot{g}_1(t) + dK_1 g_1(t) - eC_3 \dot{g}_2(t) - eK_3 g_2(t) - eC_3 \dot{g}_3(t) - eK_3 g_3(t).$$

Let

$$A_1 = C_1 + 2C_2 + 2C_3$$

$$A_2 = K_1 + 2K_2 + 2K_3$$

$$A_3 = dC_1 - 2fC_2 - 2eC_3$$

$$A_4 = dK_1 - 2fK_2 - 2eK_3$$

$$A_5 = aC_2$$

$$A_6 = aK_2$$

$$A_7 = fC_2$$

$$A_8 = fK_2$$

$$A_9 = 2a^2 C_2 + 2b^2 C_3$$

$$A_{10} = 2a^2 K_2 + 2b^2 K_3$$

$$A_{11} = d^2 C_1 + 2f^2 C_2 + 2e^2 C_3$$

$$A_{12} = d^2 K_1 + 2f^2 K_2 + 2e^2 K_3$$

$$v_1 = \dot{Z}_1$$

$$v_2 = \dot{Z}_2$$

$$v_3 = \dot{Z}_3$$

$$a = \dot{\theta}$$

$$\beta = \dot{\phi}$$

$$\begin{aligned} \text{Then } \dot{v}_1 &= \ddot{Z}_1 \\ \dot{v}_2 &= \ddot{Z}_2 \\ \dot{v}_3 &= \ddot{Z}_3 \\ \dot{a} &= \ddot{\theta} \\ \dot{\beta} &= \ddot{\phi} \end{aligned}$$

Using these results we can write a system of ten simultaneous first-order linear differential equations which will describe the motion of the model vehicle.

- (1)  $\dot{Z}_1 - v_1 = 0$
- (2)  $\dot{Z}_2 - v_2 = 0$
- (3)  $\dot{Z}_3 - v_3 = 0$
- (4)  $\dot{\theta} - a = 0$
- (5)  $\dot{\phi} - \beta = 0$
- (6)  $M_1 \dot{v}_1 + A_1 v_1 + A_2 Z_1 - C_2 v_2 - K_2 Z_2 - C_2 v_3 - K_2 Z_3 + A_3 \beta + A_4 \phi = B_1$
- (7)  $-C_2 v_1 - K_2 Z_1 + M_2 \dot{v}_2 + C_2 v_2 + K_2 Z_2 + A_5 a + A_6 \theta + A_7 \beta + A_8 \phi = 0$
- (8)  $-C_2 v_1 - K_2 Z_1 + M_2 \dot{v}_3 + C_2 v_3 + K_2 Z_3 - A_5 a - A_6 \theta + A_7 \beta + A_8 \phi = 0$
- (9)  $A_5 v_2 + A_6 Z_2 - A_5 v_3 - A_6 Z_3 + I_{xx} \dot{a} + A_9 a + A_{10} \theta = B_2$
- (10)  $A_3 v_1 + A_4 Z_1 + A_7 v_2 + A_8 Z_2 + A_7 v_3 + A_8 Z_3 + I_{yy} \dot{\beta} + A_{11} \beta + A_{12} \phi = B_3$

## 2. Computer Program for Solution of Equations of Motion

These equations can be solved for  $Z_1, Z_2, Z_3, \theta, \phi, \dot{Z}_1, \dot{Z}_2, \dot{Z}_3, \dot{\theta}, \dot{\phi}, \dot{v}_1, \dot{v}_2, \dot{v}_3, \dot{a}$  and  $\dot{\beta}$  as functions of time for given initial conditions, horizontal velocity, vehicle parameters, and descriptions of  $h_1(x), h_2(x),$  and  $h_3(x)$  [ $h'(x)$  can be approximated by  $(h(x + \Delta x) - h(x - \Delta x)) / 2 \Delta x$ ]. Since the available descriptions for  $h_1(x), h_2(x)$  and  $h_3(x)$  are in the form of elevations taken at equal intervals along the  $x$  axis, a stepwise numerical procedure can be most easily utilized. The variation of the Runge-Kutta fourth-order process due to Gill is such a procedure which is easily programmed for a digital computer.

A computer program to carry out the necessary calculations is outlined as follows.

- 1) Read and store data.
- 2) Print heading.
- 3) Set  $x = t = 0$ .
- 4) Set  $Z_1, Z_2, Z_3, \theta, \phi, v_1, v_2, v_3, a$  and  $\beta$  equal to their initial values.
- 5) Set  $t = t + \Delta x/v_0$ .
- 6) Compute  $Z_1, Z_2, Z_3, \theta, \phi, \dot{Z}_1, \dot{Z}_2, \dot{Z}_3, \dot{\theta}, \dot{\phi}, \dot{v}_1, \dot{v}_2, \dot{v}_3, \ddot{a}$  and  $\ddot{\beta}$  according to the fourth-order Runge-Kutta-Gill process.
- 7) Print out  $t$  and the responses.
- 8) Return to step 5 if more data is available; otherwise stop.

### 3. Natural Frequencies

Let us find the undamped natural frequencies of vibrations of this model. The equations of motion of the vehicle with all damping terms set equal to zero and with  $g_1(t) \equiv g_2(t) \equiv g_3(t) \equiv 0$  become

$$M_1 \ddot{Z}_1 + (K_1 + 2K_2 + 2K_3) Z_1 - K_2 Z_2 - K_2 Z_3 + (dK_1 - 2fK_2 - 2eK_3) \phi = 0$$

$$-K_2 Z_1 + M_2 \ddot{Z}_2 + K_2 Z_2 + aK_2 \theta + fK_2 \phi = 0$$

$$-K_2 Z_1 + M_2 \ddot{Z}_3 + K_2 Z_3 - aK_2 \theta + fK_2 \phi = 0$$

$$aK_2 Z_2 - aK_2 Z_3 + I_{xx} \ddot{\theta} + (2a^2 K_2 + 2b^2 K_3) \theta = 0$$

$$(dK_1 - 2fK_2 - 2eK_3) Z_1 + fK_2 Z_2 + fK_2 Z_3 + I_{yy} \ddot{\phi} + (d^2 K_1 + 2f^2 K_2 + 2e^2 K_3) \phi = 0$$

These equations have solutions of the form

$$Z_1 = D_1 e^{i\omega t}$$

$$Z_2 = D_2 e^{i\omega t}$$

$$Z_3 = D_3 e^{i\omega t}$$

$$\theta = D_4 e^{i\omega t}$$

$$\phi = D_5 e^{i\omega t}$$

When derivatives are taken

$$\ddot{Z}_1 = -\omega^2 Z_1$$

$$\ddot{Z}_2 = -\omega^2 Z_2$$

$$\ddot{Z}_3 = -\omega^2 Z_3$$

$$\ddot{\theta} = -\omega^2 \theta$$

$$\ddot{\phi} = -\omega^2 \phi$$

Substituting into the differential equations, we find

$$[-\omega^2 M_1 + (K_1 + 2K_2 + 2K_3)] Z_1 - K_2 Z_2 - K_2 Z_3 + (dK_1 - 2fK_2 - 2eK_3) \phi = 0$$

$$-K_2 Z_1 + [-\omega^2 M_2 + K_2] Z_2 + aK_2 \theta + fK_2 \phi = 0$$

$$-K_2 Z_1 + [-\omega^2 M_2 + K_2] Z_3 - aK_2 \theta + fK_2 \phi = 0$$

$$aK_2 Z_2 - aK_2 Z_3 + [-\omega^2 I_{xx} + (2a^2 K_2 + 2b^2 K_3)] \theta = 0$$

$$(dK_1 - 2fK_2 - 2eK_3) Z_1 + fK_2 Z_2 + fK_2 Z_3 + [-\omega^2 I_{yy} + (d^2 K_1 + 2f^2 K_2 + 2e^2 K_3)] \phi = 0$$

These equations can be rewritten

$$\omega^2 Z_1 = [(K_1 + 2K_2 + 2K_3) Z_1 + (-K_2) Z_2 + (-K_2) Z_3 + (0) \theta + (dK_1 - 2fK_2 - 2eK_3) \phi] / M_1$$

$$\omega^2 Z_2 = [(-K_2) Z_1 + (K_2) Z_2 + (0) Z_3 + (aK_2) \theta + (fK_2) \phi] / M_2$$

$$\omega^2 Z_3 = [(-K_2) Z_1 + (0) Z_2 + (K_2) Z_3 + (-aK_2) \theta + (fK_2) \phi] / M_2$$

$$\omega^2 \theta = [(0) Z_1 + (aK_2) Z_2 + (-aK_2) Z_3 + (2a^2 K_2 + 2b^2 K_3) \theta + (0) \phi] / I_{xx}$$

$$\omega^2 \phi = [(dK_1 - 2fK_2 - 2eK_3) Z_1 + (fK_2) Z_2 + (fK_2) Z_3 + (0) \theta + (d^2 K_1 + 2f^2 K_2 + 2e^2 K_3) \phi] / I_{yy}$$

The eigenvalues and eigenvectors of the matrix formed by the coefficients on the right hand sides of the above equations give the squares of the natural frequencies and the normal modes of vibration of the vehicle.

4. Computer Program for Obtaining Natural Frequencies

The computer program for finding the natural frequencies merely computes the coefficients of the matrix described and then obtains the eigenvalues of this matrix using the Jacobi method as described in reference 2-1.

5. Results of Computer Programs

The results of the digital computer programs are described briefly below and more completely in Appendix I. The natural frequencies and normal modes of vibration for the vehicle are given in Table 2-1.

Table 2-1

Natural Frequencies and Normal Modes of Vibration  
of Model Vehicle

Mode	Roll	Pitch	Plunge	First Flexible Wing	Second Flexible Wing
$f_n$ (cps)	0.40890	0.98783	1.43206	2.94177	4.36636
$Z_1$	0	0.19759	1.00000	1.00000	0
$Z_2$	-1.00000	1.00000	0.26345	-0.46086	1.00000
$Z_3$	1.00000	1.00000	0.26345	-0.46086	-1.00000
$\theta$	0.02286	0	0	0	0.21870
$\phi$	0	-0.01517	0.05064	-0.02892	0

The vehicle responses printed out by the computer program consist of

- (1) displacement and velocity across the nose gear damper
- (2) displacement and velocity across one main gear damper
- (3) incremental load factor and angular accelerations for the main mass  $M_1$
- (4) the displacement across one of the springs  $K_2$ .

# Contrails

These responses were computed for thirteen different forms of excitation to the model vehicle. The first input to the vehicle consisted of the profile of Sewart Air Force Base taxiway 2 (an especially rough taxiway). Three tracks down this runway were used as the inputs  $h_1(x)$ ,  $h_2(x)$  and  $h_3(x)$ . The vehicle was assumed to be moving with a constant velocity of 50 fps. The rest of the excitations to the vehicle consisted of discrete bumps of the form

$$Z = \frac{h}{2} \left( 1 + \cos \frac{2\pi x}{l} \right)$$

where  $h$  is the height of the bump and  $l$  is its length. This bump was assumed to extend across the entire width of the runway. A bump height of 0.1 ft was used for all of the data. The vehicle velocity for the first six sets of discrete bump output is 50 fps. The bump lengths were varied from 10 to 110 ft in 20-ft steps. The bump length for the second six sets of discrete bump output is 70 ft. The vehicle velocity was varied from 10 to 110 fps in steps of 20 fps.

The results of the computer study indicate that the model vehicle is conservatively designed for operation on prepared surfaces. The maximum incremental load factor was 0.06 G. The other measured responses were also rather low.

The responses of the vehicle were obtained for operation over a distance of 3600 ft on Sewart Air Force Base taxiway 2 (Project Identification No. 1129). A representative portion of the responses is plotted in Appendix I. The power spectral density of the wing deflection was computed from these data. This power spectrum is shown in Fig. 2-2. The maximum main gear deflection was approximately 0.25 ft and the maximum main gear velocity was approximately 2.5 fps. These values are low enough to justify the assumption of linear spring and damping constants. The maxima for the nose gear deflection and velocity were approximately 0.2 ft and 2.5 fps, again justifying the assumption of linear system constants. The low values of the maximum incremental load factor (0.06 G), pitch acceleration (0.08 rad/sec<sup>2</sup>) and roll acceleration (0.08 rad/sec<sup>2</sup>) indicate the conservative design of this vehicle for taxiing. The maximum wing deflection was approximately 0.3 ft. This is less than the static deflection of the wing, 0.417 ft, and if the wing is of conventional design indicates a low stress level in the wing. The following general trends were observed for the output responses to the discrete bump inputs.

## (1) Incremental load factor

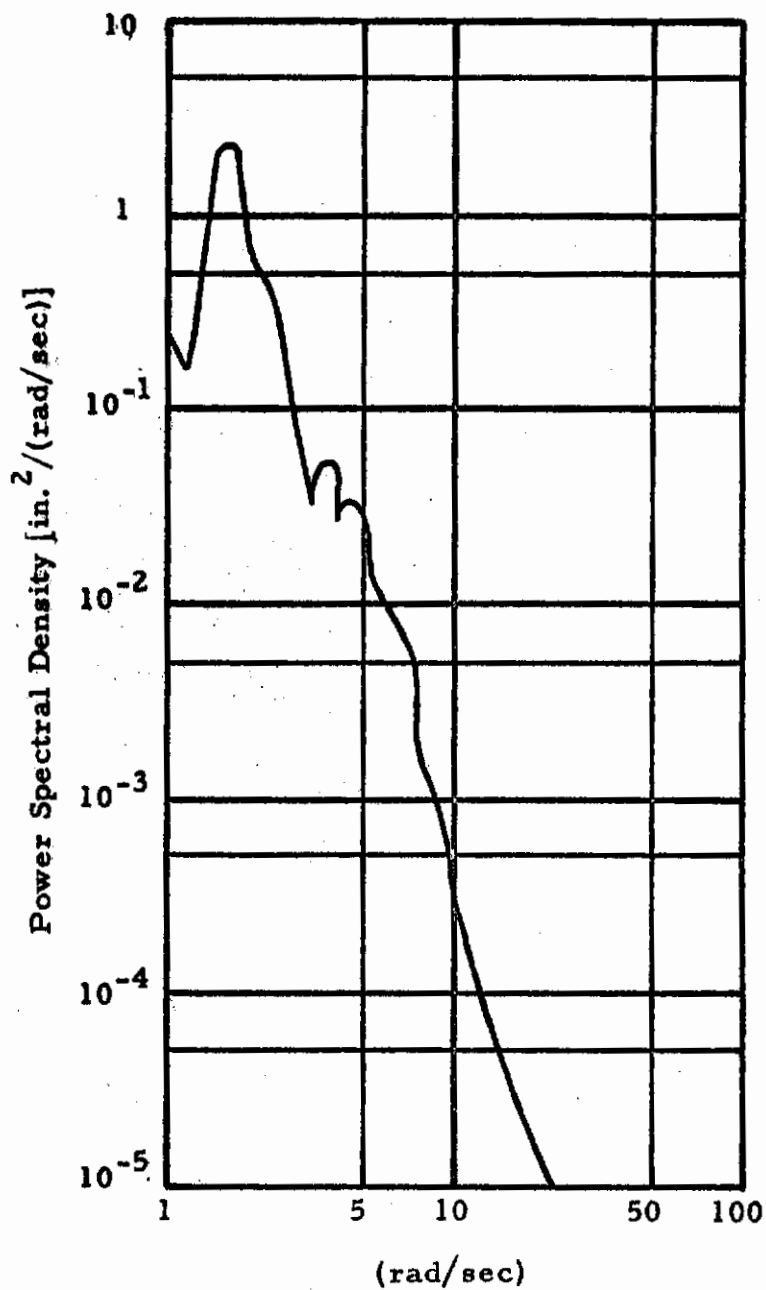
The magnitude of the peak incremental load factor decreased with increasing bump length and increased with increasing vehicle velocity.

## (2) Pitch acceleration

The magnitude of the peak pitch acceleration decreased with increasing bump length and increased with increasing vehicle velocity.

## (3) Roll acceleration

The roll acceleration was always zero since the input was always symmetrical about the centerline of the vehicle [ $h_2(x) \equiv h_3(x)$ ].



**Fig. 2-2 Power Spectrum for Wing Deflection Computed from Results of the Deterministic Analysis**



## (4) Right wing deflection

The magnitude of the peak right wing deflection increased with increasing bump length up to 70 ft. It then decreased slightly with further increase in bump length. The magnitude of the peak right wing deflection is a minimum for the lowest velocity, 10 fps, becomes a maximum at 30 and 50 fps and then decreased with further increase of velocity.

## (5) Main gear deflection

The magnitude of the peak main gear deflection decreases with increasing bump length up to a length of 90 ft. At 110 ft, the peak deflection is increased. The magnitude of the peak main gear deflection is a minimum for a velocity of 10 fps, becomes a maximum at 30 fps, decreases to a relative minimum at 50 fps and increases with further increase of velocity.

## (6) Main gear velocity

The magnitude of the peak main gear velocity decreases with increasing bump length and increases with increasing vehicle velocity.

## (7) Nose gear deflection

The peak magnitude of the nose gear deflection is a relative minimum at a bump length of 30 ft, increases to a maximum at 70 ft and then decreases with further increase in bump length. The peak magnitude of the nose gear deflection increases with increasing vehicle velocity until it reaches a maximum at 50 fps and then decreases with further increase of velocity.

## (8) Nose gear velocity

The magnitude of the peak nose gear velocity decreases with increasing bump length and increases with increasing vehicle velocity.

The spacing between the nose and main gears seemed to have very little effect on the computed responses. This can be explained by the large amount of damping in the nose gear and the geometrical isolation of the nose gear from the remainder of the vehicle. The large damping absorbs much of the energy of an input at the nose gear and the geometry of the vehicle lessens the effects of the nose gear input on the responses of the vehicle.

The output responses were sensitive to bump length and vehicle velocity. This is to be expected since the magnitude and frequency of the input are dependent on these parameters.

SECTION II

STATISTICAL ANALYSIS

The work described in this section was completed as part of an IIT Research Institute house-sponsored project. Subsection 1 is a statistical analysis of the motion of a linear one-degree-of-freedom vehicle moving on a random track. This analysis is included as an introduction to random vibration analysis. Subsection 2 gives a statistical analysis of the five-degree-of-freedom model studied in Section I.

1. Analysis of Single-Degree-of-Freedom Vehicle

In the statistical analysis of the motion of a linear one-degree-of-freedom vehicle moving on a random track, the input to the vehicle is given in terms of the power spectral density of the track. The responses computed include power spectral densities, standard deviations, and the probability density functions for the displacement, velocity and acceleration of the mass. The number of times, on the average, that the responses pass through zero per unit time and the number of times that the responses exceed a given value are also computed.

The system to be considered is shown in Figure 2-3. The mass of magnitude  $M$  is constrained to move with no rotations in the plane of the page with a constant horizontal velocity  $V_0$ . The mass is supported by a linear spring having a spring constant  $K$  and a viscous damper with damping constant  $C$ . The point follower "A" maintains constant contact with the track. The coordinates of the mass center at time  $t$  are  $[x(t), y(t)]$ . The track elevation,  $y_0(x)$ , is a sample function of a second-order stationary ergodic gaussian random process.

The equation of motion for the vehicle is

$$M\ddot{y} = -K(y - y_0) - C(\dot{y} - \dot{y}_0) \quad (2-1)$$

( $\dot{\phantom{y}}$  indicates the derivative with respect to time) where  $y$  is measured from the position of static equilibrium. This equation can be rewritten

$$\ddot{y} + \frac{C}{M} \dot{y} + \frac{K}{M} y = \frac{K}{M} y_0 + \frac{C}{M} \dot{y}_0 \quad (2-2)$$

Let

$$\omega_n^2 = \frac{K}{M}$$

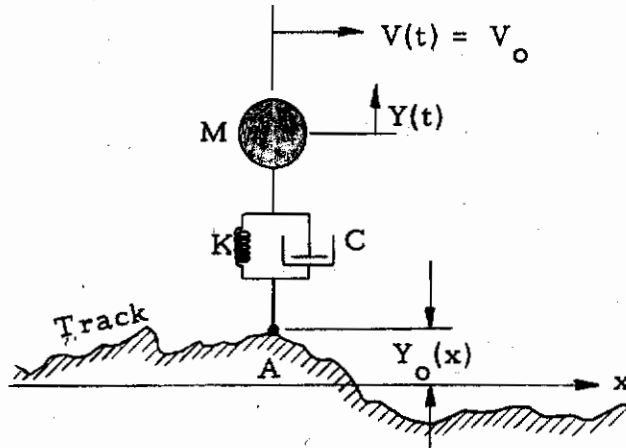


Fig. 2-3 One-Degree-of-Freedom Vehicle

and

$$\gamma = \frac{C}{C_{\text{critical}}} = \frac{C}{2\sqrt{KM}} = \frac{C}{2\omega_n M}$$

Then equation 2-2 becomes

$$\ddot{y} + 2\omega_n \gamma \dot{y} + \omega_n^2 y = \omega_n^2 y_0 + 2\omega_n \gamma \dot{y}_0 \quad (2-3)$$

Let us find the complex frequency response of this system. To do this we assume

$$y_0 = a e^{i\omega t} \quad (2-4)$$

where  $a$  is a complex constant and  $\omega$  is a real constant. Taking the derivative of (2-4) with respect to time

$$\dot{y}_0 = i\omega a e^{i\omega t}$$

Now equation (2-3) can be written

$$\ddot{y} + 2\omega_n \gamma \dot{y} + \omega_n^2 y = (\omega_n^2 + i2\omega_n \gamma \omega) a e^{i\omega t} \quad (2-5)$$

The solution of this equation is

$$y = D e^{(\lambda_1 t)} + E e^{(\lambda_2 t)} + F e^{i\omega t}$$

where

$$\lambda_1 = \omega_n (-\zeta + i\sqrt{1-\zeta^2})$$

$$\lambda_2 = \omega_n (-\zeta - i\sqrt{1-\zeta^2})$$

D and E can be evaluated from the initial conditions, and F is evaluated below. In passing we note that

$$y = D e^{(\lambda_1 t)} + E e^{(\lambda_2 t)}$$

is the transient portion of the solution and

$$y = F e^{(i\omega t)} \tag{2-6}$$

is the steady state portion of the solution. From equation (2-6)

$$\dot{y} = i\omega F e^{(i\omega t)} \tag{2-7}$$

and

$$\ddot{y} = -\omega^2 F e^{(i\omega t)} \tag{2-8}$$

Substituting equations 2-6, 2-7, and 2-8 into equation 2-5 we obtain

$$\begin{aligned} -\omega^2 F e^{(i\omega t)} + i 2\omega_n \zeta \omega F e^{(i\omega t)} + \omega_n^2 F e^{(i\omega t)} \\ = (\omega_n^2 + i 2\omega_n \zeta \omega) a e^{(i\omega t)} \end{aligned} \tag{2-9}$$

The complex frequency response  $H(\omega)$  is given by

$$H(\omega) = F/a.$$

Solving for  $H(\omega)$  from equation 2-9,

$$H(\omega) = \frac{(\omega_n^2 + i 2\omega_n \zeta \omega)}{\omega_n^2 - \omega^2 + i 2\omega_n \zeta \omega} = \frac{1}{1 - \frac{\omega^2}{\omega_n^2 + i 2\omega_n \zeta \omega}} \tag{2-10}$$

The square of the modulus of the complex frequency response is given by

$$|H(\omega)|^2 = H(\omega) \overline{H(\omega)} \tag{2-11}$$

where the bar denotes the complex conjugate.

$$\begin{aligned}
 |H(\omega)|^2 &= \left[ \frac{1}{1 - \frac{\omega^2}{\omega_n^2 + i2\omega_n\zeta\omega}} \right] \left[ \frac{1}{1 - \frac{\omega^2}{\omega_n^2 - i2\omega_n\zeta\omega}} \right] \\
 &= \frac{\omega_n^4 + (2\omega_n\zeta\omega)^2}{(\omega_n^2 - \omega^2)^2 + (2\omega_n\zeta\omega)^2} = \frac{K^2 + C^2\omega^2}{(K - M\omega^2)^2 + C^2\omega^2} \quad (2-12)
 \end{aligned}$$

The power spectrum of the vertical motion of the vehicle  $S_y(\omega)$  is related to the power spectrum of the input  $S_{y_0}(\omega)$  by

$$S_y(\omega) = |H(\omega)|^2 S_{y_0}(\omega). \quad (2-13)$$

The power spectra of the vertical velocity and acceleration of the vehicle  $S_{\dot{y}}(\omega)$  and  $S_{\ddot{y}}(\omega)$  are easily shown to be

$$S_{\dot{y}}(\omega) = |H(\omega)|^2 \omega^2 S_{y_0}(\omega) \quad (2-14)$$

$$S_{\ddot{y}}(\omega) = |H(\omega)|^2 \omega^4 S_{y_0}(\omega) \quad (2-15)$$

The power spectrum describing the track  $S'(\Omega)$  must be converted to  $S_{y_0}(\omega)$  to use equations 2-13, 2-14, and 2-15. Here  $\Omega$  is the frequency of track oscillations in radians per unit length. Since the velocity  $V_0$  of the vehicle is constant,

$$x = V_0 t$$

and

$$\Omega = \omega / V_0.$$

The autocorrelation function  $R_{y_0}'(\gamma)$  is given by

$$R_{y_0}'(\gamma) = \lim_{X \rightarrow \infty} \frac{1}{2X} \int_{-X}^X y_0(x) y_0(x + \gamma) dx \quad (2-16)$$

or, using equation 2-15 and letting

$$y_0'(t) = y_0(V_0 t),$$

$$\tau = \gamma / V_0,$$

and

$$T = X/V_o,$$

we obtain

$$\begin{aligned} R'_{y_o}(\gamma) &= \lim_{T \rightarrow \infty} \frac{1}{2V_o T} \int_{-T}^T y'_o(t) y'_o(t + \tau) V_o dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T y'_o(t) y'_o(t + \tau) dt \\ &= R_{y_o}(\tau). \end{aligned} \tag{2-17}$$

The power spectrum  $S_{y_o}(\omega)$  is given by

$$S_{y_o}(\omega) = 2 \int_{-\infty}^{\infty} R_{y_o}(\tau) e^{i\omega\tau} d\tau \tag{2-18}$$

$$\begin{aligned} S_{y_o}(\omega) &= 2 \int_{-\infty}^{\infty} R'_{y_o}(\gamma) e^{i\Omega\gamma} \frac{d\gamma}{V_o} = \frac{1}{V_o} S'_{y_o}(\Omega) \\ &= \frac{1}{V_o} S'_{y_o}(\omega/V_o). \end{aligned} \tag{2-19}$$

If  $y_o(x)$  has zero mean then it can easily be shown that  $y(t)$ ,  $\dot{y}(t)$  and  $\ddot{y}(t)$  have zero mean. The autocorrelation function  $R_y(\tau)$  is related to the output power spectrum by

$$R_y(\tau) = \frac{1}{2\pi} \int_0^{\infty} S_y(\omega) e^{-i\omega\tau} d\omega. \tag{2-20}$$

The square of the standard deviation or mean square  $\sigma_y^2$  is given by

$$\sigma_y^2 = R_y(0) = \frac{1}{2\pi} \int_0^{\infty} S_y(\omega) d\omega. \tag{2-21}$$

Since the input to our linear vehicle is a gaussian random process the output must also be gaussian. The probability density function,  $P_y(y)$ , for  $y(t)$  is therefore given by

$$P_y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} \exp(-y^2/2\sigma_y^2) \quad (2-22)$$

The number of times  $n_o^y$  that  $y(t)$  passes through zero per unit time is, on the average,

$$n_o^y = \frac{1}{\pi} \left[ \frac{\int_0^\infty \omega^2 S_y(\omega) d\omega}{\int_0^\infty S_y(\omega) d\omega} \right]^{1/2} \quad (2-23)$$

The number of times  $n_Y$  that  $y(t)$  exceeds a particular value, say  $Y$ , per unit time is, on the average,

$$n_Y = \frac{n_o^y}{2} \exp\left(-\frac{Y^2}{2\sigma_y^2}\right) \quad (2-24)$$

Equations corresponding to 2-20, 2-21, 2-22, 2-23, and 2-24 can also be written for the output velocity and acceleration. These equations are as follows.

$$R_{\dot{y}}(\tau) = \frac{1}{2\pi} \int_0^\infty S_{\dot{y}}(\omega) e^{i\omega\tau} d\omega \quad (2-25)$$

$$\sigma_{\dot{y}}^2 = R_{\dot{y}}(0) = \frac{1}{2\pi} \int_0^\infty S_{\dot{y}}(\omega) d\omega \quad (2-26)$$

$$P_{\dot{y}}(\dot{y}) = \frac{1}{\sqrt{2\pi}\sigma_{\dot{y}}} \exp\left(-\frac{\dot{y}^2}{2\sigma_{\dot{y}}^2}\right) \quad (2-27)$$

$$n_o^{\dot{y}} = \frac{1}{\pi} \left[ \frac{\int_0^\infty \omega^2 S_{\dot{y}}(\omega) d\omega}{\int_0^\infty S_{\dot{y}}(\omega) d\omega} \right]^{1/2} \quad (2-28)$$

$$n_Y^{\dot{y}} = \frac{n_o^{\dot{y}}}{2} \exp\left(-\frac{\dot{Y}^2}{2\sigma_{\dot{y}}^2}\right) \quad (2-29)$$

$$R_{\ddot{y}}(\tau) = \frac{1}{2\pi} \int_0^{\infty} S_{\ddot{y}}(\omega) e^{i\omega\tau} d\omega \quad (2-30)$$

$$\sigma_{\ddot{y}}^2 = R_{\ddot{y}}(0) = \frac{1}{2\pi} \int_0^{\infty} S_{\ddot{y}}(\omega) d\omega \quad (2-31)$$

$$P_{\ddot{y}}(\ddot{y}) = \frac{1}{\sqrt{2\pi\sigma_{\ddot{y}}^2}} \exp\left(-\frac{\ddot{y}^2}{2\sigma_{\ddot{y}}^2}\right) \quad (2-32)$$

$$n_{\ddot{y}} = \frac{1}{\pi} \left[ \frac{\int_0^{\infty} \omega^2 S_{\ddot{y}}(\omega) d\omega}{\int_0^{\infty} S_{\ddot{y}}(\omega) d\omega} \right]^{\frac{1}{2}} \quad (2-33)$$

$$n_{\ddot{y}}^{\ddot{y}} = \frac{n_{\ddot{y}}}{2} \exp\left(-\frac{\ddot{y}^2}{2\sigma_{\ddot{y}}^2}\right) \quad (2-34)$$

### Example Problem

As an example suppose a one-wheeled trailer (see Fig. 2-4) is being towed at 15 mph along a runway which has a roughness specified by the power spectral density plot given in Figure 2-5. The elevation of this runway has zero mean. The unsprung weight of the trailer is 1000 lb, the equivalent spring supporting this weight has a spring constant of 2000 lb/ft, and the damping constant of the equivalent damper is 200 lb-sec/ft. Let us find the power spectra of the vertical displacement, velocity and acceleration. The unsprung mass as well as any rotational motions of the trailer will be neglected. It will be assumed that the trailer wheel is always in contact with the ground. Figure 2-6 can be immediately obtained from Figure 2-5 using equation 2-18.

$$S_{y_0}(\omega) = \frac{1}{V_0} S'_{y_0}(\omega/V_0).$$

$$V_0 = 15 \frac{\text{mph}}{\text{hr}} \cdot 5280 \frac{\text{ft}}{\text{mi}} / 3600 \frac{\text{sec}}{\text{hr}} = 22 \text{ fps.}$$



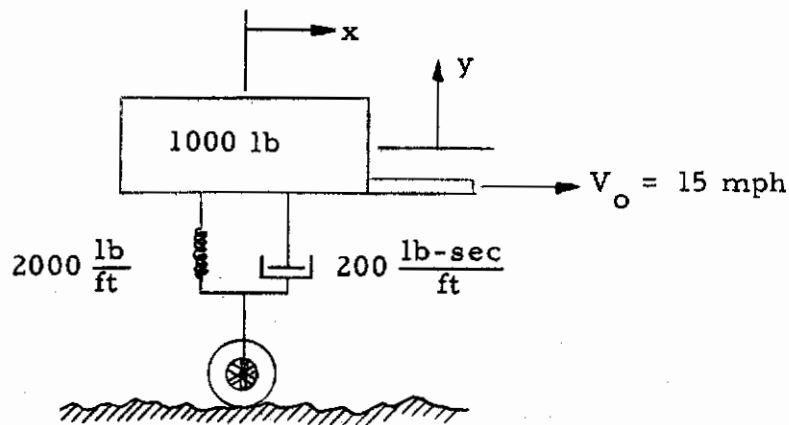


Fig. 2-4 One-Wheeled Trailer

Therefore

$$S_{y_o}(\omega) = \frac{1}{22} S'_{y_o}(\omega/22)$$

The square of the modulus of the complex frequency response is given by equation 2-12

$$\begin{aligned} |H(\omega)|^2 &= \frac{K^2 + C^2 \omega^2}{(K - M\omega)^2 + C^2 \omega^2} \\ &= \frac{(2000)^2 + (200)^2 \omega^2}{(2000 - \frac{1000}{32.2} \omega^2)^2 + (200)^2 \omega^2} \\ &= \frac{4147 + 41.47 \omega^2}{(64.4 - \omega^2)^2 + 41.47 \omega^2} \end{aligned}$$

A plot of  $|H(\omega)|^2$  versus  $\omega$  is given in Figure 2-7. The computations are shown in Table 2-2.

The power spectrum of the vertical displacement is given by equation 2-13

$$S_y(\omega) = |H(\omega)|^2 S_{y_o}(\omega)$$

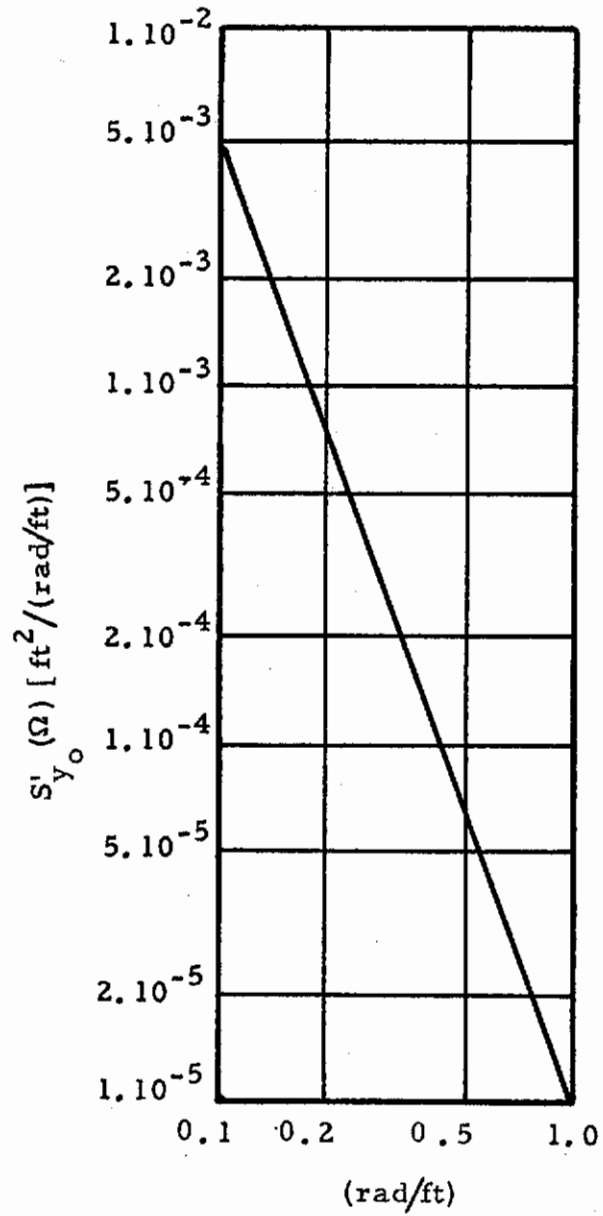


Fig. 2-5 Runway Power Spectral Density

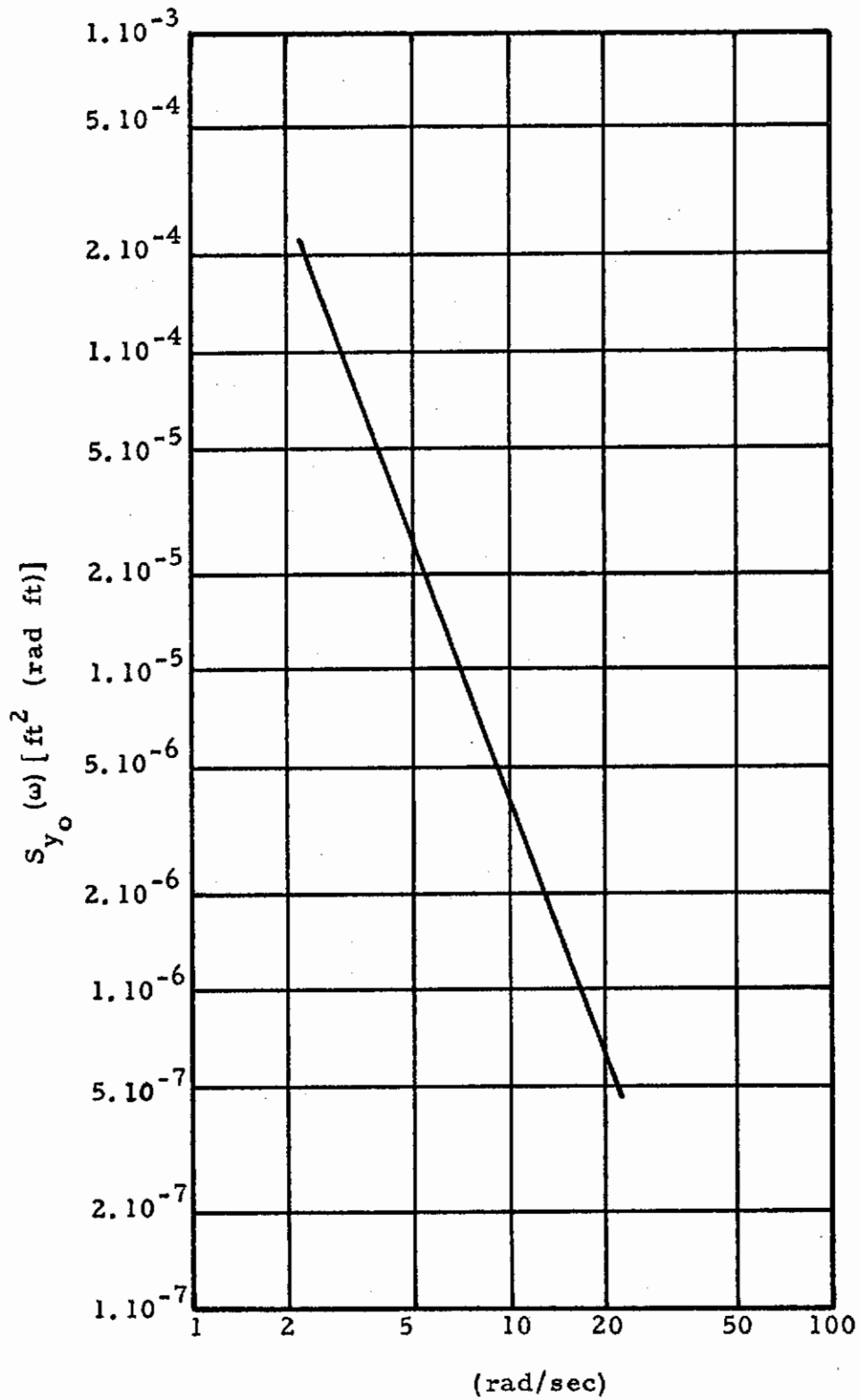


Fig. 2-6 Power Spectral Density of Input to Vehicle

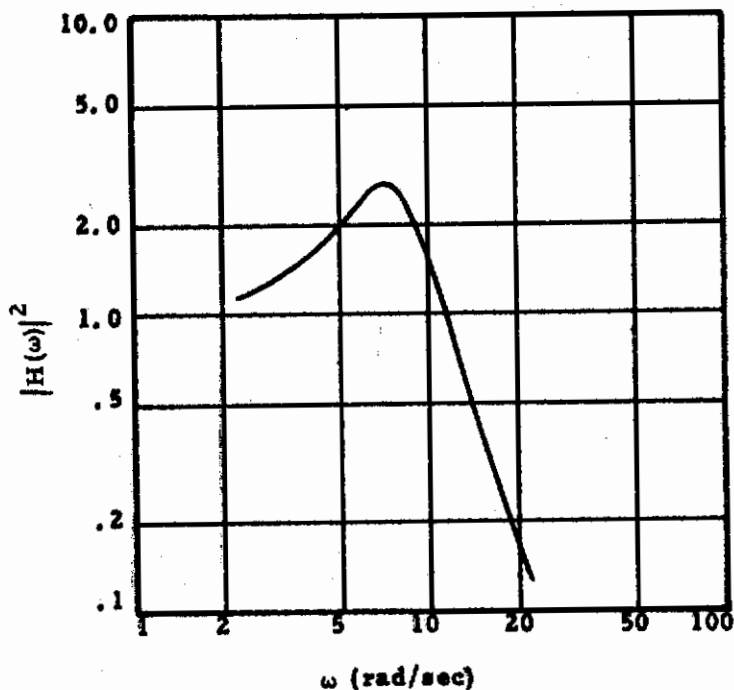


Fig. 2-7 Complex Frequency Response for Vehicle

and the power spectra of the vertical velocity and acceleration are given by equations 2-14 and 2-15

$$S_{\dot{y}}(\omega) = |H(\omega)|^2 \omega^2 S_{y_0}(\omega)$$

$$S_{\ddot{y}}(\omega) = |H(\omega)|^2 \omega^4 S_{y_0}(\omega).$$

Plots of  $S_{y_0}(\omega)$ ,  $S_{\dot{y}}(\omega)$  and  $S_{\ddot{y}}(\omega)$  are given in Figures 2-8, 2-9 and 2-10. The computations are shown in Table 2-2.

The mean squares of the outputs can be computed using equations 2-21, 2-26, and 2-31.

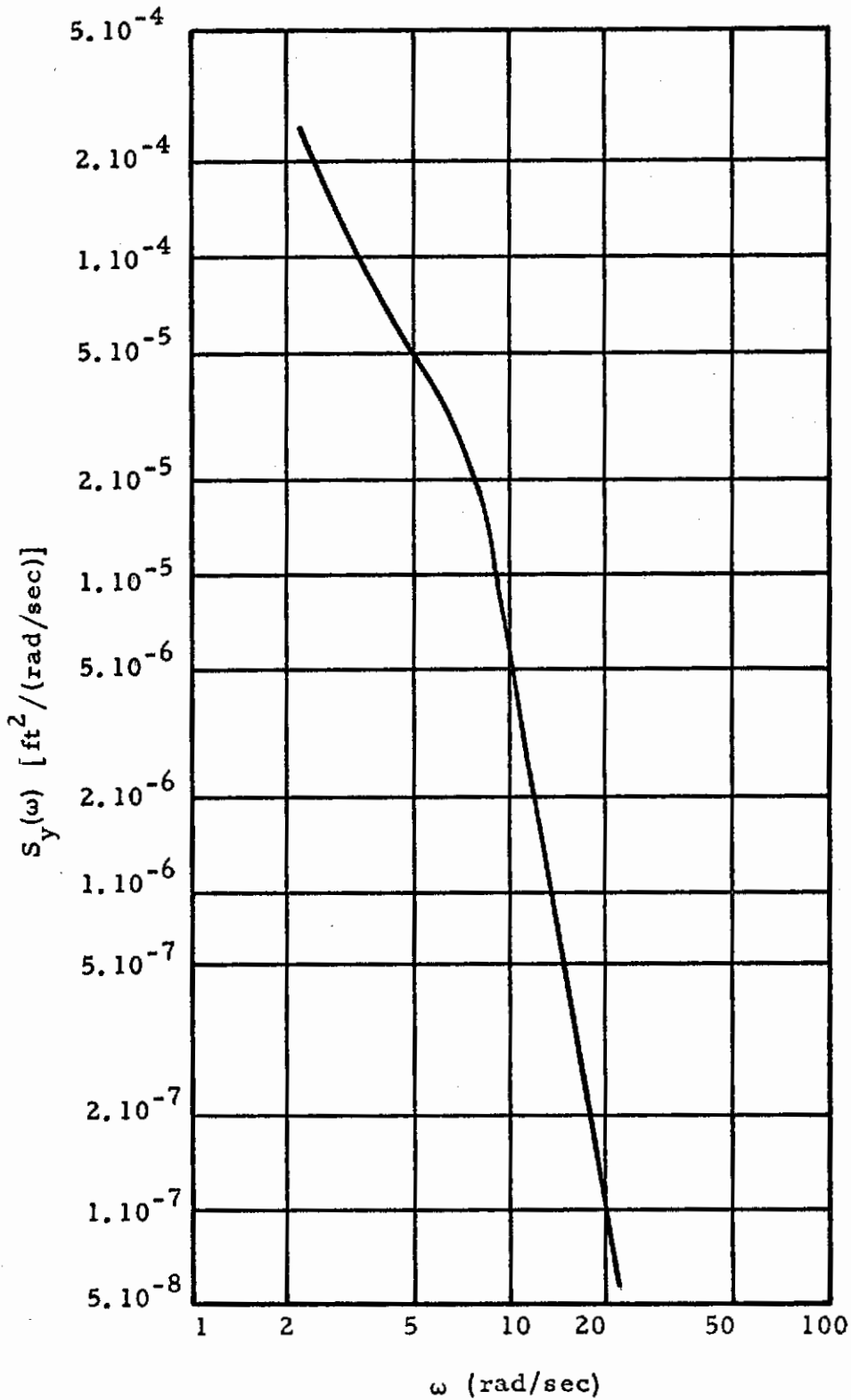


Fig. 2-8 Vehicle Displacement Power Spectrum

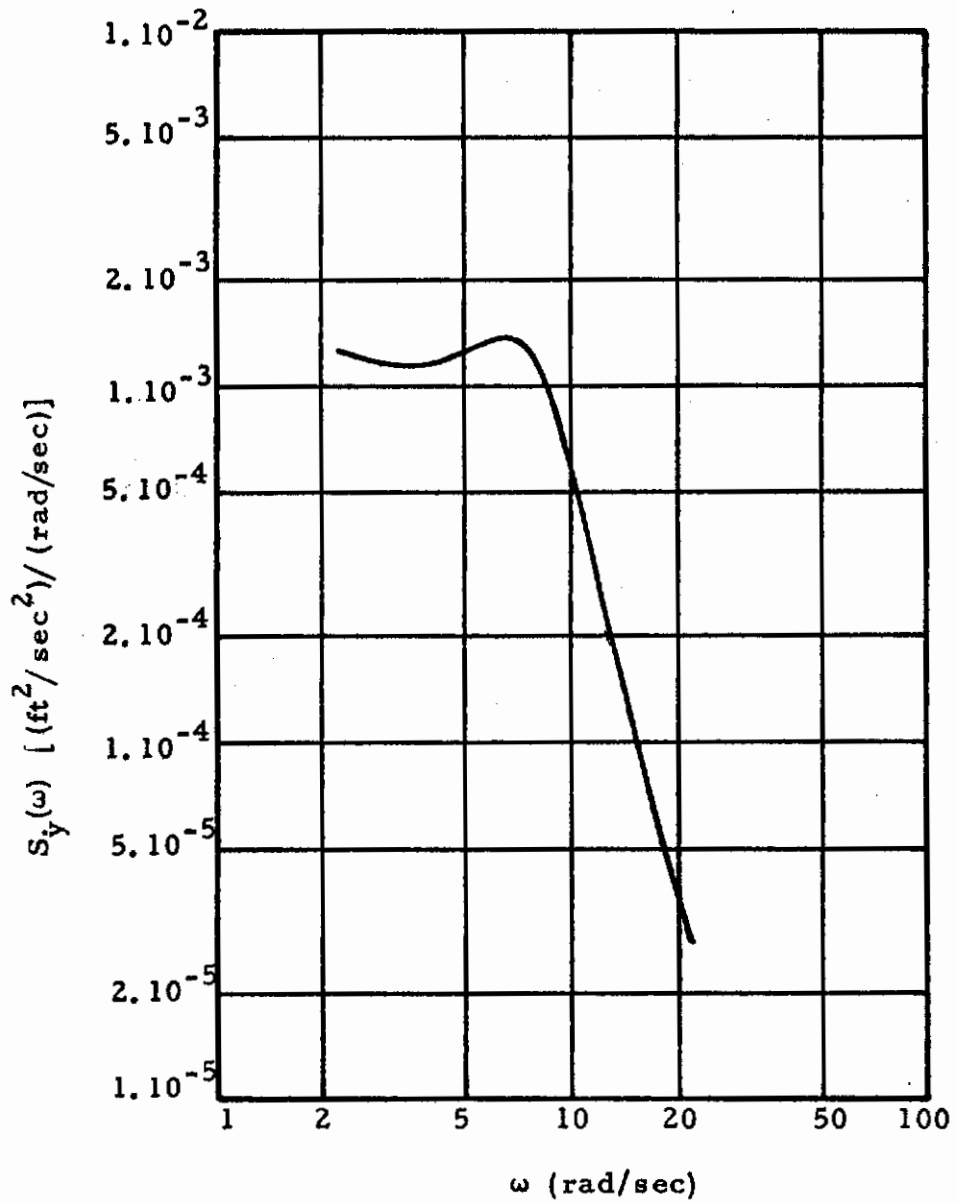


Fig. 2-9 Vehicle Velocity Power Spectrum

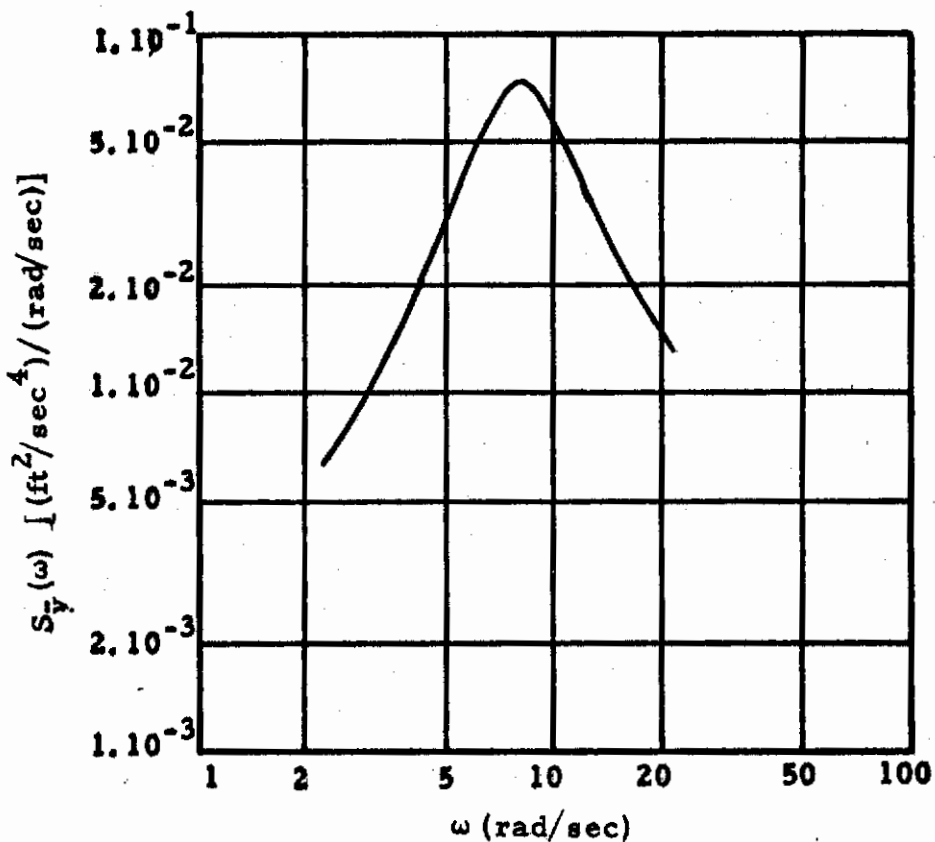


Fig. 2-10 Vehicle Acceleration Power Spectrum

$$\sigma_y^2 = \frac{1}{2\pi} \int_0^\infty S_y(\omega) d\omega \approx \frac{1}{2\pi} \int_{2.2}^{22} S_y(\omega) d\omega$$

$$\sigma_{\dot{y}}^2 = \frac{1}{2\pi} \int_0^\infty S_{\dot{y}}(\omega) d\omega \approx \frac{1}{2\pi} \int_{2.2}^{22} S_{\dot{y}}(\omega) d\omega$$

$$\sigma_{\ddot{y}}^2 = \frac{1}{2\pi} \int_0^\infty S_{\ddot{y}}(\omega) d\omega \approx \frac{1}{2\pi} \int_{2.2}^{22} S_{\ddot{y}}(\omega) d\omega$$

Table 2-2

Computation of Vehicle Responses

$\omega$	$\omega^2$	$41.47 \omega^2$	$4147 + 41.47 \omega^2$	$64.40 - \omega^2$	$(64.60 - \omega^2)^2$	$(64.40 - \omega^2)^2 + 41.47 \omega^2$	$ H(\omega) ^2$	$S_{y_0}(\omega)$	$\omega^4$	$S_{\dot{y}}(\omega)$	$S_{\ddot{y}}(\omega)$	$S_{\ddot{y}}(\omega)$
2.2	4.84	201	4348	59.56	3547	3748	1.160	$2.27 \cdot 10^{-4}$		$2.63 \cdot 10^{-4}$	$2.71 \cdot 10^{-5}$	$1.31 \cdot 10^{-2}$
3.0	9.00	373	4520	55.40	3069	3442	1.313	$9.83 \cdot 10^{-5}$	81	$1.29 \cdot 10^{-4}$	$1.16 \cdot 10^{-3}$	$1.04 \cdot 10^{-2}$
4.0	16.00	664	4811	48.40	2342	3006	1.600	$4.53 \cdot 10^{-5}$	256	$7.25 \cdot 10^{-5}$	$1.16 \cdot 10^{-3}$	$1.86 \cdot 10^{-2}$
5.0	25.00	1037	5184	39.40	1552	2589	2.002	$2.48 \cdot 10^{-5}$	625	$4.96 \cdot 10^{-5}$	$1.24 \cdot 10^{-3}$	$3.10 \cdot 10^{-2}$
6.0	36.00	1493	5640	28.40	807	2300	2.452	$1.52 \cdot 10^{-5}$	1296	$3.73 \cdot 10^{-5}$	$1.34 \cdot 10^{-3}$	$4.83 \cdot 10^{-2}$
6.5	42.25	1752	5899	22.15	491	2243	2.630	$1.22 \cdot 10^{-5}$	1785	$3.21 \cdot 10^{-5}$	$1.36 \cdot 10^{-3}$	$5.73 \cdot 10^{-2}$
7.0	49.00	2032	6179	15.40	237	2269	2.723	$1.00 \cdot 10^{-5}$	2401	$2.72 \cdot 10^{-5}$	$1.33 \cdot 10^{-3}$	$6.53 \cdot 10^{-2}$
7.5	56.25	2333	6480	8.15	66	2399	2.701	$8.30 \cdot 10^{-6}$	3164	$2.24 \cdot 10^{-5}$	$1.26 \cdot 10^{-3}$	$7.09 \cdot 10^{-2}$
8.0	64.00	2654	6801	.40	0	2654	2.563	$6.97 \cdot 10^{-6}$	4096	$1.79 \cdot 10^{-5}$	$1.15 \cdot 10^{-3}$	$7.33 \cdot 10^{-2}$
9.0	81.00	3359	7506	-16.60	276	3635	2.065	$5.07 \cdot 10^{-6}$	6561	$1.05 \cdot 10^{-5}$	$8.51 \cdot 10^{-4}$	$6.89 \cdot 10^{-2}$
10.0	100.00	4147	8294	-35.60	1267	5414	1.532	$3.82 \cdot 10^{-6}$	10000	$5.85 \cdot 10^{-6}$	$5.85 \cdot 10^{-4}$	$5.85 \cdot 10^{-2}$
12.0	144.00	5972	10119	-79.60	6336	12308	.822	$2.33 \cdot 10^{-6}$	20736	$1.92 \cdot 10^{-6}$	$2.76 \cdot 10^{-4}$	$3.98 \cdot 10^{-2}$
14.0	196.00	8128	12275	-131.60	17319	25447	.482	$1.54 \cdot 10^{-6}$	38416	$7.42 \cdot 10^{-7}$	$1.45 \cdot 10^{-4}$	$2.85 \cdot 10^{-2}$
16.0	256.00	10616	14763	-191.60	36711	47327	.312	$1.07 \cdot 10^{-6}$	65536	$3.34 \cdot 10^{-7}$	$8.55 \cdot 10^{-5}$	$2.19 \cdot 10^{-2}$
18.0	324.00	13436	17583	-259.60	67392	80828	.218	$7.81 \cdot 10^{-7}$	104976	$1.70 \cdot 10^{-7}$	$5.51 \cdot 10^{-5}$	$1.78 \cdot 10^{-2}$
20.0	400.00	16588	20735	-335.60	112627	129215	.160	$5.88 \cdot 10^{-7}$	160000	$9.41 \cdot 10^{-8}$	$3.76 \cdot 10^{-5}$	$1.51 \cdot 10^{-2}$
22.0	484.00	20071	24218	-419.60	176064	196135	.123	$4.55 \cdot 10^{-7}$	234256	$5.60 \cdot 10^{-8}$	$2.71 \cdot 10^{-5}$	$1.31 \cdot 10^{-2}$



These integrals are evaluated numerically in Table 2-3. Using these results we obtain

$$\sigma_y^2 = \frac{1}{2\pi} (4.52 \cdot 10^{-4}) = 0.0000720 \text{ ft}^2$$

$$\sigma_y = 0.00849 \text{ ft}$$

$$\sigma_{\dot{y}}^2 = \frac{1}{2\pi} (10.77 \cdot 10^{-3}) = 0.00171 \text{ ft}^2/\text{sec}^2$$

$$\sigma_{\dot{y}} = 0.0414 \text{ fps}$$

$$\sigma_{\ddot{y}}^2 = \frac{1}{2\pi} (66.59 \cdot 10^{-2}) = 0.106 \text{ ft}^2/\text{sec}^4$$

$$\sigma_{\ddot{y}} = 0.326 \text{ ft}/\text{sec}^2.$$

The probability density functions are given by equations 2-22, 2-27 and 2-32 to be

$$P_y(y) = \frac{1}{\sqrt{2\pi} \sigma_y} \exp\left(-\frac{y^2}{2\sigma_y^2}\right) = 47.0 \exp(-6940 y^2) \text{ ft}^{-1}$$

$$P_{\dot{y}}(\dot{y}) = \frac{1}{\sqrt{2\pi} \sigma_{\dot{y}}} \exp\left(-\frac{\dot{y}^2}{2\sigma_{\dot{y}}^2}\right) = 9.65 \exp(-292 \dot{y}^2) \text{ sec}/\text{ft}$$

$$P_{\ddot{y}}(\ddot{y}) = \frac{1}{\sqrt{2\pi} \sigma_{\ddot{y}}} \exp\left(-\frac{\ddot{y}^2}{2\sigma_{\ddot{y}}^2}\right) = 1.22 \exp(-4.72 \ddot{y}^2) \text{ sec}^2/\text{ft}.$$

The number of times that  $y(t)$  passes through zero is, on the average, given by equation 2-23 to be

$$n_o^y = \frac{1}{\pi} \left[ \frac{\int_0^\infty \omega^2 S_y(\omega) d\omega}{\int_0^\infty S_y(\omega) d\omega} \right]^{\frac{1}{2}} \approx \frac{1}{\pi} \left[ \frac{\int_{2.2}^{22} \omega^2 S_y(\omega) d\omega}{\int_{2.2}^{22} S_y(\omega) d\omega} \right]^{\frac{1}{2}}$$

Evaluating the integrals numerically in Table 2-3, we find

$$n_o^y = \frac{1}{\pi} \left[ \frac{11.27 \cdot 10^{-3}}{4.52 \cdot 10^{-4}} \right]^{\frac{1}{2}} = 1.59 \text{ times}/\text{sec}$$

Table 2-3

Numerical Evaluation of Integrals

$\omega$	$\Delta\omega$	$S_y(\omega)$ av.	$S_y(\omega)$ av.	$S_y(\omega)$ av.	$S_y(\omega)$ av.	$A(\omega) \cdot S_y(\omega)$ av.	$\Delta\omega \cdot S_y(\omega)$ av.	$\Delta\omega \cdot S_y(\omega)$ av.	$\omega^2$ av.	$\omega^2 \cdot \Delta\omega$ $\cdot S_y(\omega)$ av.	$\omega^2 \cdot \Delta\omega$ $\cdot S_y(\omega)$ av.	$\omega^2 \cdot \Delta\omega$ $\cdot S_y(\omega)$ av.
2.2	.8	$1.96 \cdot 10^{-4}$	$1.22 \cdot 10^{-3}$	$8.28 \cdot 10^{-3}$	$1.57 \cdot 10^{-4}$	$.98 \cdot 10^{-3}$	$.66 \cdot 10^{-2}$	6.920	$1.09 \cdot 10^{-3}$	$6.8 \cdot 10^{-3}$	0.05	
3.0	1.0	$1.01 \cdot 10^{-4}$	$1.16 \cdot 10^{-3}$	$1.45 \cdot 10^{-2}$	$1.01 \cdot 10^{-4}$	$1.16 \cdot 10^{-3}$	$1.45 \cdot 10^{-2}$	12.500	$1.26 \cdot 10^{-3}$	$14.5 \cdot 10^{-3}$	0.18	
4.0	1.0	$6.11 \cdot 10^{-5}$	$1.20 \cdot 10^{-3}$	$2.48 \cdot 10^{-2}$	$6.11 \cdot 10^{-5}$	$1.20 \cdot 10^{-3}$	$2.48 \cdot 10^{-2}$	20.500	$1.25 \cdot 10^{-3}$	$24.6 \cdot 10^{-3}$	0.51	
5.0	1.0	$4.35 \cdot 10^{-5}$	$1.29 \cdot 10^{-3}$	$3.97 \cdot 10^{-2}$	$4.35 \cdot 10^{-5}$	$1.29 \cdot 10^{-3}$	$3.97 \cdot 10^{-2}$	30.500	$1.33 \cdot 10^{-3}$	$39.3 \cdot 10^{-3}$	1.21	
6.0	0.5	$3.47 \cdot 10^{-5}$	$1.35 \cdot 10^{-3}$	$5.28 \cdot 10^{-2}$	$1.74 \cdot 10^{-5}$	$0.68 \cdot 10^{-3}$	$2.64 \cdot 10^{-2}$	39.125	$6.81 \cdot 10^{-4}$	$26.4 \cdot 10^{-3}$	1.03	
6.5	0.5	$2.97 \cdot 10^{-5}$	$1.35 \cdot 10^{-3}$	$6.13 \cdot 10^{-2}$	$1.49 \cdot 10^{-5}$	$0.68 \cdot 10^{-3}$	$3.07 \cdot 10^{-2}$	45.625	$6.80 \cdot 10^{-4}$	$30.8 \cdot 10^{-3}$	1.40	
7.0	0.5	$2.48 \cdot 10^{-5}$	$1.30 \cdot 10^{-3}$	$6.81 \cdot 10^{-2}$	$1.24 \cdot 10^{-5}$	$0.65 \cdot 10^{-3}$	$3.41 \cdot 10^{-2}$	52.625	$6.53 \cdot 10^{-4}$	$34.2 \cdot 10^{-3}$	1.79	
7.5	0.5	$2.02 \cdot 10^{-5}$	$1.21 \cdot 10^{-3}$	$7.21 \cdot 10^{-2}$	$1.01 \cdot 10^{-5}$	$0.61 \cdot 10^{-3}$	$3.61 \cdot 10^{-2}$	60.125	$6.07 \cdot 10^{-4}$	$36.4 \cdot 10^{-3}$	2.17	
8.0	1.0	$1.42 \cdot 10^{-5}$	$1.00 \cdot 10^{-3}$	$7.11 \cdot 10^{-2}$	$1.42 \cdot 10^{-5}$	$1.00 \cdot 10^{-3}$	$7.11 \cdot 10^{-2}$	72.500	$1.03 \cdot 10^{-3}$	$72.5 \cdot 10^{-3}$	5.15	
9.0	1.0	$8.18 \cdot 10^{-6}$	$7.18 \cdot 10^{-4}$	$6.37 \cdot 10^{-2}$	$8.18 \cdot 10^{-6}$	$0.72 \cdot 10^{-3}$	$6.37 \cdot 10^{-2}$	90.500	$7.40 \cdot 10^{-4}$	$65.0 \cdot 10^{-3}$	5.76	
10.0	2.0	$3.89 \cdot 10^{-6}$	$4.31 \cdot 10^{-4}$	$4.92 \cdot 10^{-2}$	$7.78 \cdot 10^{-6}$	$0.86 \cdot 10^{-3}$	$9.84 \cdot 10^{-2}$	122.000	$9.49 \cdot 10^{-4}$	$104.9 \cdot 10^{-3}$	12.00	
12.0	2.0	$1.33 \cdot 10^{-6}$	$2.11 \cdot 10^{-4}$	$3.42 \cdot 10^{-2}$	$2.66 \cdot 10^{-6}$	$0.42 \cdot 10^{-3}$	$6.84 \cdot 10^{-2}$	170.000	$4.52 \cdot 10^{-4}$	$71.7 \cdot 10^{-3}$	11.63	
14.0	2.0	$5.38 \cdot 10^{-7}$	$1.15 \cdot 10^{-4}$	$2.52 \cdot 10^{-2}$	$1.08 \cdot 10^{-6}$	$0.23 \cdot 10^{-3}$	$5.04 \cdot 10^{-2}$	226.000	$2.44 \cdot 10^{-4}$	$52.0 \cdot 10^{-3}$	11.39	
16.0	2.0	$2.52 \cdot 10^{-7}$	$7.03 \cdot 10^{-5}$	$1.99 \cdot 10^{-2}$	$5.04 \cdot 10^{-7}$	$0.14 \cdot 10^{-3}$	$3.98 \cdot 10^{-2}$	290.000	$1.46 \cdot 10^{-4}$	$40.9 \cdot 10^{-3}$	11.54	
18.0	2.0	$1.32 \cdot 10^{-7}$	$4.64 \cdot 10^{-5}$	$1.65 \cdot 10^{-2}$	$2.64 \cdot 10^{-7}$	$0.09 \cdot 10^{-3}$	$3.30 \cdot 10^{-2}$	362.000	$9.56 \cdot 10^{-5}$	$33.6 \cdot 10^{-3}$	11.95	
20.0	2.0	$7.51 \cdot 10^{-8}$	$3.24 \cdot 10^{-5}$	$1.41 \cdot 10^{-2}$	$1.50 \cdot 10^{-7}$	$0.06 \cdot 10^{-3}$	$2.82 \cdot 10^{-2}$	442.000	$6.63 \cdot 10^{-5}$	$28.6 \cdot 10^{-3}$	12.46	
22.0					$4.52 \cdot 10^{-4}$	$10.77 \cdot 10^{-3}$	$66.59 \cdot 10^{-2}$	TOTALS	$11.27 \cdot 10^{-3}$	$682.2 \cdot 10^{-3}$	90.22	

The number of times that  $\dot{y}(t)$  passes through zero is, on the average, given by equation 2-28 to be

$$n_{\dot{y}} = \frac{1}{\pi} \left[ \frac{\int_0^{\infty} \omega^2 S_{\dot{y}}(\omega) d\omega}{\int_0^{\infty} S_{\dot{y}}(\omega) d\omega} \right]^{\frac{1}{2}} \approx \frac{1}{\pi} \left[ \frac{\int_{2.2}^{22} \omega^2 S_{\dot{y}}(\omega) d\omega}{\int_{2.2}^{22} S_{\dot{y}}(\omega) d\omega} \right]^{\frac{1}{2}}$$

Evaluating the integrals numerically in Table 2-3, we find

$$n_{\dot{y}} = \frac{1}{\pi} \left[ \frac{682.2 \cdot 10^{-3}}{10.77 \cdot 10^{-3}} \right]^{\frac{1}{2}} = 2.54 \text{ times/sec}$$

The number of times that  $\ddot{y}(t)$  passes through zero is, on the average, given by equation 2-33 to be

$$n_{\ddot{y}} = \frac{1}{\pi} \left[ \frac{\int_0^{\infty} \omega^2 S_{\ddot{y}}(\omega) d\omega}{\int_0^{\infty} S_{\ddot{y}}(\omega) d\omega} \right]^{\frac{1}{2}} \approx \frac{1}{\pi} \left[ \frac{\int_{2.2}^{22} \omega^2 S_{\ddot{y}}(\omega) d\omega}{\int_{2.2}^{22} S_{\ddot{y}}(\omega) d\omega} \right]^{\frac{1}{2}}$$

Evaluating the integrals numerically in Table 2-3, we find

$$n_{\ddot{y}} = \frac{1}{\pi} \left[ \frac{90.22}{66.59 \cdot 10^{-2}} \right]^{\frac{1}{2}} = 3.71 \text{ times/sec}$$

The number of times that  $y(t)$  exceeds  $Y$  is, on the average, given by equation 2-24 to be

$$\begin{aligned} n_Y &= \frac{n_{\dot{y}}}{2} \exp\left(-\frac{Y^2}{2\sigma_y^2}\right) = \frac{1.59}{2} \exp\left(-\frac{Y^2}{2 \cdot 0.0000720}\right) \\ &= 0.795 \exp(-6940 Y^2) \text{ times/sec} \end{aligned}$$

The number of times that  $\dot{y}(t)$  exceeds  $\dot{Y}$  is, on the average, given by equation 2-29 to be

$$\begin{aligned} n_{\dot{Y}} &= \frac{n_{\dot{y}}}{2} \exp\left(-\frac{\dot{Y}^2}{2\sigma_{\dot{y}}^2}\right) = \frac{2.54}{2} \exp\left(-\frac{\dot{Y}^2}{2 \cdot 0.00171}\right) \\ &= 1.27 \exp(-292 \dot{Y}^2) \text{ times/sec} \end{aligned}$$

The number of times that  $\ddot{y}(t)$  exceed  $\ddot{Y}$  is, on the average, given by equation 2-34 to be

$$n_{\ddot{Y}} = \frac{n_0}{2} \exp\left(-\frac{\ddot{Y}^2}{2\sigma_{\ddot{y}}^2}\right) = \frac{3.71}{2} \exp\left(-\frac{\ddot{Y}^2}{2 \cdot 0.106}\right)$$

$$= 1.86 \exp(-4.72 \ddot{Y}^2) \text{ times/sec}$$

It is of particular interest to note the average number of times that  $y(t)$  exceeds the static deflection of the spring. Comparison of this number with  $n_0^y$  gives some notion of the amount of time that the runway must exert a downward force on the trailer wheel, or roughly, since the runway does not actually exert any downward force, some notion of the amount of time the trailer wheel will lose contact with the runway.

$$\delta = W K = 1000 \cdot 2000 = 0.5 \text{ ft}$$

$$n_{\delta} = 0.795 \exp(6940(0.5)^2) = 2.5 \cdot 10^{-754} \text{ times/sec.}$$

It is evident that the assumption that the wheel is always in contact with the ground is quite good.

Another interesting quantity might be the number of times the velocity exceeds one foot per second.

$$n_1 = 1.27 \exp(-292(1)^2) = 1.94 \cdot 10^{-127} \text{ times/sec.}$$

It might also be of interest to find the average number of times the acceleration exceeds one G.

$$n_{32.2} = 1.86 \exp(-4.72(32.2)^2) = 2.2 \cdot 10^{-2126} \text{ times/sec.}$$

## 2. Analysis of Five-Degree-of-Freedom Vehicle

In the statistical analysis of the five-degree-of-freedom vehicle (Fig. 2-1) the input to the vehicle is given in terms of the power spectra of the three runway tracks traversed by the "wheels" of this vehicle. The power spectrum of the deflection of spring  $K_2$  is computed as a response. Mass  $M_1$  is constrained to move with constant horizontal velocity along the  $x$  axis. Masses  $M_2$  are considered to be point masses having no mass moments of inertia. The profile followers are assumed to remain in constant contact with the runway. The runway profiles  $h_1(x)$ ,  $h_2(x)$  and  $h_3(x)$ , are considered to be sample functions of second order stationary ergodic gaussian random processes.  $h_1(x)$ ,  $h_2(x)$  and  $h_3(x)$  are assumed to be completely uncorrelated. Let  $H_j(\omega)$  be the

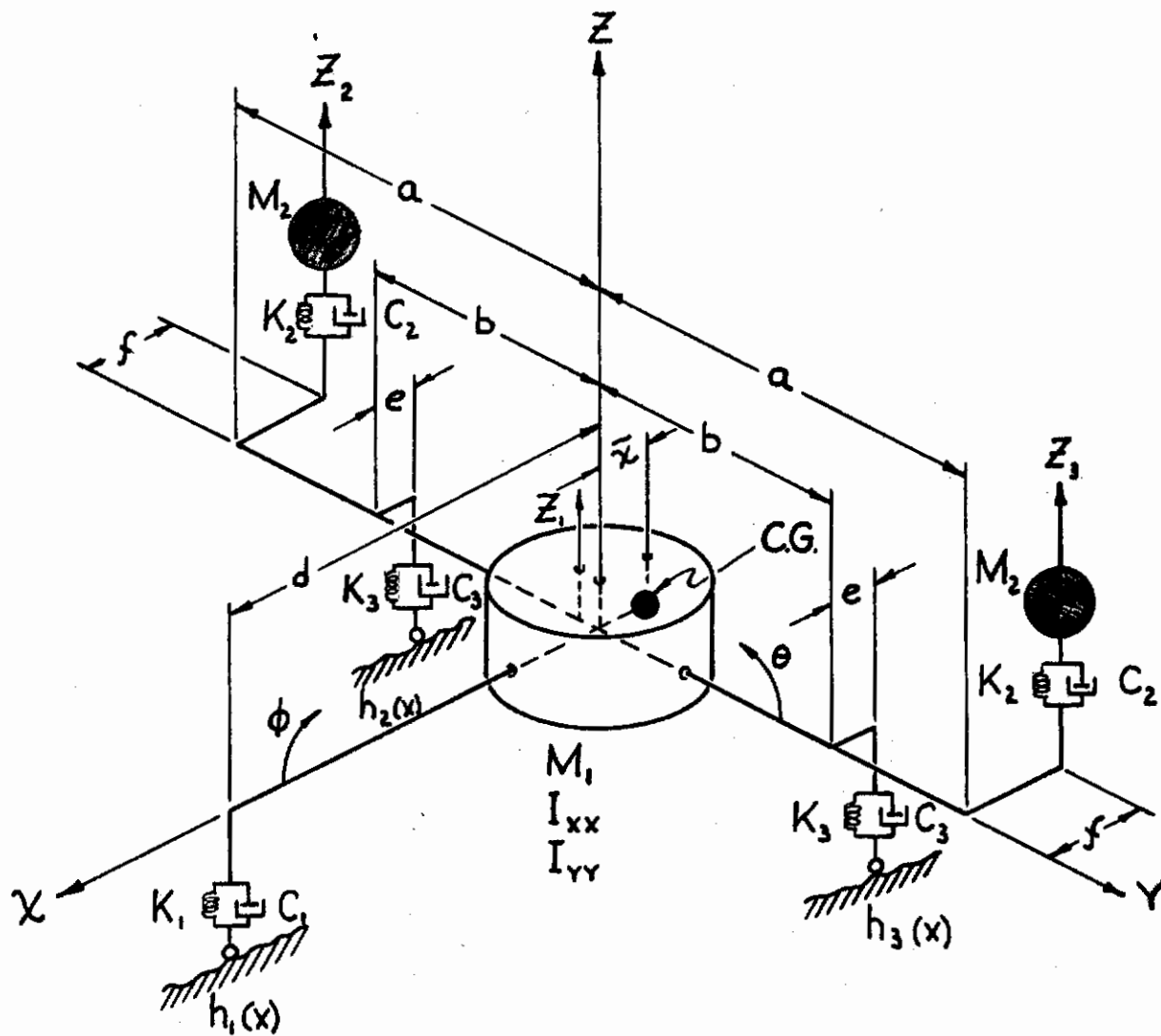


Fig. 2-1 (bis) Five-Degree-of-Freedom Model Vehicle

complex frequency response relating a response,  $z$ , to an input at support  $j$ ,  $j=1, 2, 3$ . Let  $h_j(t)$  be the corresponding unit impulse responses.  $H_j(\omega)$  and  $h_j(t)$  are related by

$$H_j(\omega) = \int_{-\infty}^{\infty} h_j(t) \exp(-i\omega t) dt \quad (2-35)$$

$$h_j(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_j(\omega) \exp(i\omega t) d\omega \quad (2-36)$$

The response,  $z$ , is given by

$$z(t) = \sum_{j=1}^3 \frac{1}{2\pi} \int_{-\infty}^{\infty} H_j(\omega) F_j(\omega) \exp(i\omega t) d\omega \quad (2-37)$$

where  $F_j(\omega)$  is the Fourier transform of the excitation at support  $j$ ,  $f_j(t)$ .

$$F_j(\omega) = \int_{-\infty}^{\infty} f_j(t) \exp(-i\omega t) dt \quad (2-38)$$

Also  $z(t)$  is given by

$$z(t) = \sum_{j=1}^3 \int_{-\infty}^t f_j(\tau) h_j(t-\tau) d\tau \quad (2-39)$$

$$= \sum_{j=1}^3 \int_0^{\infty} h_j(\tau) f_j(t-\tau) d\tau \quad (2-40)$$

Let us consider the excitations  $f_j(t)$  to be sample functions of stationary random processes.

The most important statistical properties of the excitation random process are the mean, which for simplicity will be taken to be zero, the variance or mean square

$$\overline{f_j^2} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f_j^2(t) dt, \quad (2-41)$$

the covariance or autocorrelation function

$$R_{jj}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f_j(t) f_j(t + \tau) dt \quad (2-42)$$

the cross correlation function

$$R_{jb}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f_j(t) f_b(t + \tau) dt, \quad (2-43)$$

the power spectral density

$$S_{jj}(\omega) = 2 \int_{-\infty}^{\infty} R_{jj}(\tau) \exp(-i\omega\tau) d\tau, \quad (2-44)$$

and the cross power spectral density

$$S_{jb}(\omega) = 2 \int_{-\infty}^{\infty} R_{jb}(\tau) \exp(-i\omega\tau) d\tau. \quad (2-45)$$

Also we note that

$$R_{jj}(\tau) = \frac{1}{4\pi} \int_{-\infty}^{\infty} S_{jj}(\omega) \exp(i\omega\tau) d\omega$$

$$R_{jb}(\tau) = \frac{1}{4\pi} \int_{-\infty}^{\infty} S_{jk}(\omega) \exp(i\omega\tau) d\omega$$

When the  $f_j(t)$  are real, we may write

$$S_{jj}(\omega) = 4 \int_0^{\infty} R_{jj}(\tau) \exp(-i\omega\tau) d\tau$$

$$R_{jj}(\tau) = \frac{1}{2\pi} \int_0^{\infty} S_{jj}(\omega) \exp(i\omega\tau) d\omega$$

We also notice that

$$R_{jj}(0) = \overline{f_j^2} = \frac{1}{2\pi} \int_0^{\infty} S_{jj}(\omega) d\omega$$

Let us now consider the statistical properties of the response,  $z(t)$ .

If the  $f_j(t)$  have zero mean then so does  $z(t)$

$$\begin{aligned} E[z(t)] &= E \left[ \sum_{j=1}^3 \int_0^{\infty} h_j(\tau) f_j(t-\tau) d\tau \right] \\ &= \sum_{j=1}^3 \int_0^{\infty} h_j(\tau) E[f_j(t-\tau)] d\tau \\ &= 0. \end{aligned}$$

The autocorrelation function for the response is

$$R_z(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T z(t) z(t+\tau) dt \quad (2-46)$$

when equation 2-40 is substituted into equation 2-46

$$\begin{aligned} R_z(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[ \sum_{j=1}^3 \int_0^{\infty} h_j(\tau_1) f_j(t-\tau_1) d\tau_1 \right] \\ &\quad \left[ \sum_{b=1}^3 \int_0^{\infty} h_b(\tau_2) f_b(t+\tau-\tau_2) d\tau_2 \right] dt \end{aligned}$$

Interchanging the order of these operations

$$\begin{aligned} R_z(\tau) &= \sum_{j=1}^3 \left\{ \sum_{b=1}^3 \left[ \int_0^{\infty} h_j(\tau_1) d\tau_1 \int_0^{\infty} h_b(\tau_2) d\tau_2 \right. \right. \\ &\quad \left. \left. \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f_j(t-\tau_1) f_b(t+\tau-\tau_2) dt \right] \right\} \quad (2-47) \end{aligned}$$

We recognize from equations 2-42 and 2-43 that equation 2-47 can be rewritten

$$\begin{aligned} R_z(\tau) &= \sum_{j=1}^3 \left\{ \sum_{b=1}^3 \left[ \int_0^{\infty} h_j(\tau_1) d\tau_1 \int_0^{\infty} h_b(\tau_2) R_{jb}(\tau+\tau_1-\tau_2) d\tau_2 \right] \right\} \quad (2-48) \end{aligned}$$



# Contrails

The power spectral density of the response is defined by

$$S_z(\omega) = 2 \int_{-\infty}^{\infty} R_z(\tau) \exp(-i\omega\tau) d\tau. \quad (2-49)$$

When (2-48) is substituted into (2-49),

$$S_z(\omega) = 2 \int_{-\infty}^{\infty} \exp(-i\omega\tau) d\tau \sum_{j=1}^3 \left\{ \sum_{b=1}^3 \left[ \int_0^{\infty} h_j(\tau_1) d\tau_1 \int_0^{\infty} h_b(\tau_2) R_{jb}(\tau + \tau_1 - \tau_2) d\tau_2 \right] \right\} \quad (2-50)$$

Since  $h_j(t)$  is identically zero for  $t < 0$  the lower limits of the last two integrals in equation 2-50 can be extended to  $-\infty$ . Doing this and changing the order of the operations

$$S_z(\omega) = \sum_{j=1}^3 \left\{ \sum_{b=1}^3 \left[ 2 \int_{-\infty}^{\infty} h_j(\tau_1) d\tau_1 \int_{-\infty}^{\infty} h_b(\tau_2) d\tau_2 \int_{-\infty}^{\infty} \exp(-i\omega\tau) R_{jb}(\tau + \tau_1 - \tau_2) d\tau \right] \right\} \quad (2-51)$$

Now let

$$\tau_3 = \tau + \tau_1 - \tau_2 \quad (2-52)$$

When (2-52) is substituted into (2-51) and the exponentials are rearranged,

$$S_z(\omega) = \sum_{j=1}^3 \left\{ \sum_{b=1}^3 \left[ 2 \int_{-\infty}^{\infty} h_j(\tau_1) \exp(i\omega\tau_1) d\tau_1 \int_{-\infty}^{\infty} h_b(\tau_2) \exp(-i\omega\tau_2) d\tau_2 \int_{-\infty}^{\infty} R_{jb}(\tau_3) \exp(-i\omega\tau_3) d\tau_3 \right] \right\}$$

If the cross correlations are all zero

$$S_z(\omega) = \sum_{j=1}^3 \left[ 2 \int_{-\infty}^{\infty} h_j(\tau_1) \exp(i\omega\tau_1) d\tau_1 \int_{-\infty}^{\infty} h_j(\tau_2) \exp(-i\omega\tau_2) d\tau_2 \int_{-\infty}^{\infty} R_{jj}(\tau_3) \exp(-i\omega\tau_3) d\tau_3 \right] \quad (2-53)$$

Utilizing equations 2-35 and 2-44 we rewrite (2-53)

$$S_z(\omega) = \sum_{j=1}^3 H_j(-\omega) H_j(\omega) S_{jj}(\omega)$$

$$= \sum_{j=1}^3 \left[ |H_j(\omega)|^2 S_{jj}(\omega) \right].$$

Let us compute the power spectrum of the deflection of the right wing of the model vehicle. Then the response  $z$  is given by

$$z = z_2 - z_1 + a\theta + f\phi.$$

The equations of motion of the vehicle are

$$M_1 \ddot{z}_1 + A_1 \dot{z}_1 + A_2 z_1 - C_2 \dot{z}_2 - K_2 z_2 - C_2 \dot{z}_3 - K_2 z_3 + A_3 \ddot{\phi} + A_4 \dot{\phi} = B_1$$

$$-C_2 \dot{z}_1 - K_2 z_1 + M_2 \ddot{z}_2 + C_2 \dot{z}_2 + K_2 z_2 + A_5 \ddot{\theta} + A_6 \dot{\theta} + A_7 \ddot{\phi} + A_8 \dot{\phi} = 0$$

$$-C_2 \dot{z}_1 - K_2 z_1 + M_2 \ddot{z}_3 + C_2 \dot{z}_3 + K_2 z_3 - A_5 \ddot{\theta} - A_6 \dot{\theta} + A_7 \ddot{\phi} + A_8 \dot{\phi} = 0$$

$$A_5 \ddot{z}_2 + A_6 z_2 - A_5 \ddot{z}_3 - A_6 z_3 + I_{xx} \ddot{\theta} + A_9 \dot{\theta} + A_{10} \theta = B_2$$

$$A_3 \dot{z}_1 + A_4 z_1 = a_7 \dot{z}_2 + A_8 z_2 + A_7 \dot{z}_3 + A_8 z_3 + I_{yy} \ddot{\phi} = a_{11} \ddot{\phi} + A_{12} \dot{\phi} = B_3$$

where

$$A_1 = C_1 + 2C_2 + 2C_3$$

$$A_2 = K_1 + 2K_2 + 2K_3$$

$$A_3 = dC_1 - 2fC_2 - 2eC_3$$

$$A_4 = dK_1 - 2fK_2 - 2eK_3$$

$$A_5 = aC_2$$

$$A_6 = aK_2$$

$$A_7 = fC_2$$

$$A_8 = fK_2$$

$$A_9 = 2a^2 C_2 + 2b^2 C_3$$

$$A_{10} = 2a^2 K_2 + 2b^2 K_3$$

# Contrails

$$A_{11} = d^2 C_1 + 2f^2 C_2 + 2e^2 C_3$$

$$A_{12} = d^2 K_1 + 2f^2 K_2 + 2e^2 K_3$$

$$B_1 = C_1 \dot{g}_1(t) + K_1 g_1(t) + C_3 \dot{g}_2(t) + K_3 g_2(t) + C_3 \dot{g}_3(t) + K_3 g_3(t)$$

$$B_2 = -b C_3 \dot{g}_2(t) - b K_3 g_2(t) + b C_3 \dot{g}_3(t) + b K_3 g_3(t)$$

$$B_3 = d C_1 \dot{g}_1(t) + d K_1 g_1(t) - e C_3 \dot{g}_2(t) - e K_3 g_2(t) - e C_3 \dot{g}_3(t) - e K_3 g_3(t).$$

$g_1(t)$ ,  $g_2(t)$  and  $g_3(t)$  are the input displacements at supports 1, 2 and 3. We must compute  $H_1(\omega)$ ,  $H_2(\omega)$  and  $H_3(\omega)$  by finding the response  $z$  due to a set of forcing functions

$$g_1(t) = G_1 e^{i\omega t}$$

$$g_2(t) = G_2 e^{i\omega t}$$

$$g_3(t) = G_3 e^{i\omega t}$$

Now

$$B_1 = [(K_1 + i\omega C_1) G_1 + (K_3 + i\omega C_3) (G_2 + G_3)] e^{i\omega t}$$

$$B_2 = [b(K_3 + i\omega C_3) (-G_2 + G_3)] e^{i\omega t}$$

$$B_3 = [d(K_1 + i\omega C_1) G_1 - e(K_3 + i\omega C_3) (G_2 + G_3)] e^{i\omega t}$$

The solutions of the set of differential equations will have the form

$$z_1 = R_1 e^{i\omega t}$$

$$z_2 = R_2 e^{i\omega t}$$

$$z_3 = R_3 e^{i\omega t}$$

$$\theta = R_4 e^{i\omega t}$$

$$\phi = R_5 e^{i\omega t}$$

where the  $R_n$  are complex constants. When these equations are substituted into the equations of motion

# Contrails

$$\begin{aligned} & (-\omega^2 M_1 + i\omega A_1 + A_2) R_1 e^{(i\omega t)} + (-i\omega C_2 - K_2) R_2 e^{(i\omega t)} \\ & \quad + (-i\omega C_2 - K_2) R_3 e^{(i\omega t)} + (i\omega A_3 + A_4) R_5 e^{(i\omega t)} \\ & = [(K_1 + i\omega C_1) G_1 + (K_3 + i\omega C_3) (G_2 + G_3)] e^{(i\omega t)} \end{aligned}$$

$$\begin{aligned} & (-i\omega C_2 - K_2) R_1 e^{(i\omega t)} + (-\omega^2 M_2 + i\omega C_2 + K_2) R_2 e^{(i\omega t)} \\ & \quad + (i\omega A_5 + A_6) R_4 e^{(i\omega t)} + (i\omega A_7 + A_8) R_5 e^{(i\omega t)} = 0 \end{aligned}$$

$$\begin{aligned} & (-i\omega C_2 - K_2) R_1 e^{(i\omega t)} + (-\omega^2 M_2 + i\omega C_2 + K_2) R_3 e^{(i\omega t)} \\ & \quad - (i\omega A_5 + A_6) R_4 e^{(i\omega t)} + (i\omega A_7 + A_8) R_5 e^{(i\omega t)} = 0 \end{aligned}$$

$$\begin{aligned} & (i\omega A_5 + A_6) R_2 e^{(i\omega t)} + (-i\omega A_5 - A_6) R_3 e^{(i\omega t)} \\ & \quad + (-\omega^2 I_{xx} + i\omega A_9 + A_{10}) R_4 e^{(i\omega t)} \\ & = [b(K_3 + i\omega C_3) (-G_2 + G_3)] e^{(i\omega t)} \end{aligned}$$

$$\begin{aligned} & (i\omega A_3 + A_4) R_1 e^{(i\omega t)} + (i\omega A_7 + A_8) R_2 e^{(i\omega t)} \\ & \quad + (i\omega A_7 + A_8) R_3 e^{(i\omega t)} + (-\omega^2 I_{yy} + i\omega A_{11} + A_{12}) R_5 e^{(i\omega t)} \\ & = [d(K_1 + i\omega C_1) G_1 - e (K_3 + i\omega C_3) (G_2 + G_3)] e^{(i\omega t)} \end{aligned}$$

Solving for the  $R_n$  we obtain

$$R_1 = \frac{1}{\Delta} \begin{pmatrix} (K_1 + i\omega C_1)G_1 + (K_3 + i\omega C_3)(G_2 + G_3) & (-i\omega C_2 - K_2) & (-i\omega C_2 - K_2) & (0) & (i\omega A_3 + A_4) \\ (0) & (-\omega^2 M_2 + i\omega C_2 + K_2) & (0) & (i\omega A_5 + A_6) & (i\omega A_7 + A_8) \\ (0) & (0) & (-\omega^2 M_2 + i\omega C_2 + K_2) & (-i\omega A_5 - A_6) & (i\omega A_7 + A_8) \\ b(K_3 + i\omega C_3)(-G_2 + G_3) & (i\omega A_5 + A_6) & (-i\omega A_5) & (-\omega^2 I_{xx} + i\omega A_9 + A_{10}) & (0) \\ d(K_1 + i\omega C_1)G_1 - e(K_3 + i\omega C_3)(G_2 + G_3) & (i\omega A_7 + A_8) & (i\omega A_7 + A_8) & (0) & (-\omega^2 I_{yy} + i\omega A_{11} + A_{12}) \end{pmatrix}$$

$$R_2 = \frac{1}{\Delta} \begin{pmatrix} (-\omega^2 M_1 + i\omega A_1 + A_2) & (K_1 + i\omega C_1)G_1 + (K_3 + i\omega C_3)(G_2 + G_3) & (-i\omega C_2 - K_2) & (0) & (i\omega A_3 + A_4) \\ (-i\omega C_2 - K_2) & (0) & (0) & (i\omega A_5 + A_6) & (i\omega A_7 + A_8) \\ (-i\omega C_2 - K_2) & (0) & (-\omega^2 M_2 + i\omega C_2 + K_2) & (-i\omega A_5 - A_6) & (i\omega A_7 + A_8) \\ (0) & b(K_3 + i\omega C_3)(-G_2 + G_3) & (-i\omega A_5 - A_6) & (-\omega^2 I_{xx} + i\omega A_9 + A_{10}) & (0) \\ (i\omega A_3 + A_4) & d(K_1 + i\omega C_1)G_1 - e(K_3 + i\omega C_3)(G_2 + G_3) & (i\omega A_7 + A_8) & (0) & (-\omega^2 I_{yy} + i\omega A_{11} + A_{12}) \end{pmatrix}$$

$$R_3 = \frac{1}{\Delta} \begin{pmatrix} (-\omega^2 M_1 + i\omega A_1 + A_2) & (-i\omega C_2 - K_2) & (K_1 + i\omega C_1)G_1 + (K_3 + i\omega C_3)(G_2 + G_3) & (0) & (i\omega A_3 + A_4) \\ (-i\omega C_2 - K_2) & (-\omega^2 M_2 + i\omega C_2 + K_2) & (0) & (i\omega A_5 + A_6) & (i\omega A_7 + A_8) \\ (-i\omega C_2 - K_2) & (0) & (0) & (-i\omega A_5 - A_6) & (i\omega A_7 + A_8) \\ (0) & (i\omega A_5 + A_6) & b(K_3 + i\omega C_3)(-G_2 + G_3) & (-\omega^2 I_{xx} + i\omega A_9 + A_{10}) & (0) \\ (i\omega A_3 + A_4) & (i\omega A_7 + A_8) & d(K_1 + i\omega C_1)G_1 - e(K_3 + i\omega C_3)(G_2 + G_3) & (0) & (-\omega^2 I_{yy} + i\omega A_{11} + A_{12}) \end{pmatrix}$$

$$R_4 = \frac{1}{\Delta} \begin{pmatrix} (-\omega^2 M_1 + i\omega A_1 + A_2) & (-i\omega C_2 - K_2) & (-i\omega C_2 - K_2) & (K_1 + i\omega C_1)G_1 + (K_3 + i\omega C_3)(G_2 + G_3) & (i\omega A_3 + A_4) \\ (-i\omega C_2 - K_2) & (-\omega^2 M_2 + i\omega C_2 + K_2) & (0) & (0) & (i\omega A_7 + A_8) \\ (-i\omega C_2 - K_2) & (0) & (-\omega^2 M_2 + i\omega C_2 + K_2) & (0) & (i\omega A_7 + A_8) \\ (0) & (i\omega A_5 + A_6) & (-i\omega A_5 - A_6) & b(K_3 + i\omega C_3)(-G_2 + G_3) & (0) \\ (i\omega A_3 + A_4) & (i\omega A_7 + A_8) & (i\omega A_7 + A_8) & d(K_1 + i\omega C_1)G_1 - e(K_3 + i\omega C_3)(G_2 + G_3) & (-\omega^2 I_{yy} + i\omega A_{11} + A_{12}) \end{pmatrix}$$

$$R_5 = \frac{1}{\Delta} \begin{pmatrix} (-\omega^2 M_1 + i\omega A_1 + A_2) & (-i\omega C_2 - K_2) & (-i\omega C_2 - K_2) & (0) & (K_1 + i\omega C_1)G_1 + (K_3 + i\omega C_3)(G_2 + G_3) \\ (-i\omega C_2 - K_2) & (-\omega^2 M_2 + i\omega C_2 + K_2) & (0) & (i\omega A_5 + A_6) & (0) \\ (-i\omega C_2 - K_2) & (0) & (-\omega^2 M_2 + i\omega C_2 + K_2) & (-i\omega A_5 - A_6) & (0) \\ (0) & (i\omega A_5 + A_6) & (i\omega A_5 + A_6) & (-\omega^2 I_{xx} + i\omega A_9 + A_{10}) & b(K_3 + i\omega C_3)(-G_2 + G_3) \\ (i\omega A_3 + A_4) & (i\omega A_7 + A_8) & (i\omega A_7 + A_8) & (0) & d(K_1 + i\omega C_1)G_1 - e(K_3 + i\omega C_3)(G_2 + G_3) \end{pmatrix}$$

where  $\Delta$  is given by

$$\Delta = \begin{pmatrix} (-\omega^2 M_1 + i\omega A_1 + A_2) & (-i\omega C_2 - K_2) & (-i\omega C_2 - K_2) & (0) & (i\omega A_3 + A_4) \\ (-i\omega C_2 - K_2) & (-\omega^2 M_2 + i\omega C_2 + K_2) & (0) & (i\omega A_5 + A_6) & (i\omega A_7 + A_8) \\ (-i\omega C_2 - K_2) & (0) & (-\omega^2 M_2 + i\omega C_2 + K_2) & (-i\omega A_5 - A_6) & (i\omega A_7 + A_8) \\ (0) & (i\omega A_5 + A_6) & (-i\omega A_5 - A_6) & (-\omega^2 I_{xx} + i\omega A_9 + A_{10}) & (0) \\ (i\omega A_3 + A_4) & (i\omega A_7 + A_8) & (i\omega A_7 + A_8) & (0) & (-\omega^2 I_{yy} + i\omega A_{11} + A_{12}) \end{pmatrix}$$

The response  $z$  is given by

$$z = (R_2 - R_1 + aR_4 + fR_5) e^{i\omega t} = R e^{i\omega t}$$

The complex frequency responses  $H_1(\omega)$ ,  $H_2(\omega)$  and  $H_3(\omega)$  can now be obtained in the following manner. Set  $G_1$  equal to unity and  $G_2$  and  $G_3$ , equal to zero, then  $H_1(\omega)$  is equal to  $R$ . Set  $G_2$  equal to unity and  $G_1$  and  $G_3$  equal to zero, then  $H_2(\omega)$  is equal to  $R$ . Set  $G_3$  equal to unity and  $G_1$  and  $G_2$  equal to zero, then  $H_3(\omega)$  is equal to  $R$ .

### Example Problem

As an example, let us find the power spectrum of the deflection of the right spring of the vehicle analyzed in Section I. The physical constants for this vehicle are

$a = 40.0$ ft	$M_1 g = 150,000$ lb
$b = 10.0$ ft	$M_2 g = 75,000$ lb
$d = 35.0$ ft	$I_{xx} = 932,488$ lb-sec <sup>2</sup> -ft
$e = 15.0$ ft	$I_{yy} = 2,797,464$ lb-sec <sup>2</sup> -ft
$f = 20.0$ ft	
$K_1 = 10,000$ lb/in.	$C_1 = 2,220$ lb-sec/ft
$K_2 = 15,000$ lb/in.	$C_2 = 1,260$ lb-sec/ft
$K_3 = 25,000$ lb/in.	$C_3 = 2,220$ lb-sec/ft.

The input to the vehicle will be the power spectra for taxiway 2 of Sewart Air Force Base. The power spectra are reproduced in Figure 2-11. The horizontal velocity of the vehicle is 50 fps. A flow diagram for a digital computer program to compute the complex frequency response is given in Figure 2-12.

Table 2-4 gives the calculations necessary to compute the power spectrum of the spring deflection. The power spectra of the input was taken from Figure 2-11. The values of the complex frequency response were obtained from a digital computer program according to the flow diagram in Figure 2-12. A plot of the power spectra of the response is given in Figure 2-13.

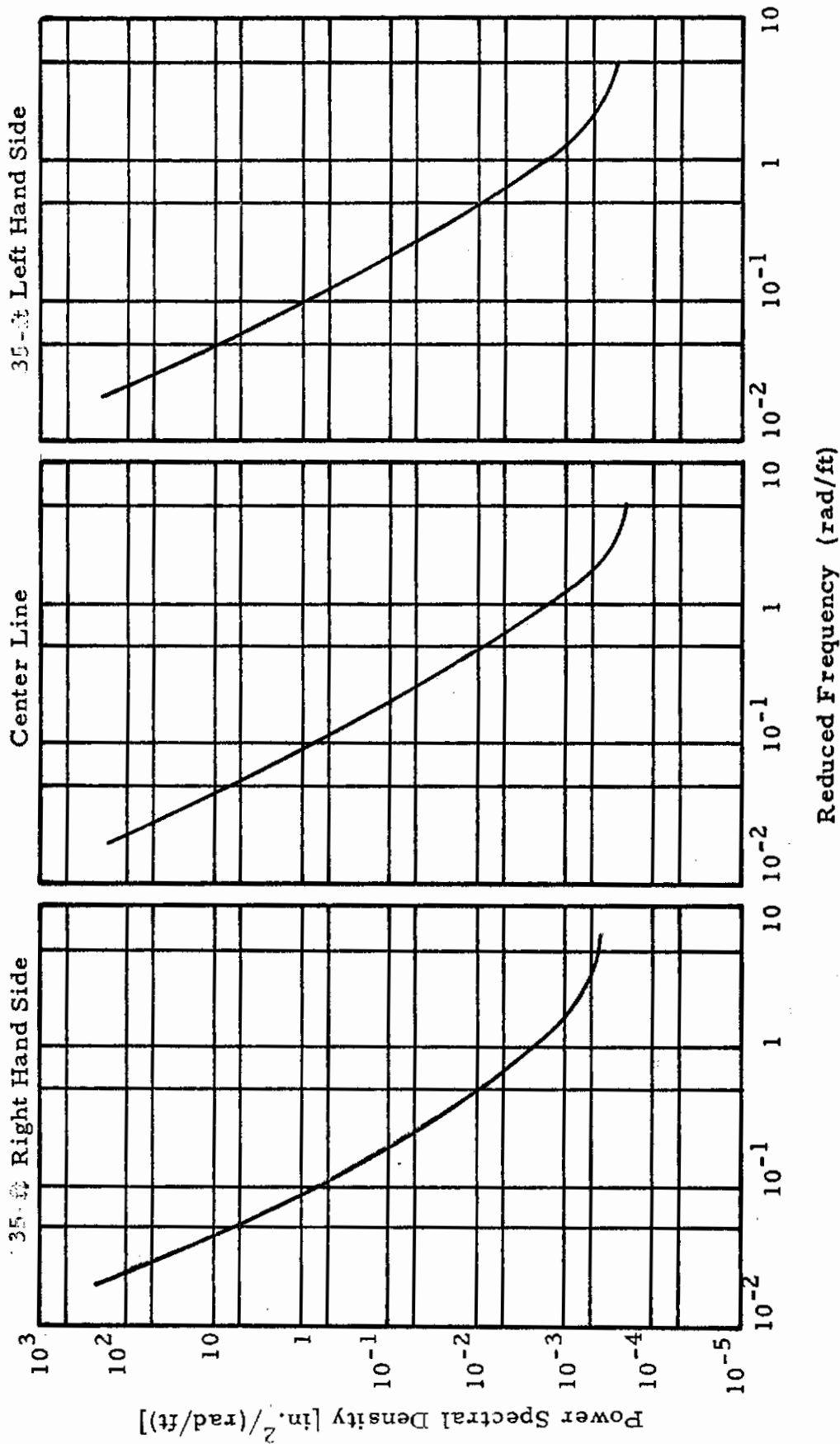


Fig. 2-11 Power Spectral Densities for Sewart Air Force Base Taxiway 2

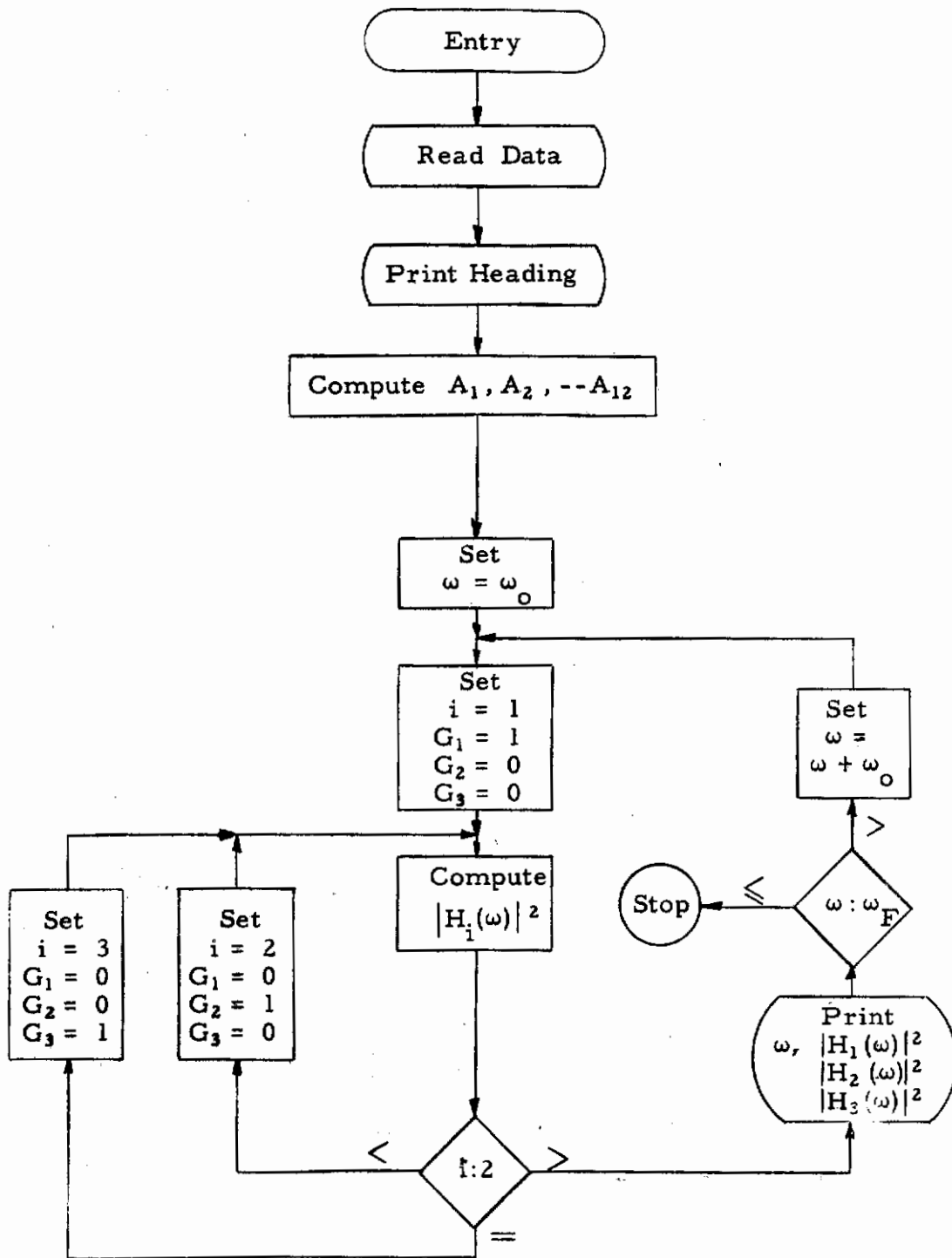


Fig. 2-12 Flow Diagram for Computer Program



Table 2-4 Compilation Calculations, Power Spectrum of the Spring Deflection

$\omega$ rad/sec	$\Omega$ rad/sec	$\phi_{1,1}(\Omega)$ $\frac{\text{in.}^2}{\text{rad/ft}}$	$S_{1,1}(\omega)$ $\frac{\text{in.}^2}{\text{rad/sec}}$	$ H_1(\omega) ^2$	$\phi_{2,2}(\Omega)$ $\frac{\text{in.}^2}{\text{rad/ft}}$	$S_{2,2}(\omega)$ $\frac{\text{in.}^2}{\text{rad/sec}}$	$ H_2(\omega) ^2$	$\phi_{3,3}(\Omega)$ $\frac{\text{in.}^2}{\text{rad/ft}}$	$S_{3,3}(\omega)$ $\frac{\text{in.}^2}{\text{rad/sec}}$	$ H_3(\omega) ^2$	$S_x(\omega)$ $\frac{\text{in.}^2}{\text{rad/sec}}$
1.00	0.020	85.0	1.70	0.0006754	150.0	3.00	0.06883	220.0	4.40	0.2583	1.3441
1.25	0.024	49.0	0.98	0.002443	78.0	1.56	0.007222	120.0	2.40	0.2572	0.6310
1.50	0.026	44.0	0.88	0.003412	66.0	1.32	0.005468	100.0	2.00	0.2824	0.5750
1.75	0.030	39.0	0.78	0.004812	59.0	1.18	0.009760	88.0	1.76	0.3194	0.5774
2.00	0.032	35.0	0.50	0.02212	34.0	0.68	0.1821	52.0	1.04	0.7037	0.8667
2.25	0.034	31.0	0.42	0.05402	27.0	0.54	0.5929	42.4	0.848	1.2676	1.4178
2.50	0.035	27.0	0.344	0.1374	21.2	0.424	1.7805	33.6	0.672	2.4539	2.4512
2.75	0.036	23.0	0.330	0.2005	19.0	0.380	2.7428	30.0	0.600	3.1558	3.5975
3.00	0.038	19.0	0.290	0.2455	17.2	0.344	3.4990	27.0	0.540	3.4289	3.1265
3.25	0.040	15.0	0.248	0.2106	14.2	0.284	3.1650	22.5	0.450	2.3618	2.0139
3.50	0.045	10.5	0.210	0.1416	12.0	0.240	2.1907	19.0	0.38	1.3099	1.0533
3.75	0.050	7.5	0.150	0.06984	7.7	0.154	1.1445	12.4	0.248	0.4609	0.3011
4.00	0.055	5.4	0.108	0.04782	5.4	0.108	0.8512	8.6	0.172	0.2694	0.1434
4.25	0.060	4.2	0.084	0.02672	3.8	0.076	0.7445	6.0	0.120	0.2032	0.08320
4.50	0.065	3.2	0.064	0.01548	2.8	0.056	0.7044	4.4	0.088	0.1757	0.05590
4.75	0.070	2.5	0.050	0.009045	2.1	0.042	0.6999	3.3	0.066	0.1666	0.04084
5.00	0.075	2.0	0.04	0.005534	1.70	0.034	0.7212	2.6	0.052	0.1697	0.03357
5.25	0.080	1.6	0.032	0.003581	1.34	0.0268	0.7638	2.05	0.041	0.1833	0.02810
5.50	0.085	1.3	0.026	0.002446	1.10	0.0220	0.8246	1.65	0.0330	0.2069	0.02503
5.75	0.090	1.1	0.022	0.001757	0.90	0.018	0.8983	1.37	0.0274	0.2405	0.02276
6.00	0.095	0.93	0.0186	0.001325	0.75	0.015	0.9739	1.18	0.0236	0.2829	0.02131
6.25	0.100	0.80	0.0160	0.001047	0.63	0.0126	1.0305	0.990	0.0198	0.3285	0.01950
6.50	0.110	0.70	0.0140	0.0008633	0.55	0.011	1.0407	0.840	0.0168	0.3666	0.01762
6.75	0.120	0.52	0.0104	0.0006376	0.40	0.008	0.8752	0.610	0.0122	0.3749	0.01158
7.00	0.130	0.40	0.008	0.0004832	0.31	0.0062	0.6135	0.470	0.0094	0.3024	0.00650
7.25	0.140	0.31	0.0062	0.0003674	0.25	0.0050	0.4340	0.370	0.0074	0.2208	0.003805
7.50	0.150	0.25	0.0050	0.0002843	0.197	0.00394	0.3399	0.300	0.0060	0.1577	0.002287
7.75	0.160	0.20	0.0040	0.0002253	0.160	0.0032	0.2882	0.240	0.0048	0.1115	0.001458
8.00	0.170	0.17	0.0034	0.0001827	0.130	0.0026	0.2503	0.200	0.0040	0.07755	0.0009616
8.25	0.180	0.14	0.0028	0.0001512	0.112	0.00224	0.2152	0.166	0.00312	0.05313	0.0006482
8.50	0.190	0.120	0.0024	0.0001273	0.095	0.0019	0.1817	0.140	0.0028	0.03617	0.0004468
8.75	0.200	0.100	0.0020	0.0001088	0.083	0.00166	0.1513	0.120	0.0024	0.02477	0.0003108
9.00	0.210	0.087	0.00174	0.00009411	0.072	0.00144	0.1253	0.105	0.0021	0.01722	0.0002167
9.25	0.220	0.075	0.00150	0.00008230	0.064	0.00128	0.1040	0.092	0.00184	0.0122	0.0001557
9.50	0.230	0.066	0.00132	0.00007264	0.056	0.00112	0.08677	0.082	0.00164	0.00869	0.0001118
9.75	0.240	0.059	0.00118	0.00006464	0.050	0.0010	0.07296	0.073	0.00146	0.006590	0.00008265
10.00	0.250	0.052	0.00104	0.00005794	0.045	0.00090	0.06190	0.065	0.0013	0.005006	0.00006508
10.25	0.260	0.046	0.00088	0.00005395	0.031	0.00062	0.03497	0.042	0.00084	0.002013	0.00001691
10.50	0.270	0.040	0.00074	0.00004887	0.0232	0.000464	0.02217	0.030	0.00060	0.001015	0.000006092
10.75	0.280	0.034	0.00060	0.00004464	0.0180	0.00036	0.01527	0.022	0.00044	0.0005953	0.000002619
11.00	0.290	0.029	0.00050	0.00004148	0.0142	0.000284	0.01117	0.017	0.00034	0.0003871	0.000001316
11.25	0.300	0.024	0.00040	0.00003871	0.0115	0.000230	0.008545	0.013	0.00026	0.0002710	0.0000007044
11.50	0.310	0.020	0.00032	0.00003616	0.0095	0.000190	0.006760	0.0105	0.00021	0.0002003	0.0000004204

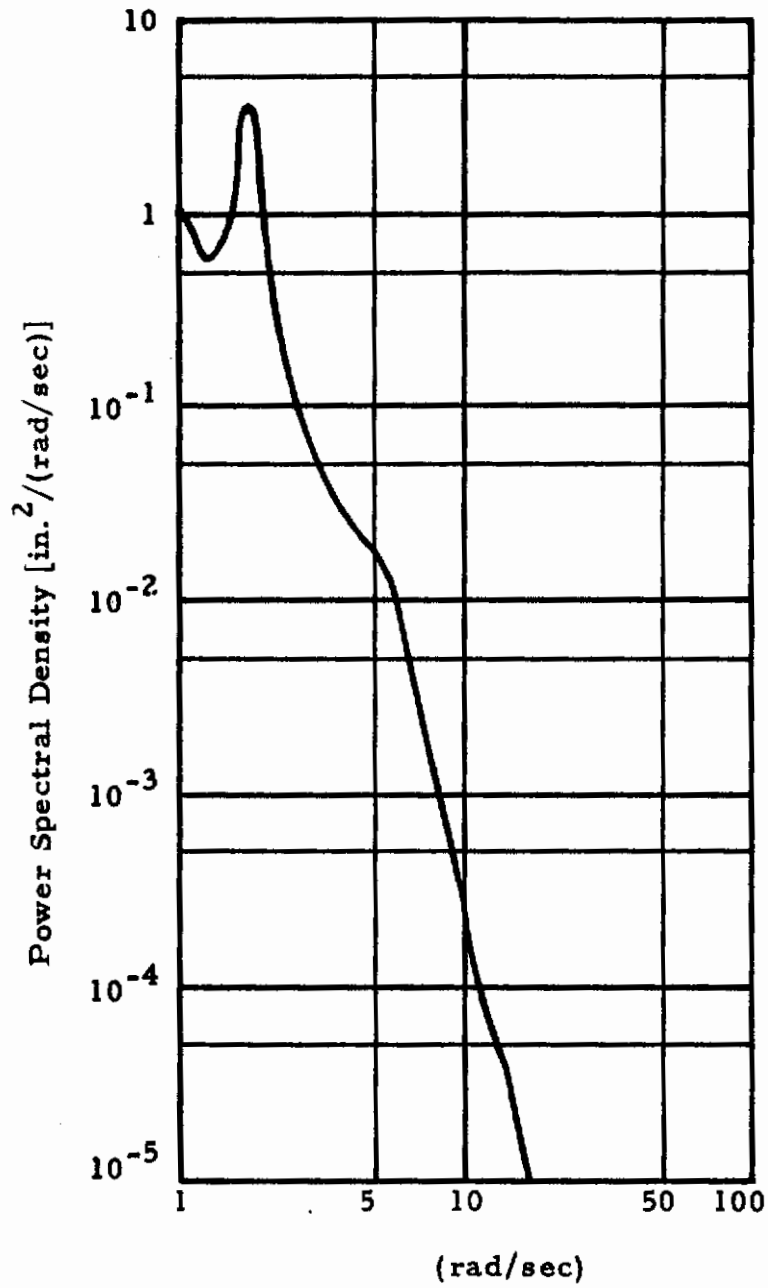


Fig. 2-13 Power Spectrum for Wing Deflection Computed Using the Statistical Analysis

# Contrails

The excellent match between the power spectral density obtained from this statistical analysis and the power spectrum obtained in Figure 2-2 from the deterministic analysis indicate the accuracy of the assumptions used in the statistical analysis as well as the accuracy of the analysis itself.

As is evident from the preceding sections the statistical analysis has a great advantage in ease of computations over the deterministic analysis. The running time on the IBM 7090 digital computer was four times as long for the deterministic analysis as for the statistical analysis. Further computing time was then necessary to analyze the great amount of data which results from the deterministic analysis. Although it is true that a greater amount of information and more exact information can be obtained for operation on any particular runway from a deterministic analysis, we can only expect averages to hold true for operation of the vehicle on a runway with similar roughness characteristics. It is far easier to obtain these averages from a statistical analysis as is demonstrated in this section.

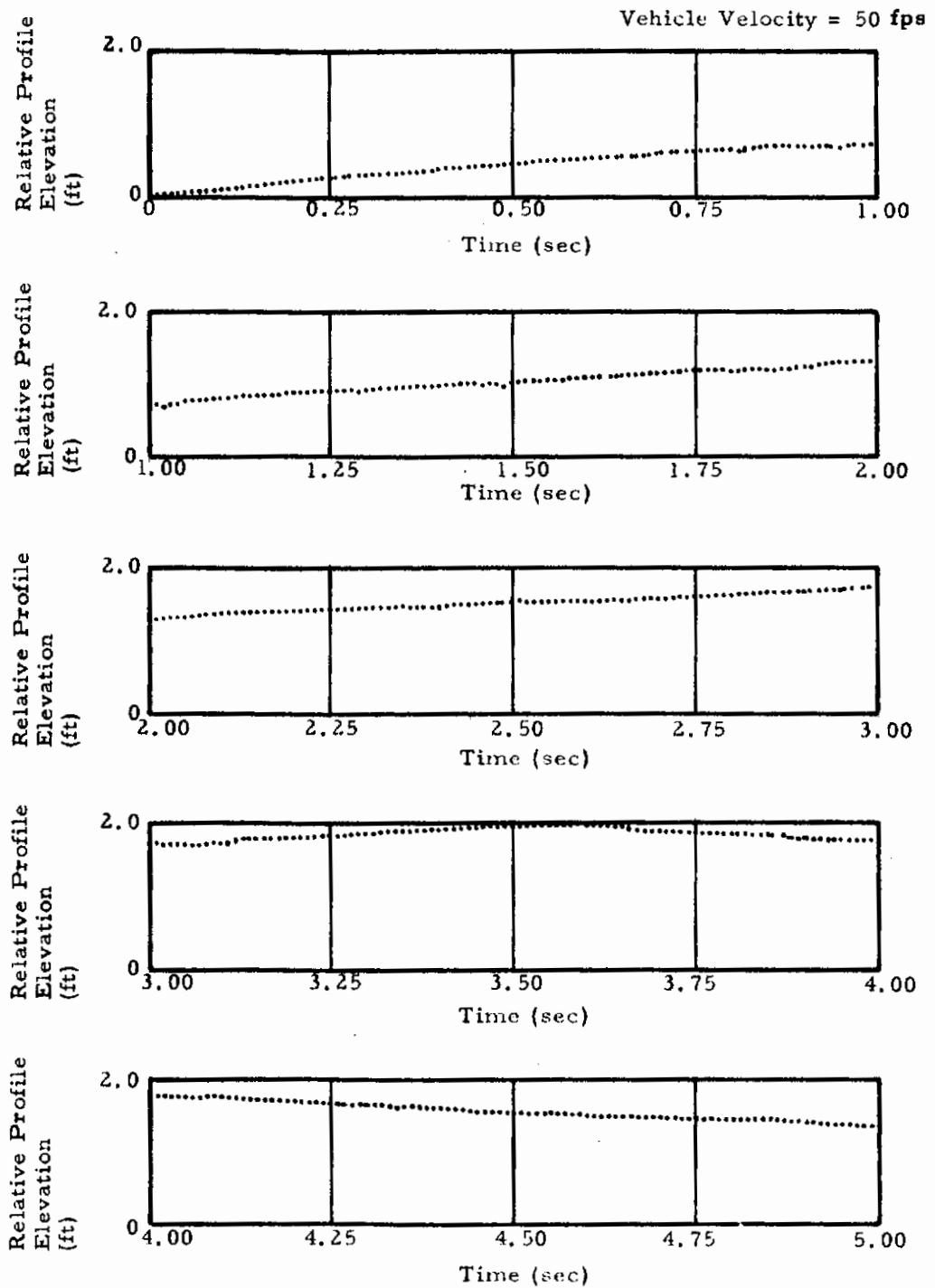
References

- 2-1 Ralston, A., and Wilf, H. S., Mathematical Methods  
for Digital Computers, John Wiley and Sons, Inc.  
N. Y. N. Y. (1960).

APPENDIX I

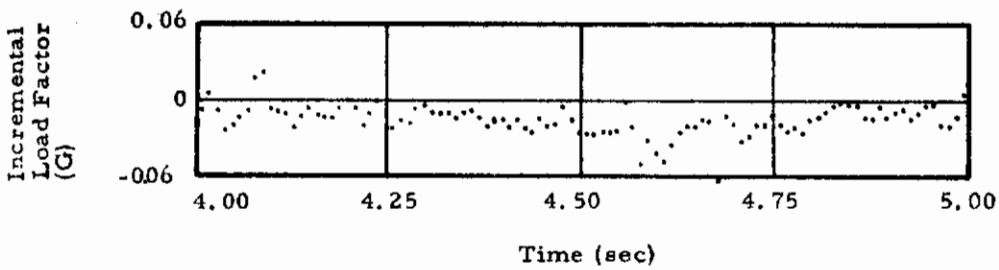
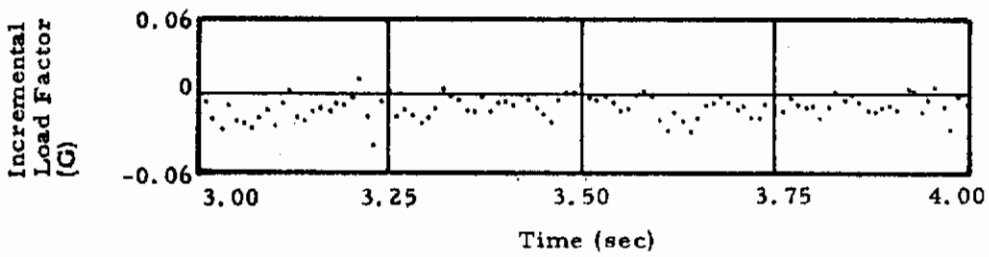
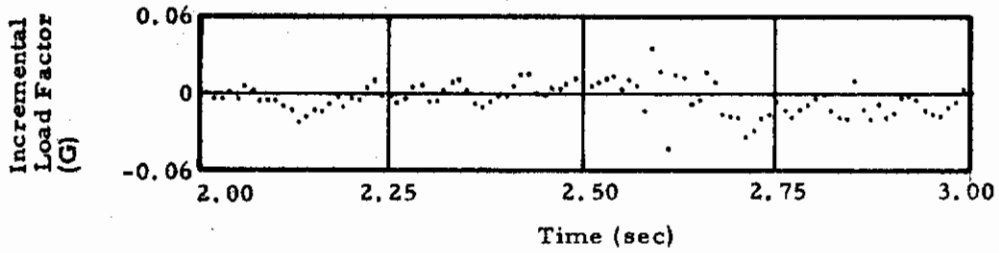
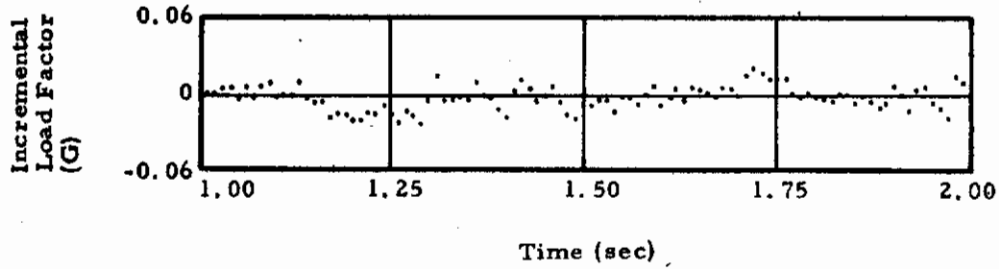
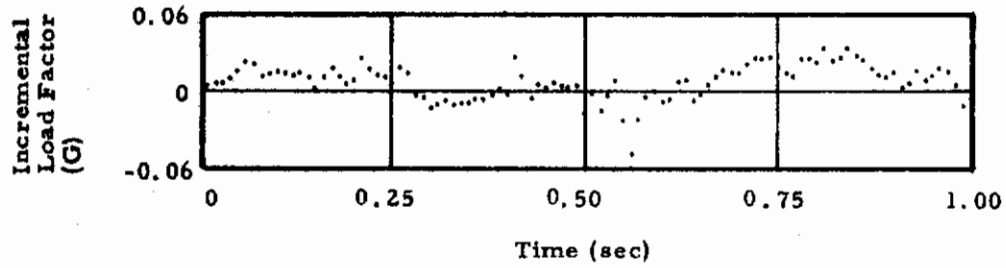
MODEL VEHICLE RESPONSES

## Response of Vehicle to Sewart Air Force Base Taxiway 2 Taxiway Profile



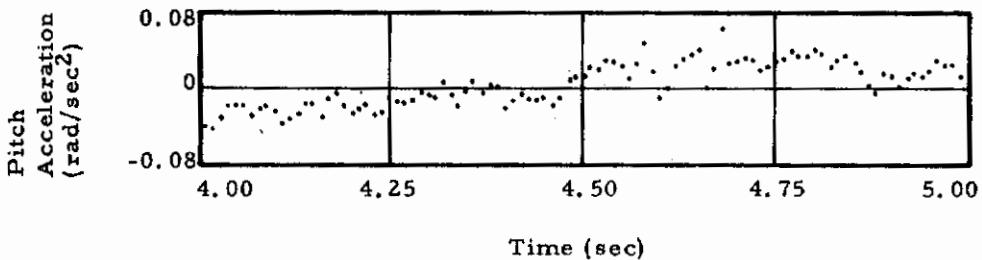
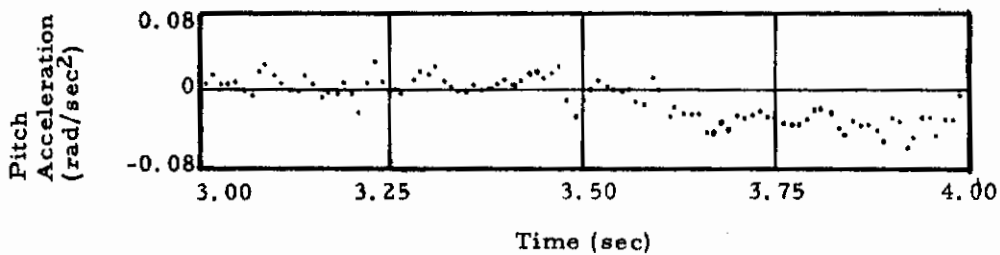
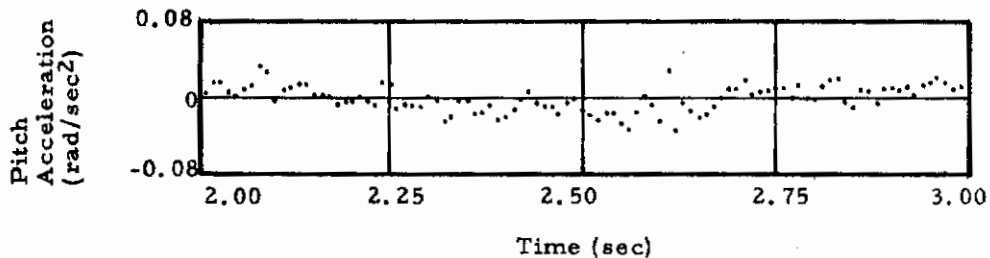
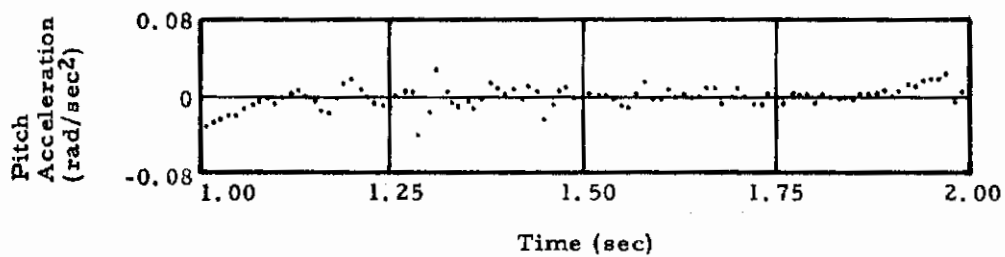
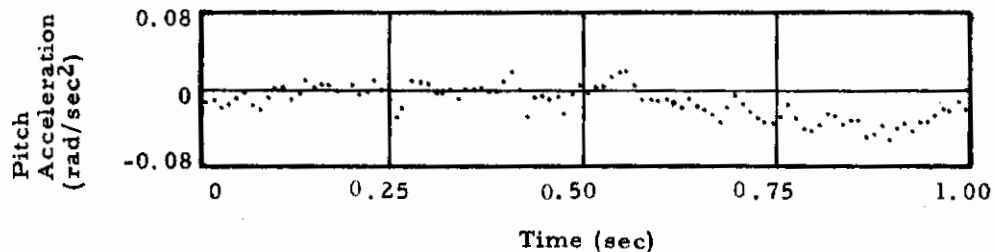
## Incremental Load Factor

Vehicle velocity = 50 fps



## Pitch Acceleration

Vehicle velocity = 50 fps

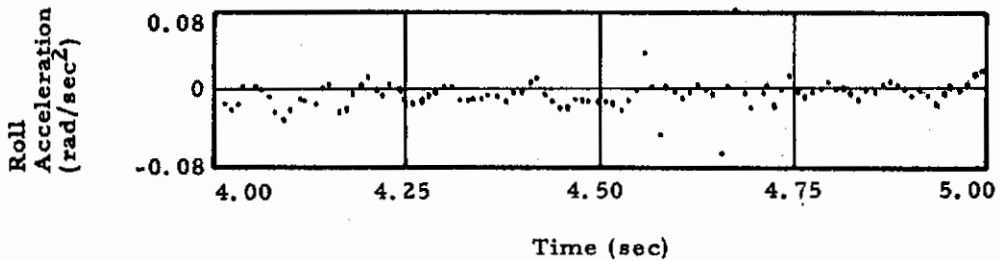
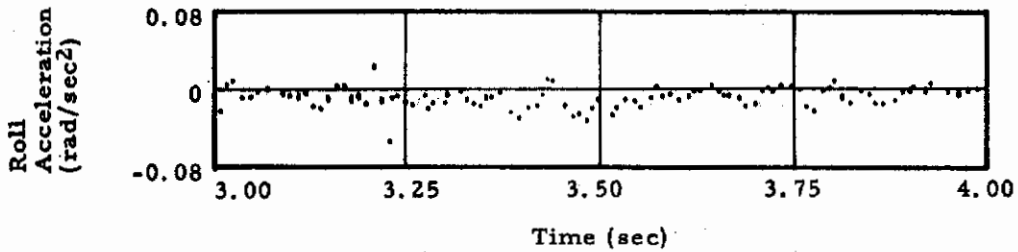
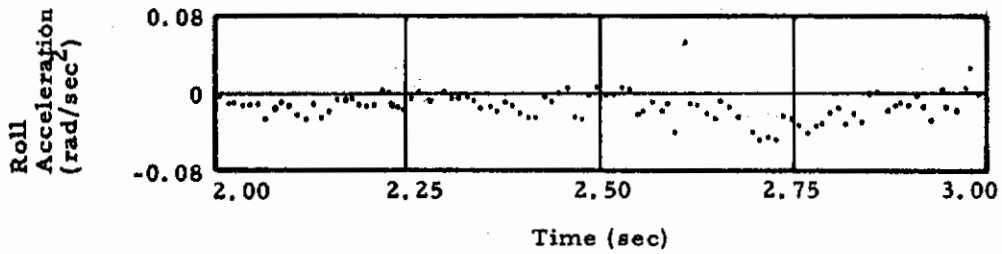
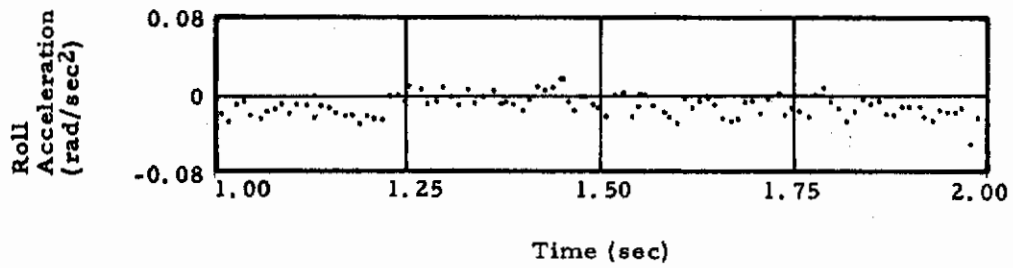
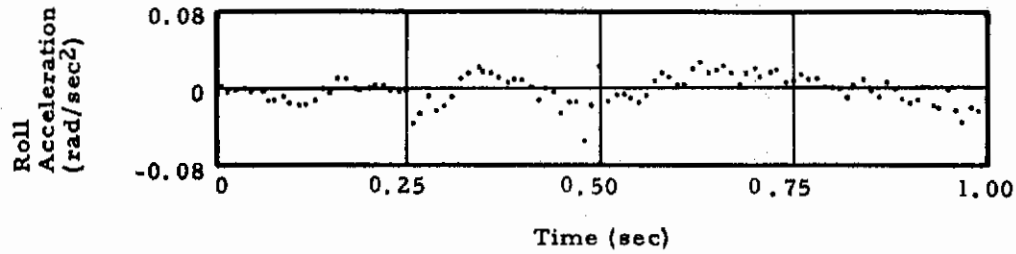




# Contrails

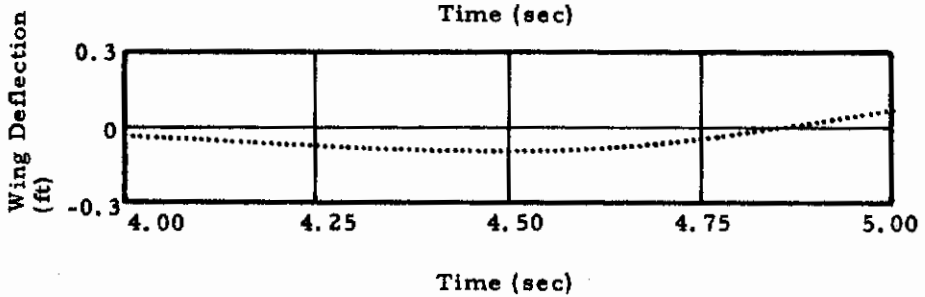
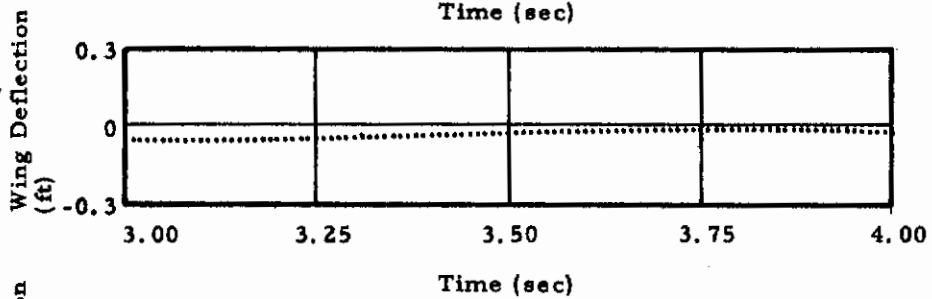
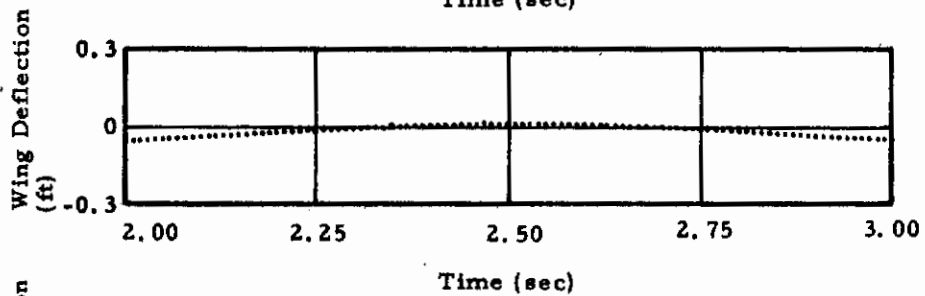
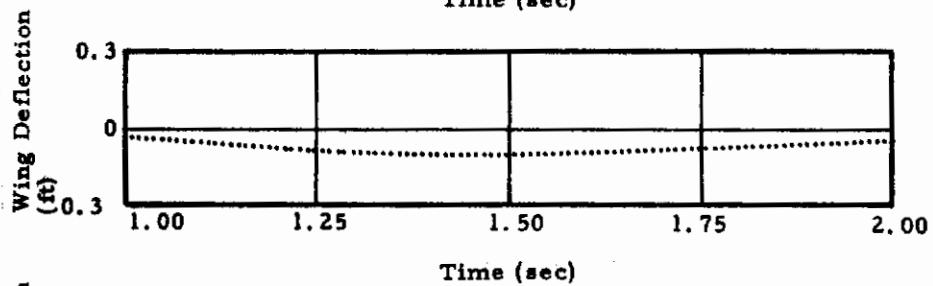
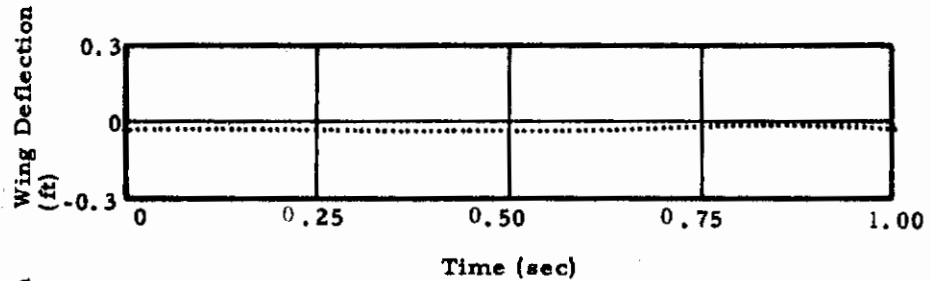
## Roll Acceleration

Vehicle velocity = 50 fps



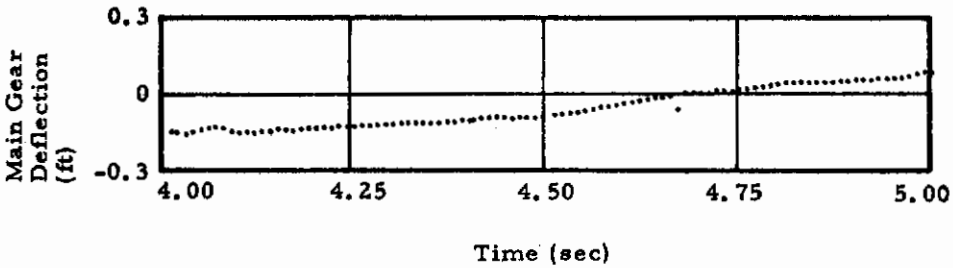
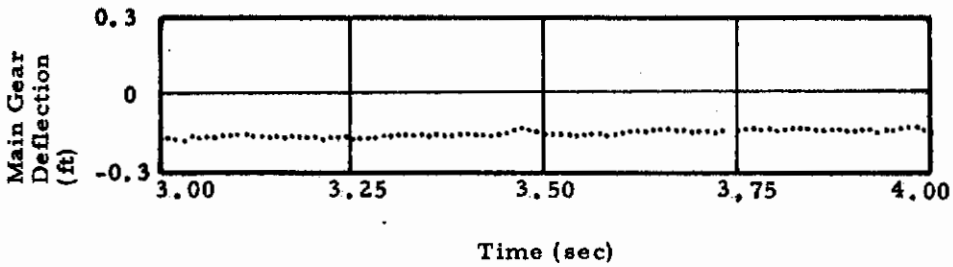
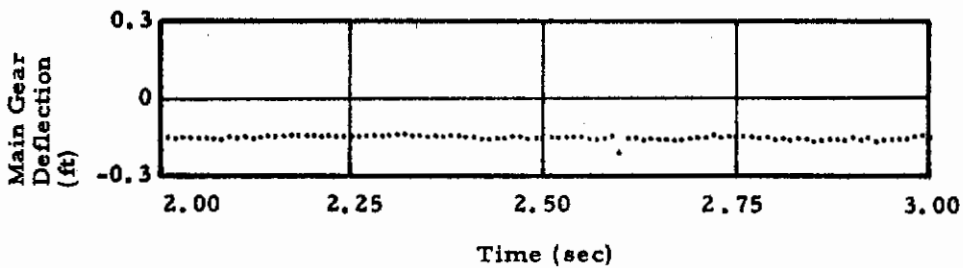
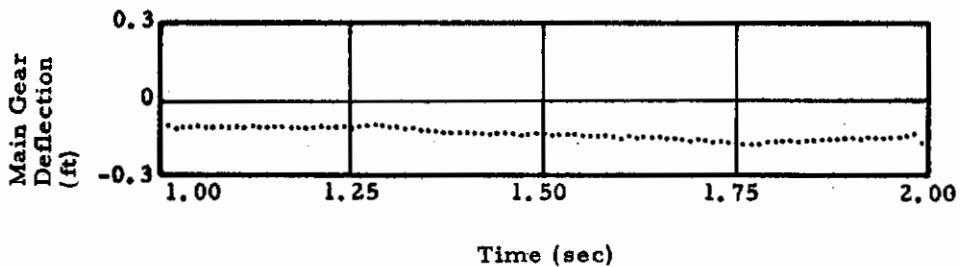
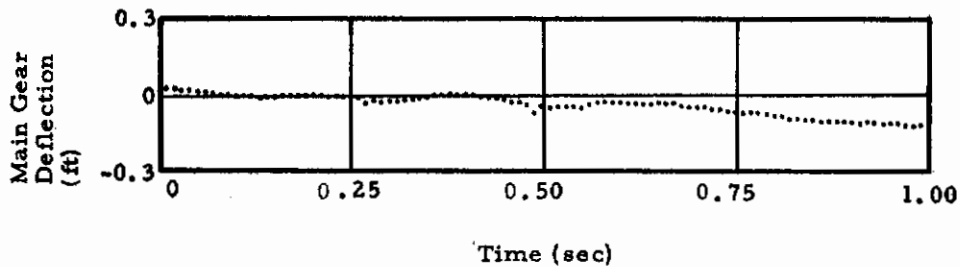
## Right Wing Deflection

Vehicle velocity = 50 fps



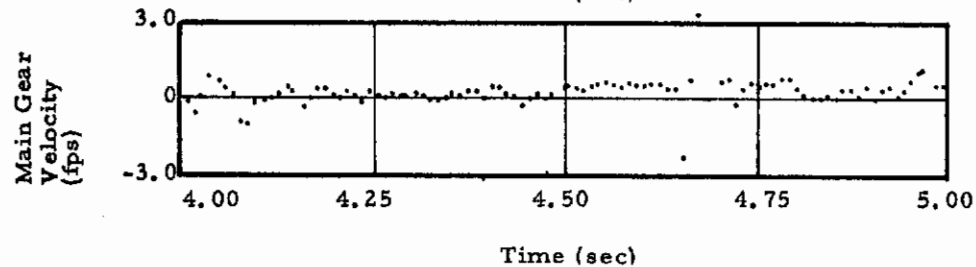
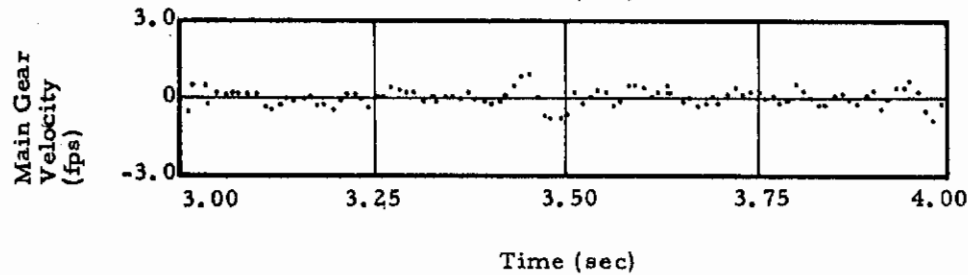
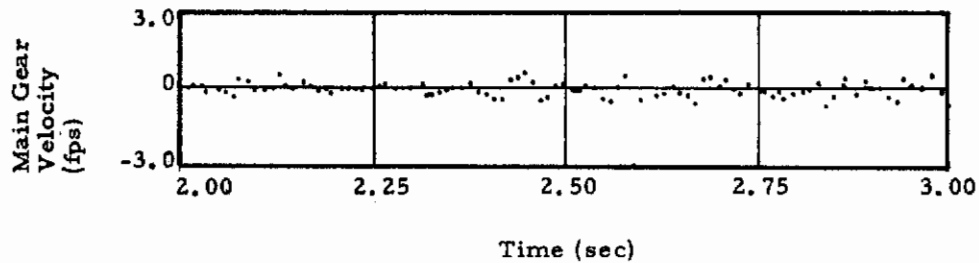
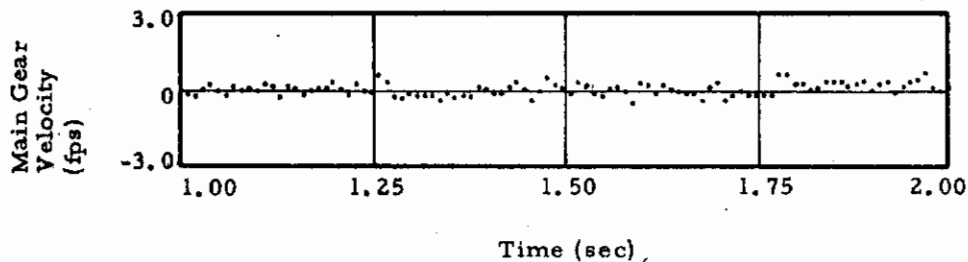
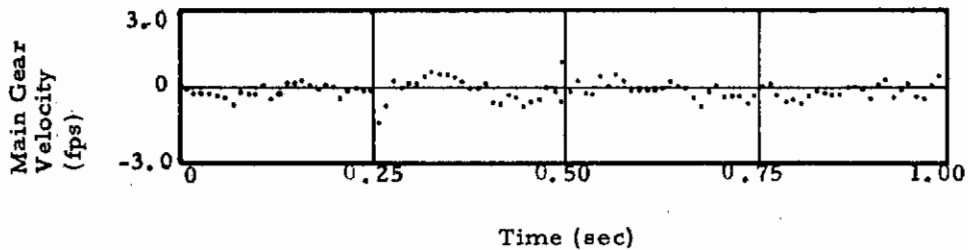
## Main Gear Deflection

Vehicle velocity = 50 fps



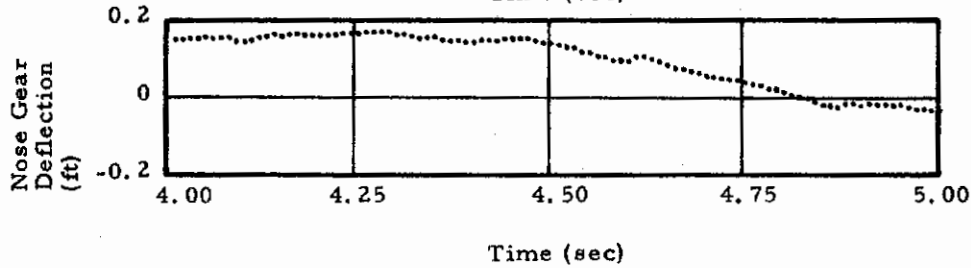
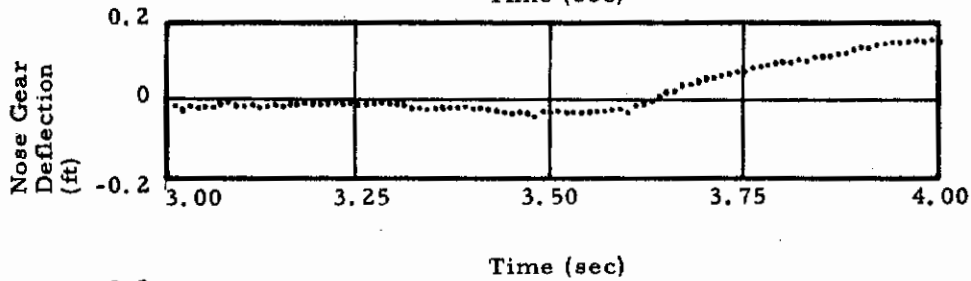
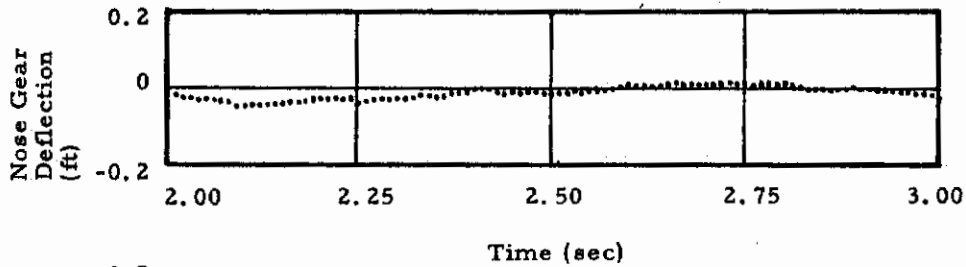
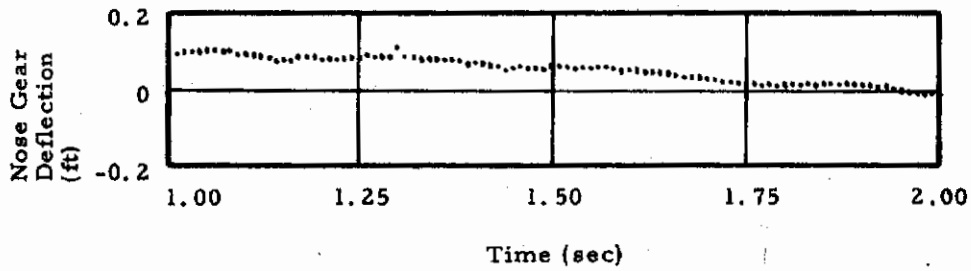
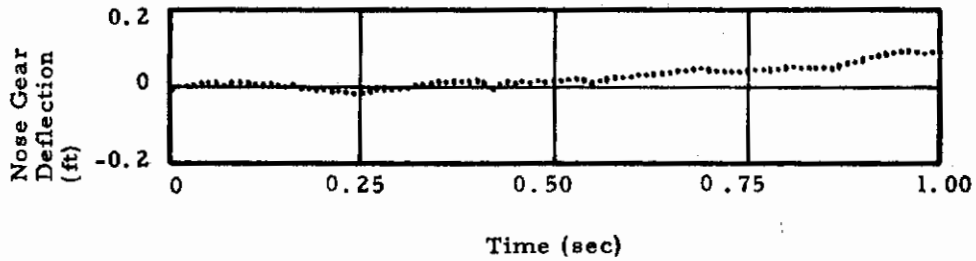
## Main Gear Velocity

Vehicle velocity = 50 fps



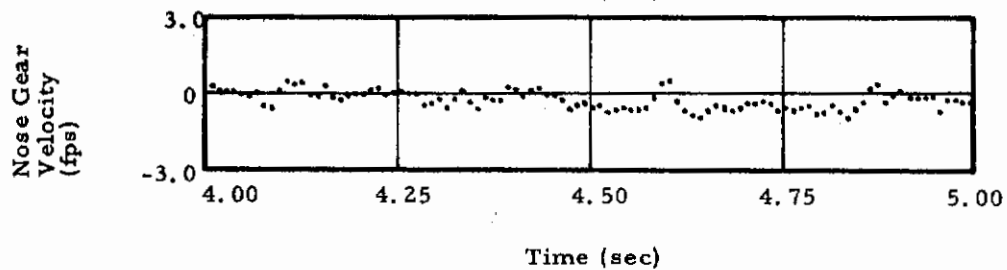
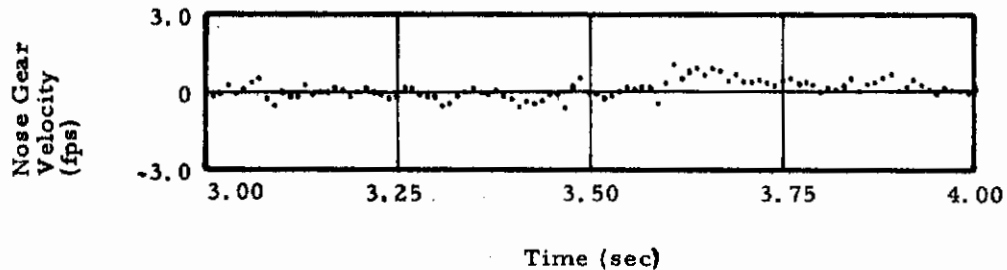
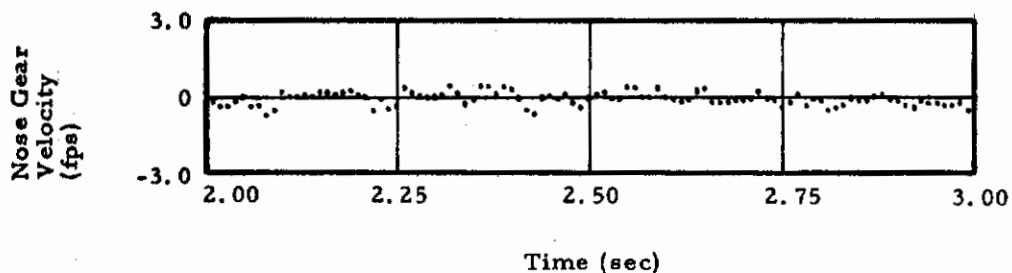
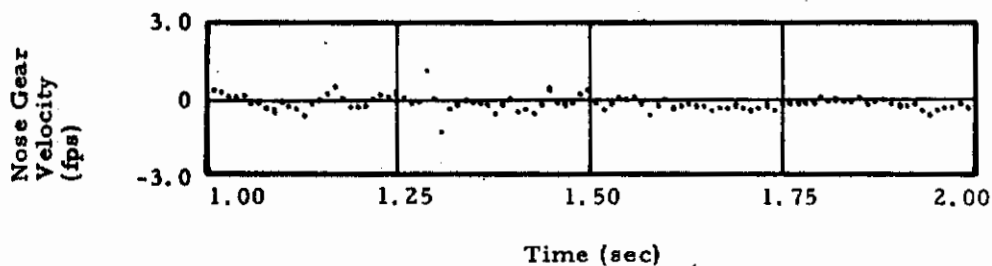
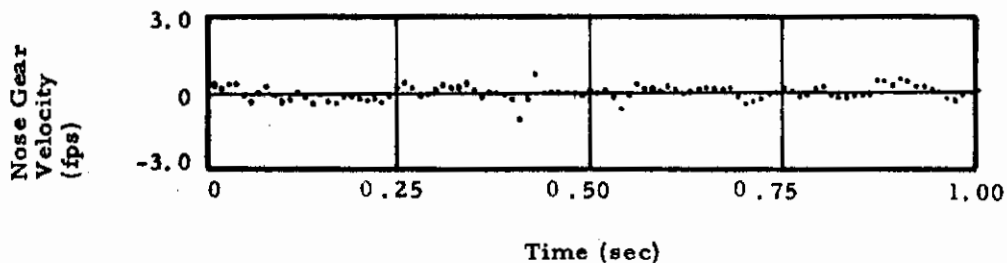
## Nose Gear Velocity

Vehicle velocity = 50 fps



## Nose Gear Velocity

Vehicle velocity = 50 fps

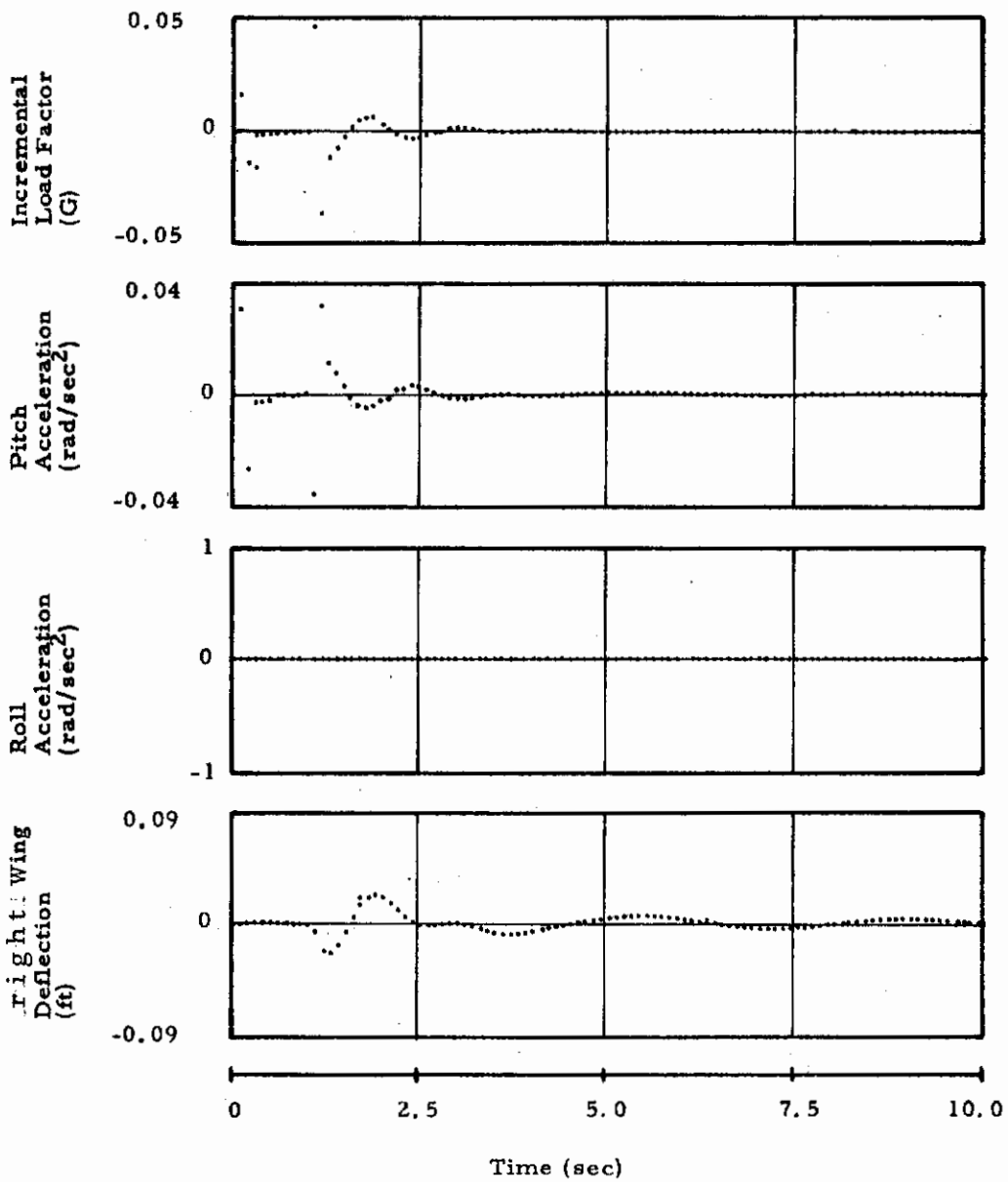


# Contrails

## Vehicle Responses

Bump Length = 10 ft

Velocity = 50 fps

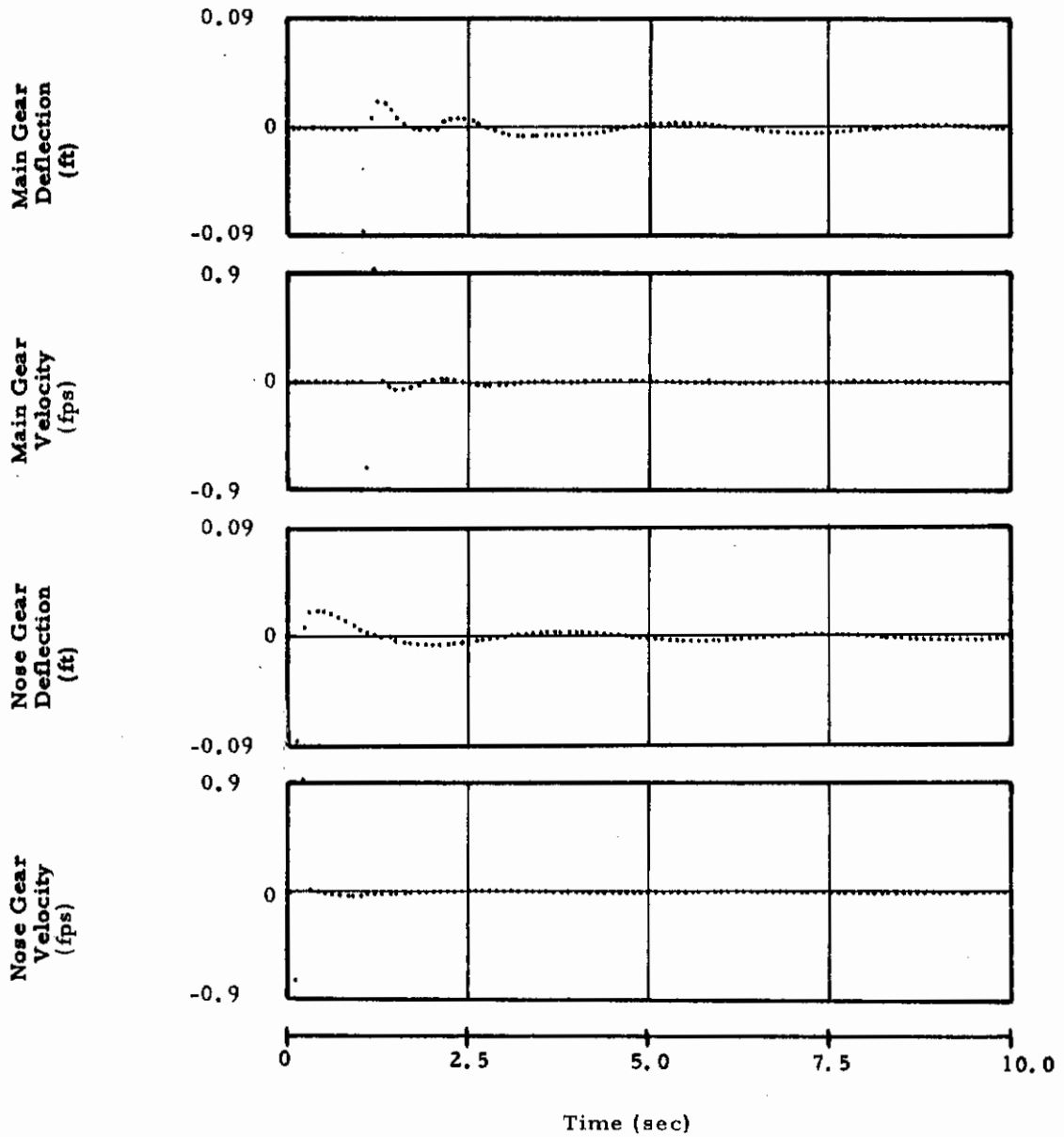


# Contrails

## Vehicle Responses

Bump Length = 10 ft

Velocity = 50 fps



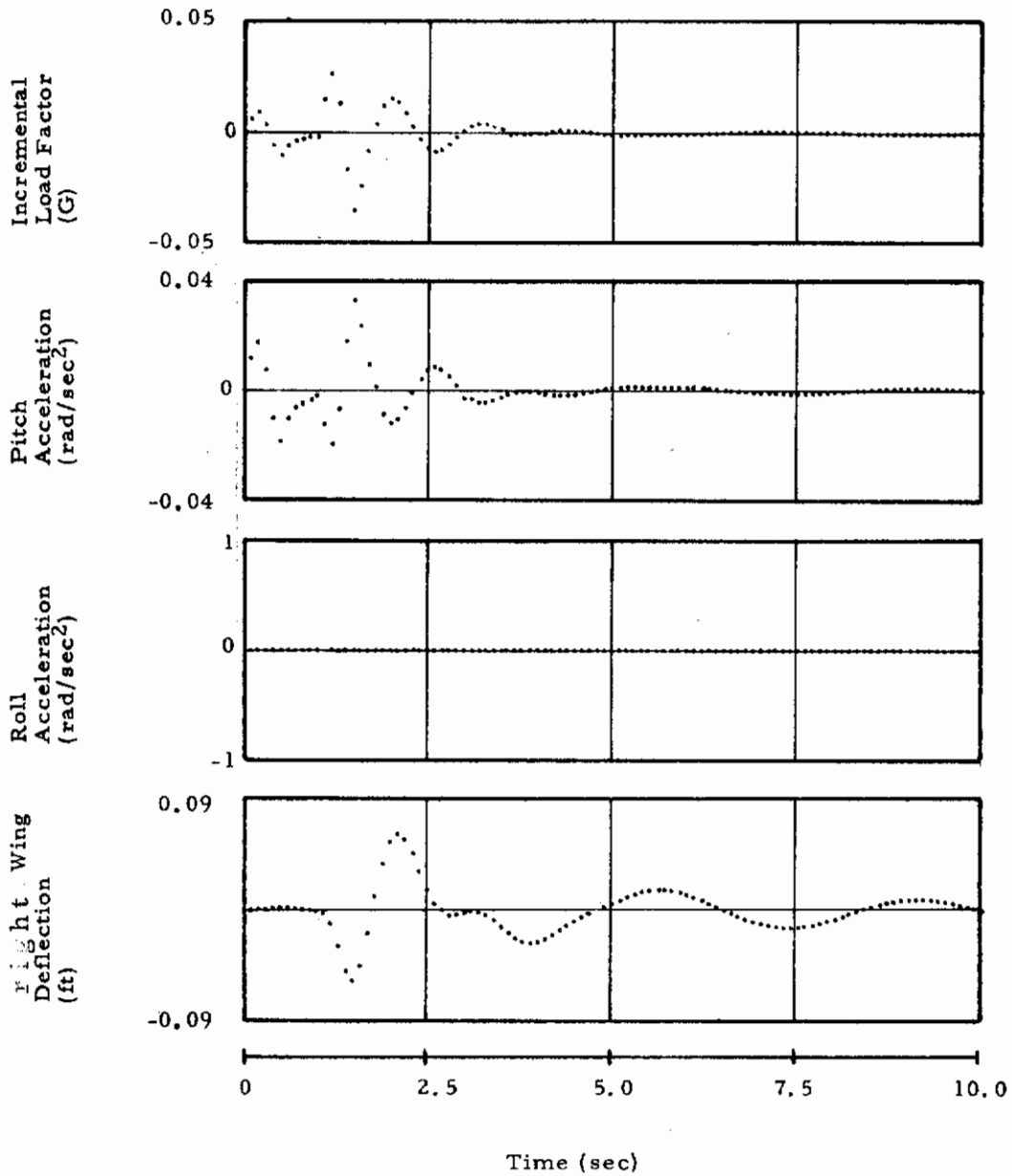


# Contrails

## Vehicle Responses

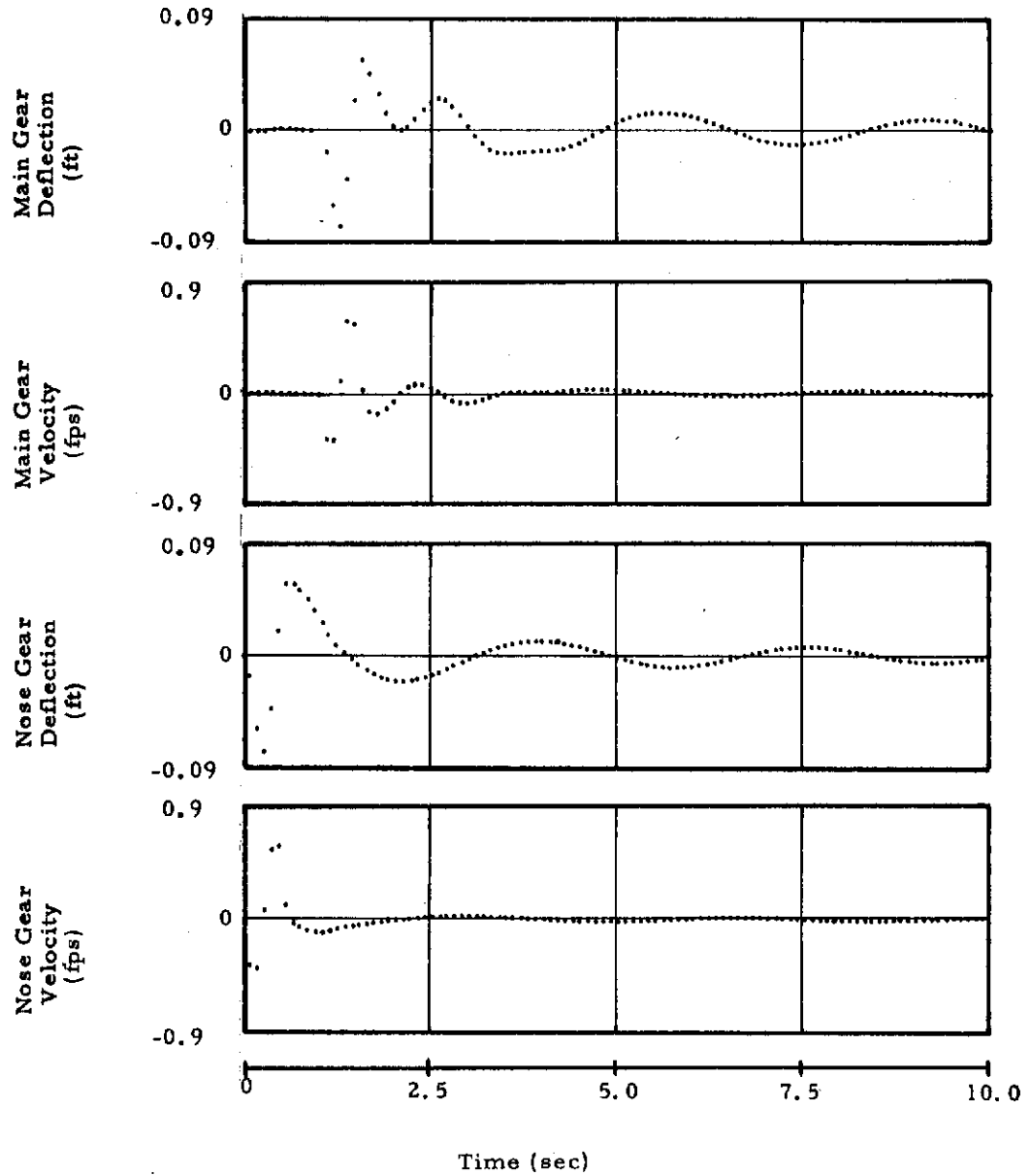
Bump Length = 30 ft

Velocity = 50 fps



## Vehicle Responses

Bump Length = 30 ft  
Velocity = 50 fps

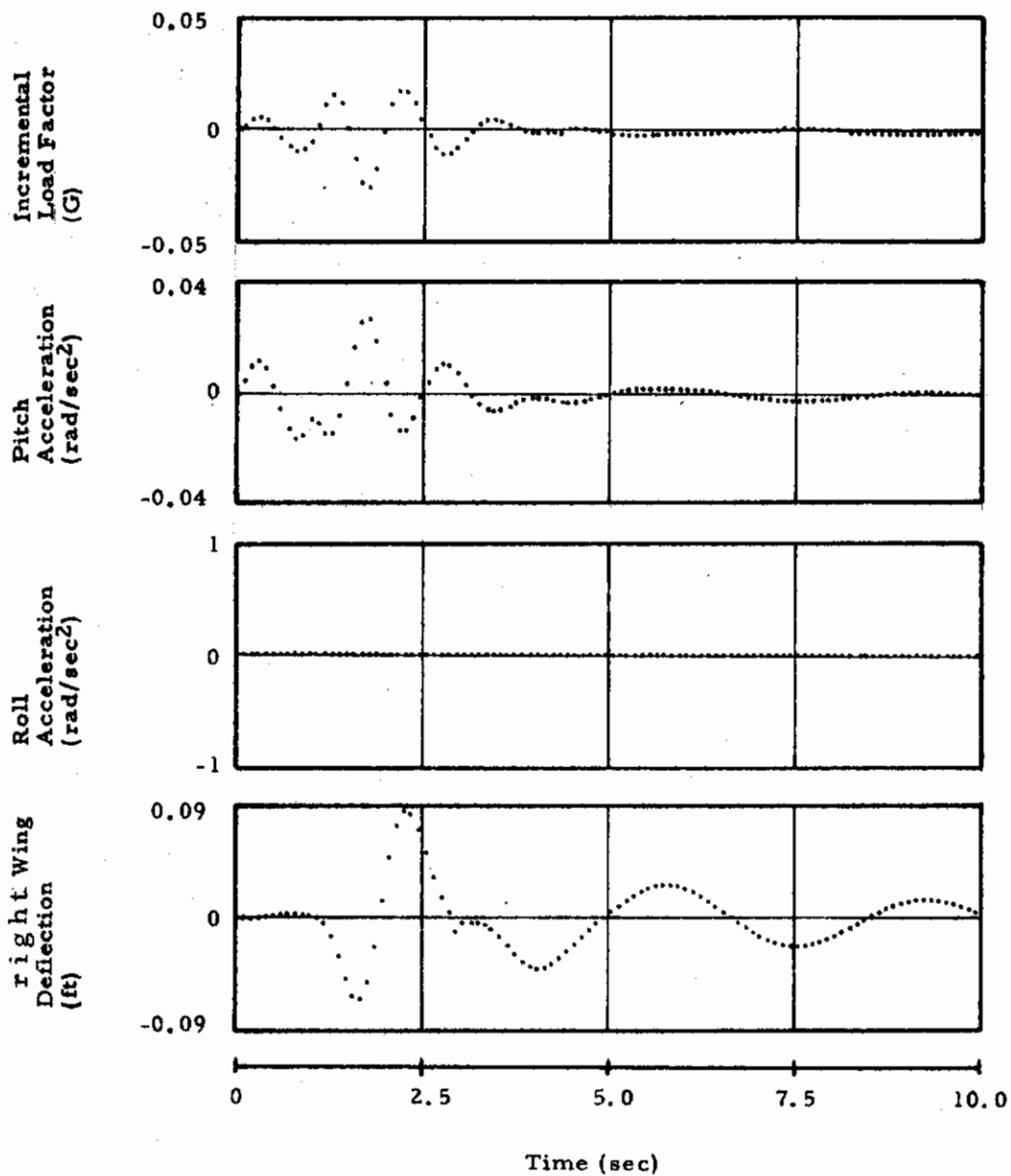


# Contrails

## Vehicle Responses

Bump Length = 50 ft

Velocity = 50 fps

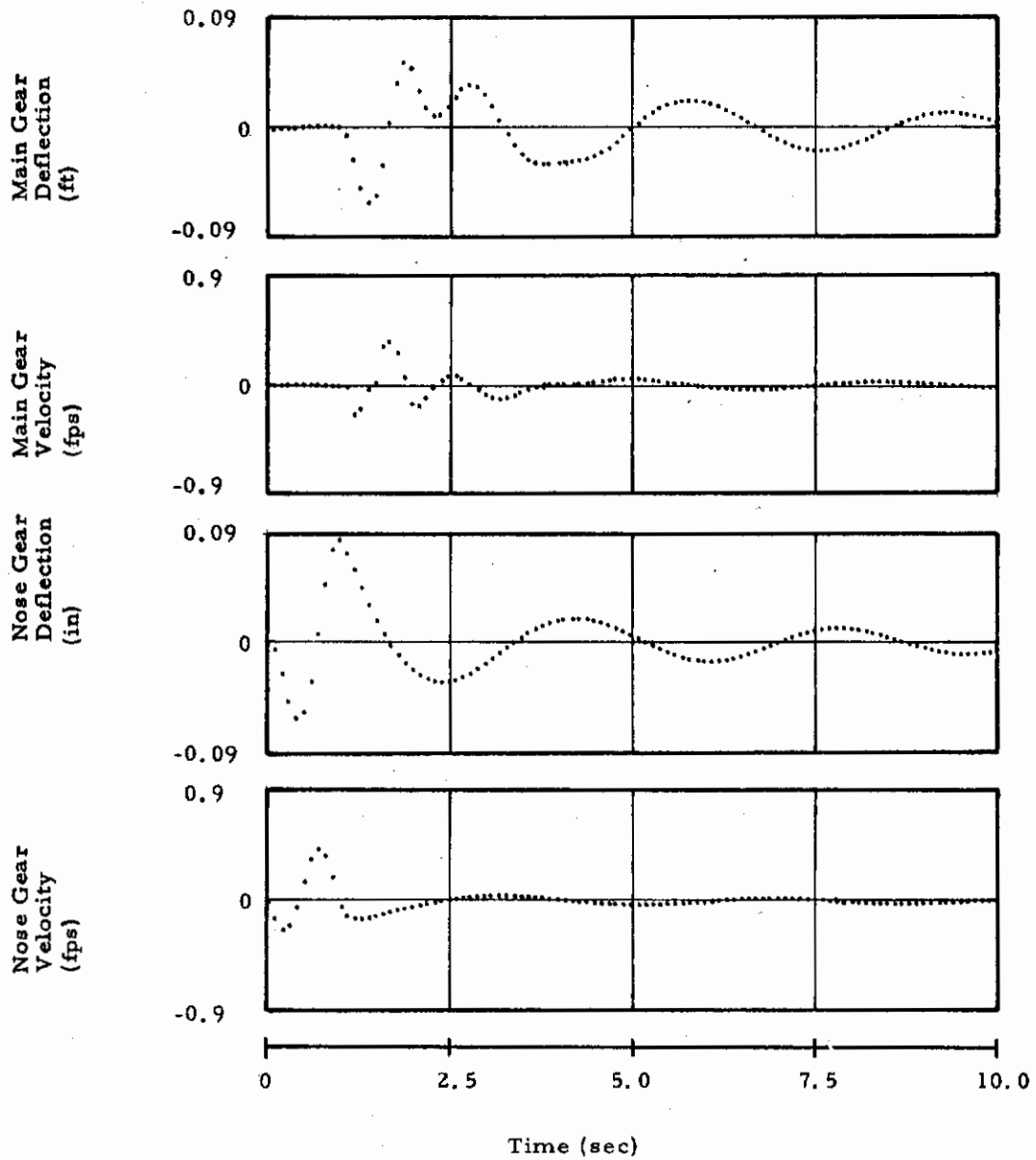


# Contrails

## Vehicle Responses

Bump Length = 50ft

Velocity = 50fps

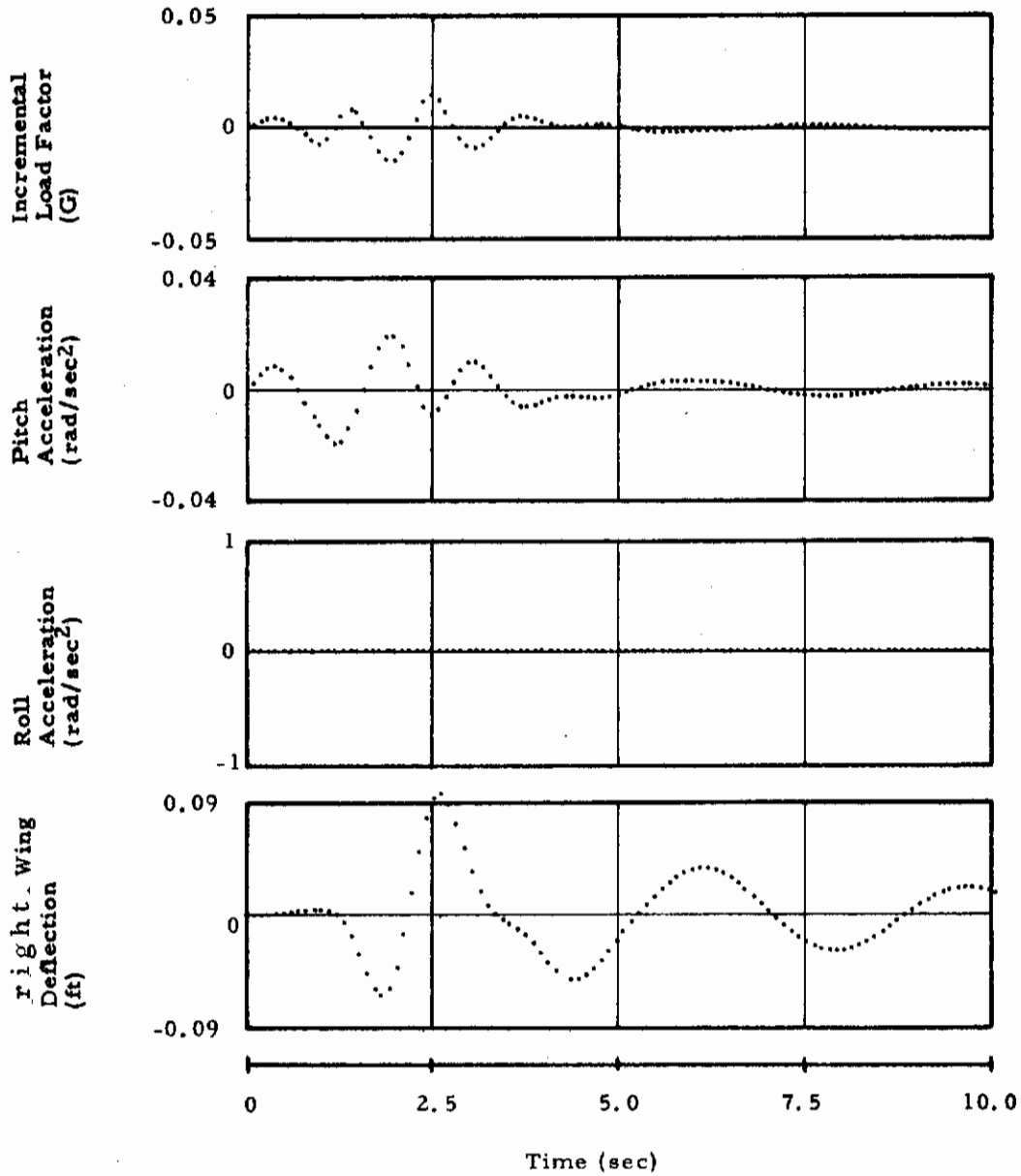


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 50 fps

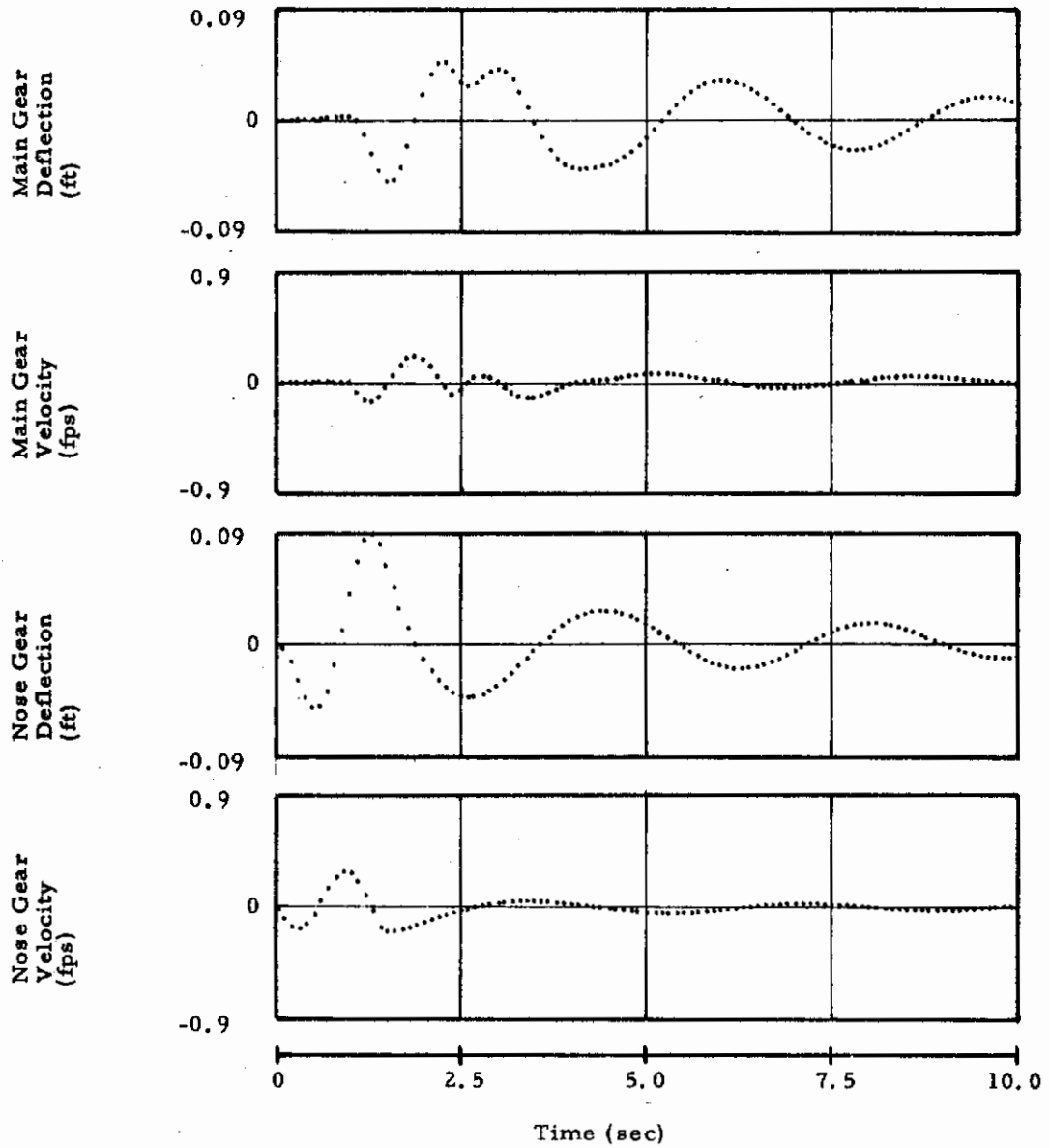


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 50fps

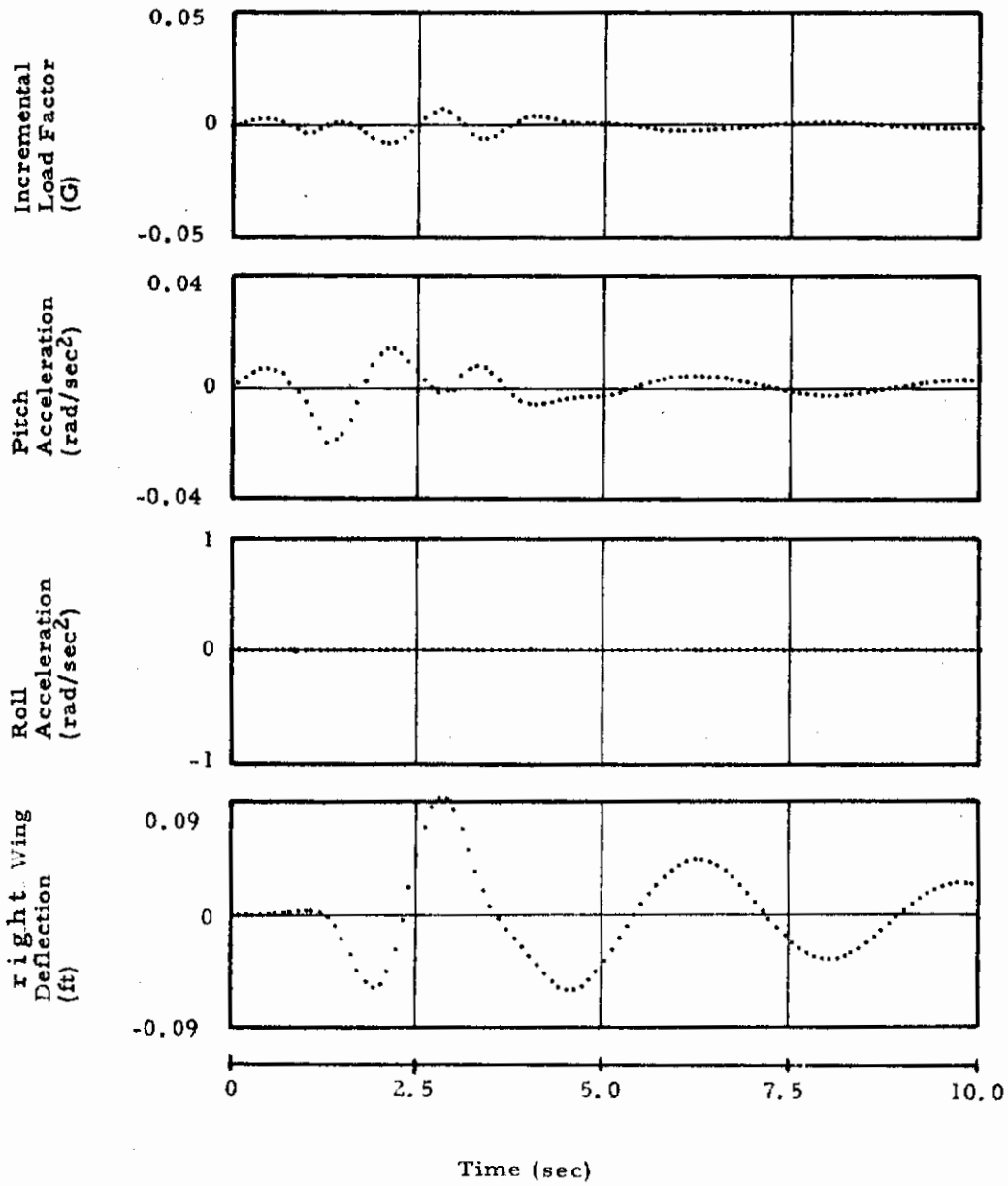


# Contrails

## Vehicle Responses

Bump Length = 90 ft

Velocity = 50 fps

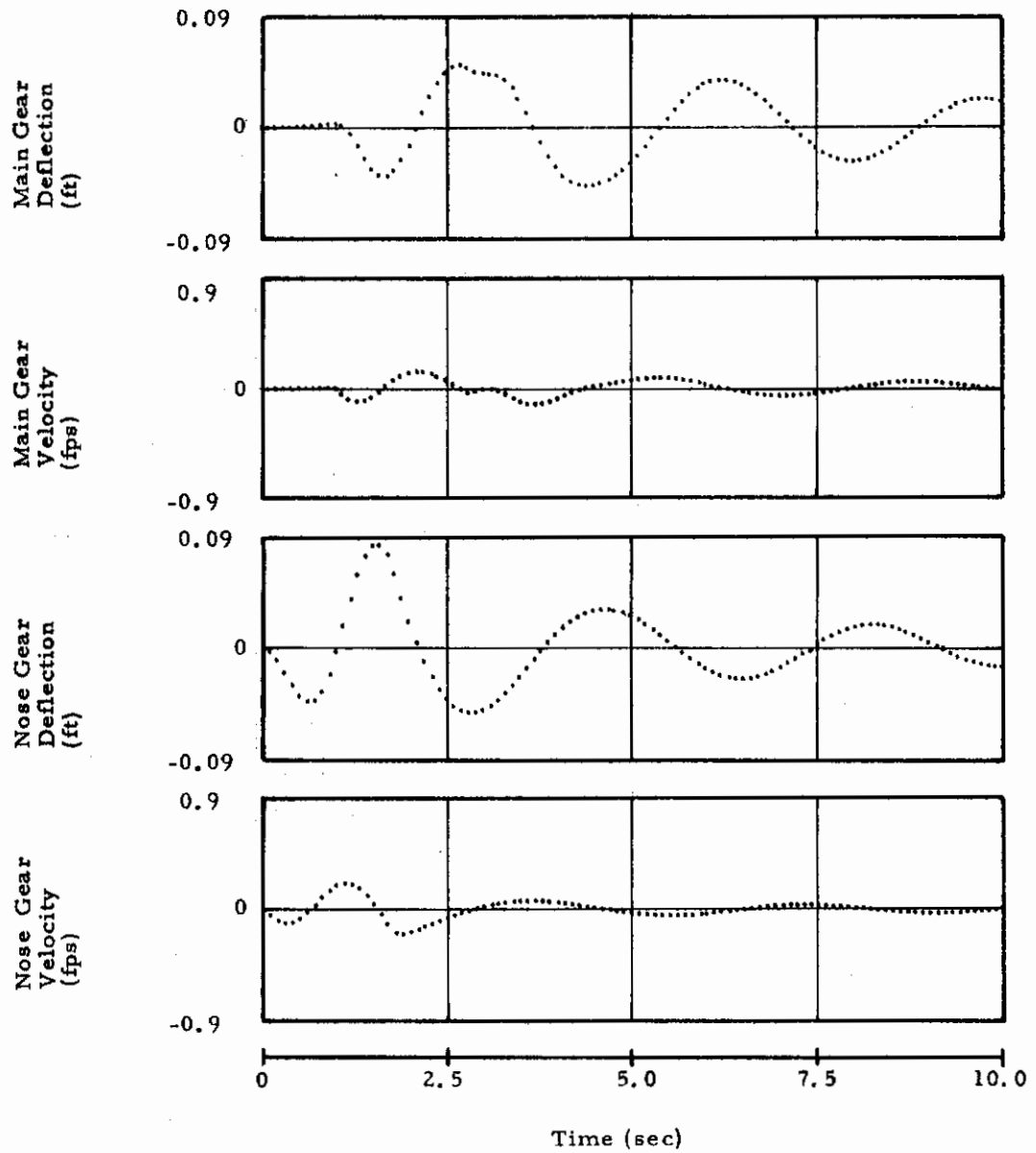


# Contrails

## Vehicle Responses

Bump Length = 90 ft

Velocity = 50 fps

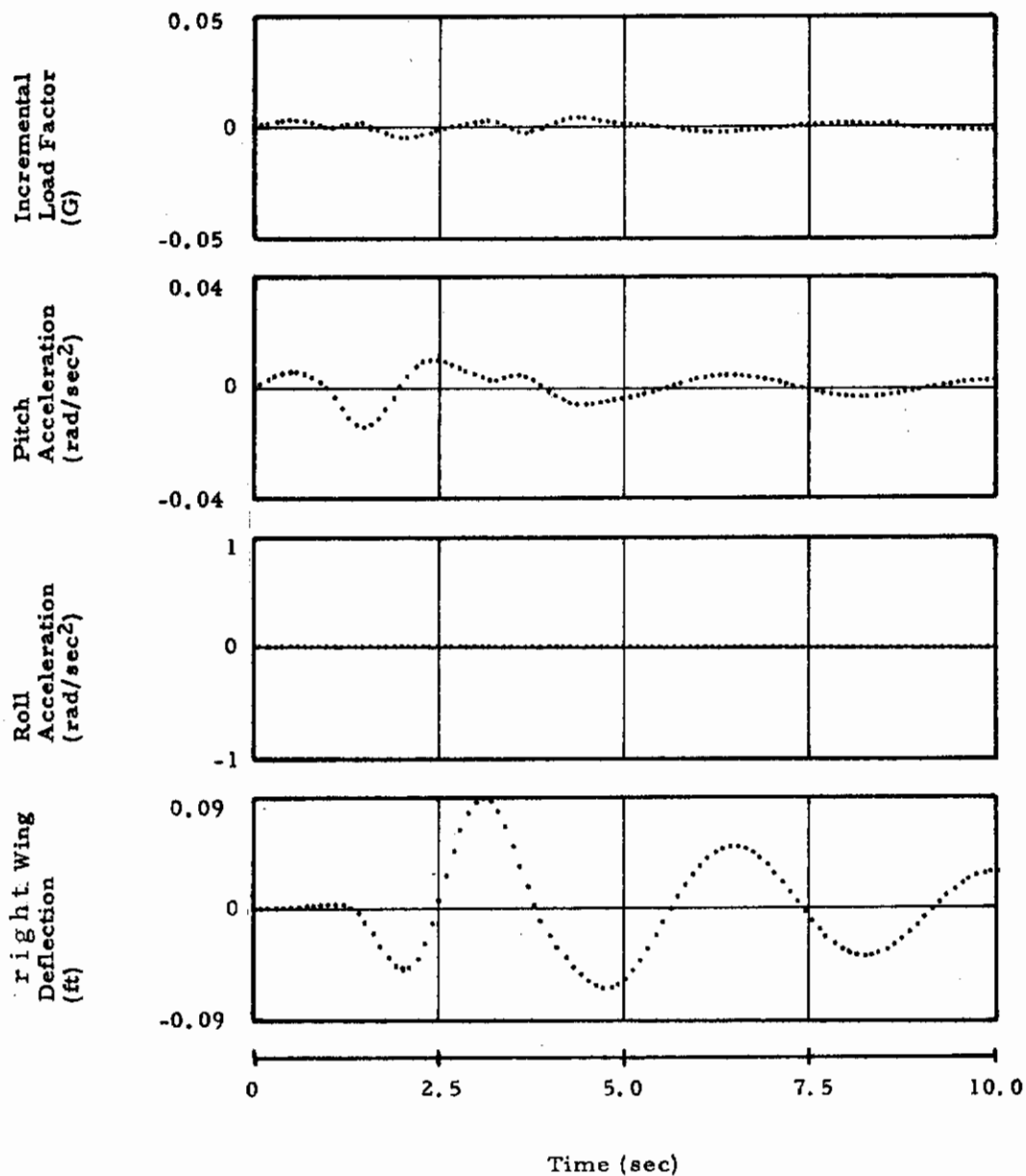




# Contrails

## Vehicle Responses

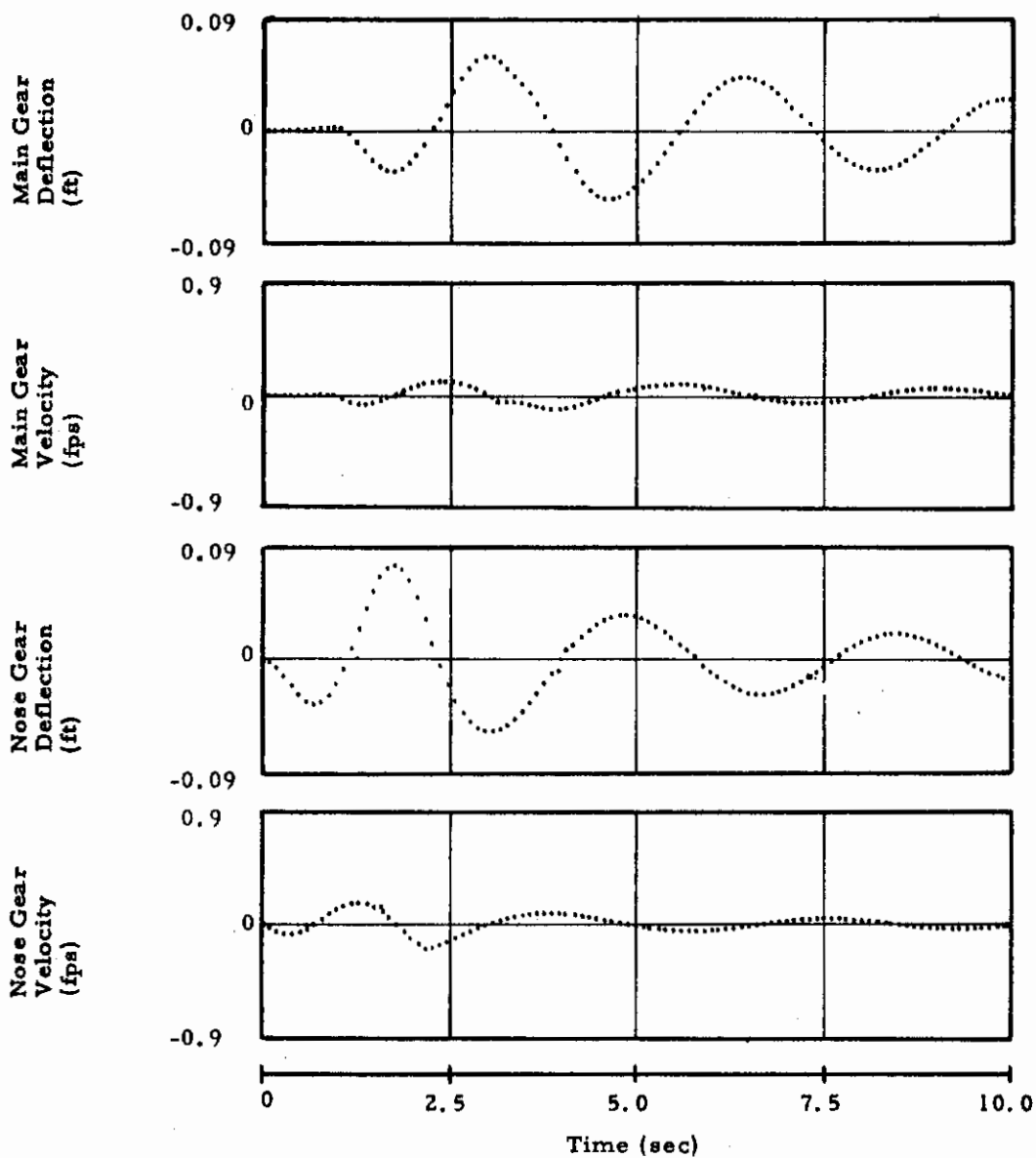
Bump Length = 110 ft  
Velocity = 50 fps



## Vehicle Responses

Bump Length = 110ft

Velocity = 50 fps

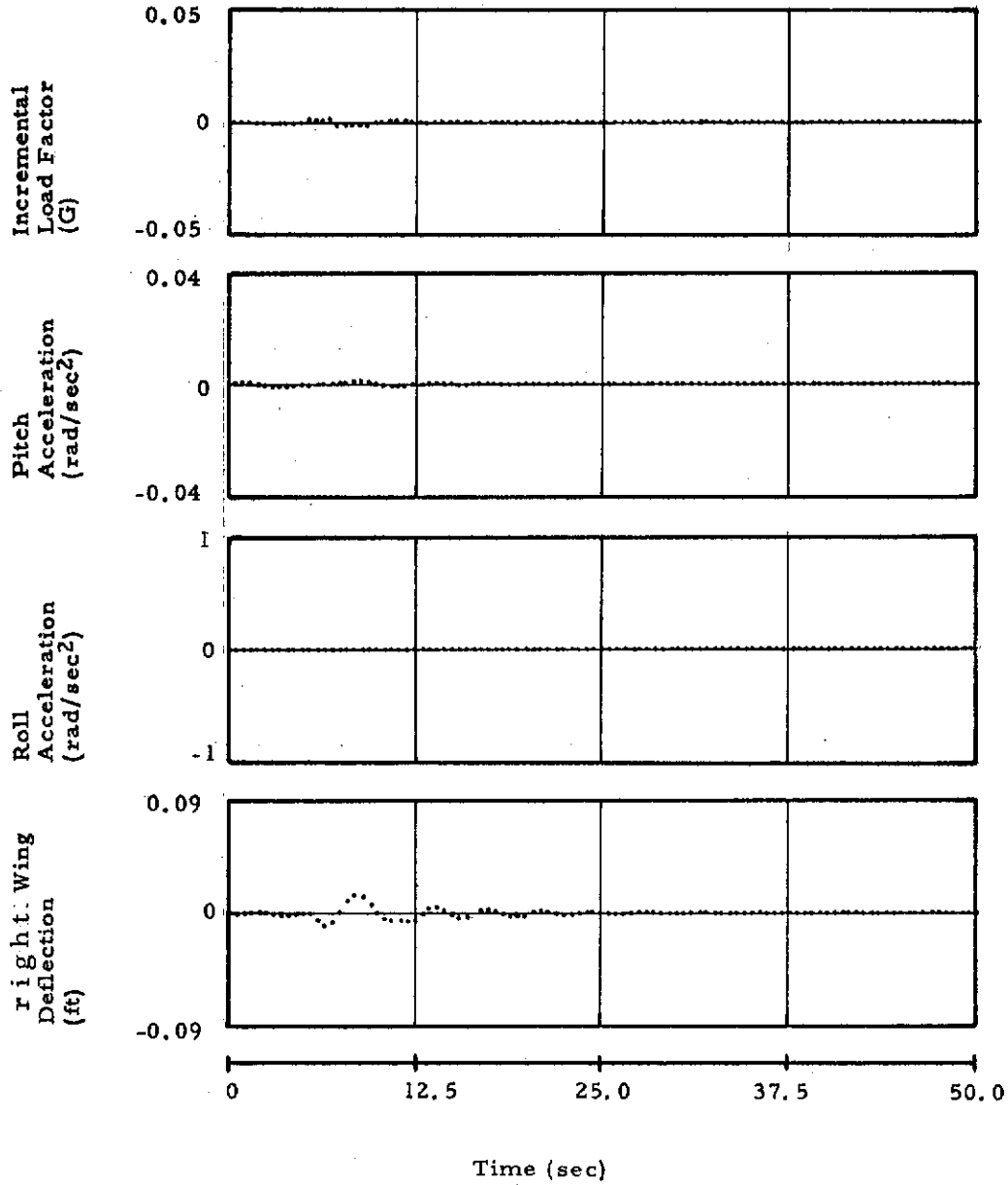


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 10fps

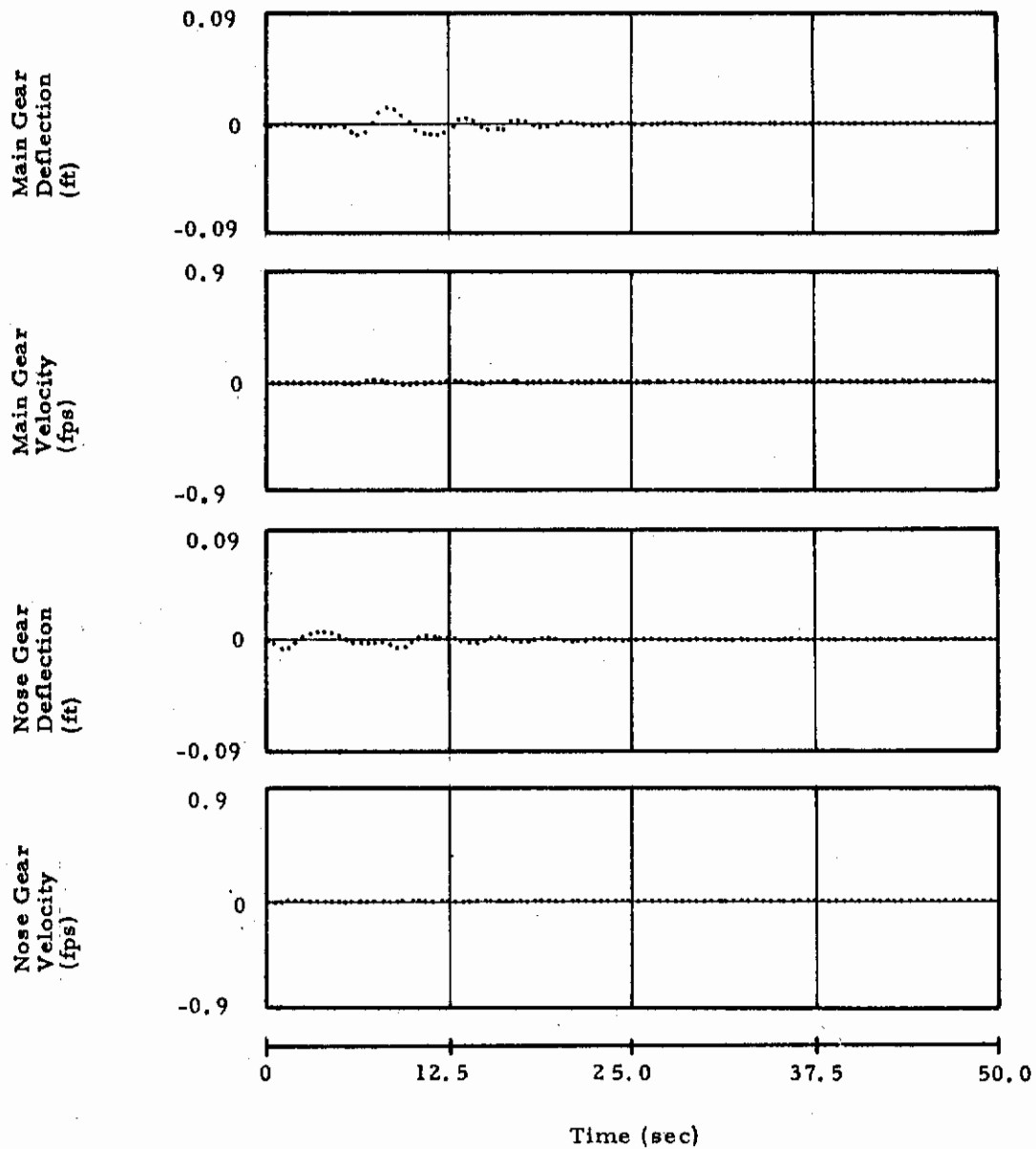


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 10 fps

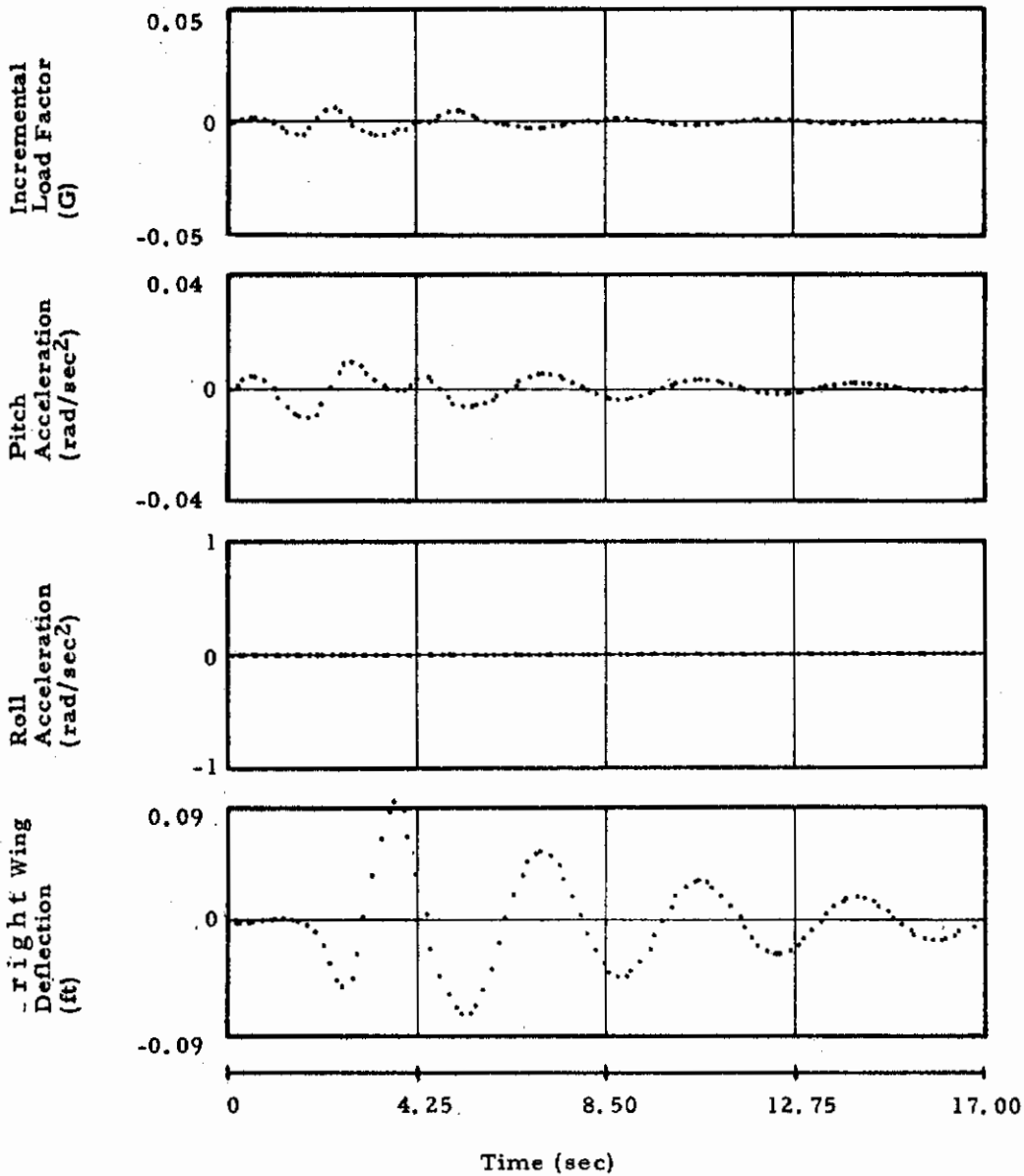


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 30 fps

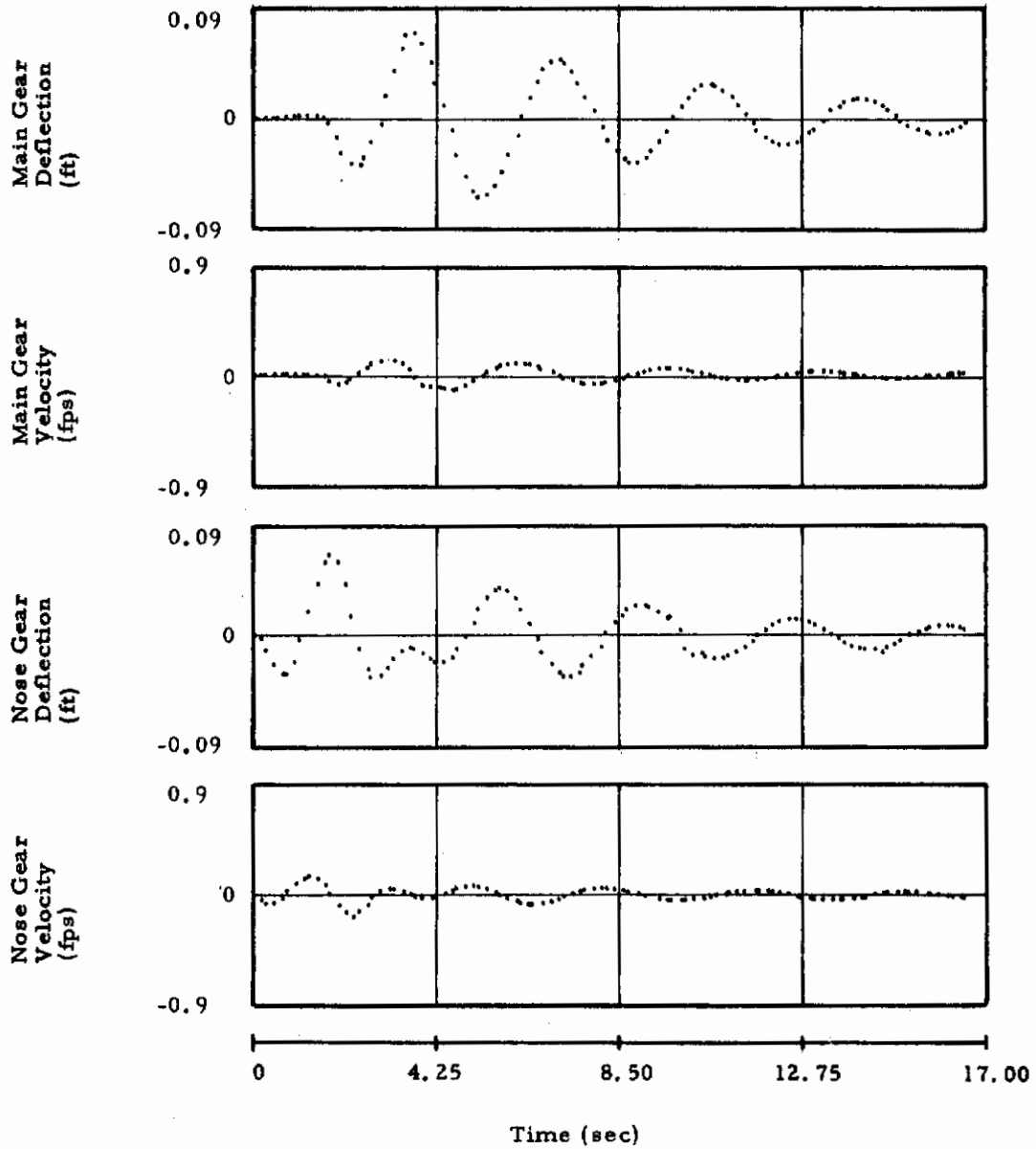


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 30 fps

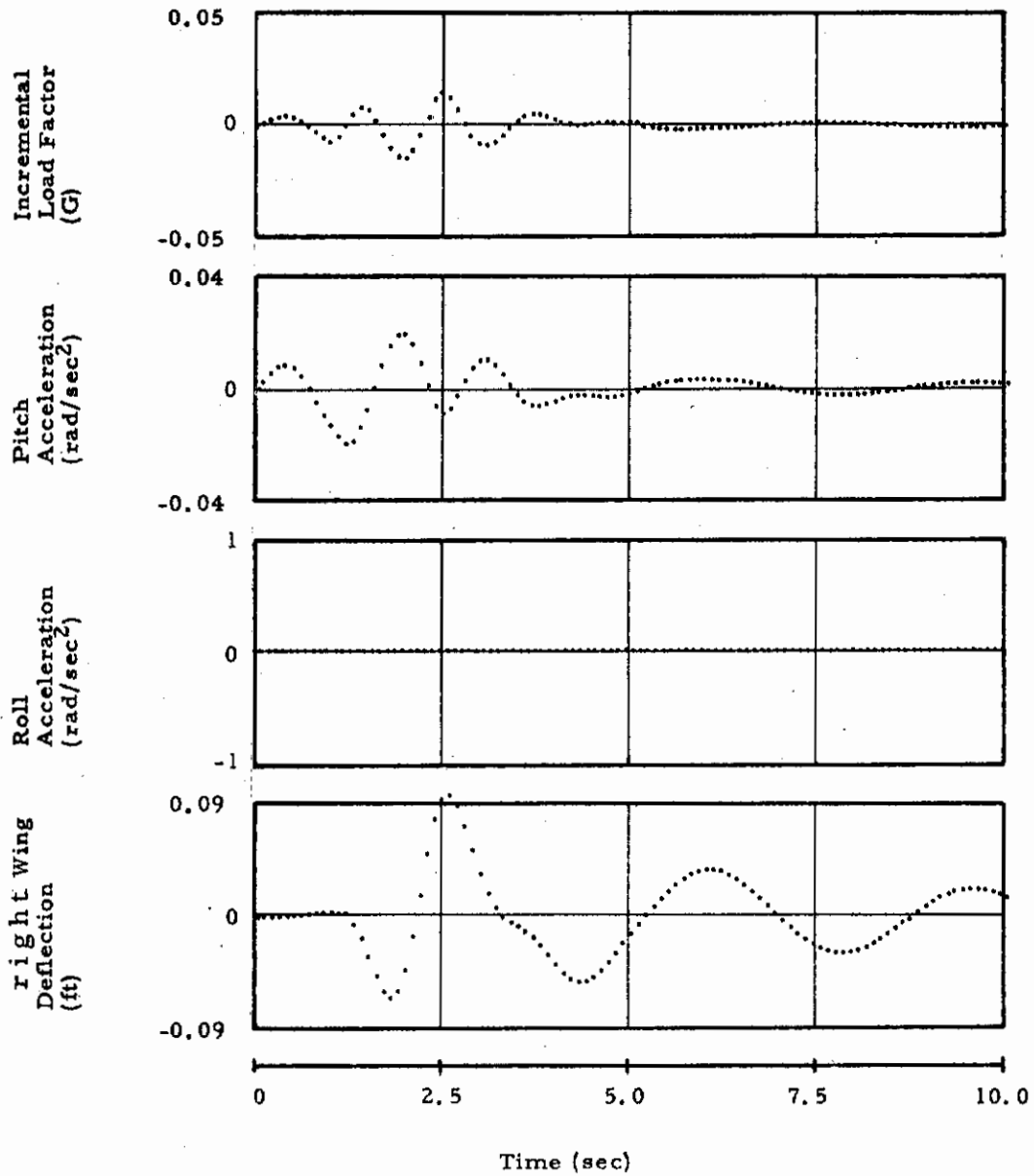


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 50 fps

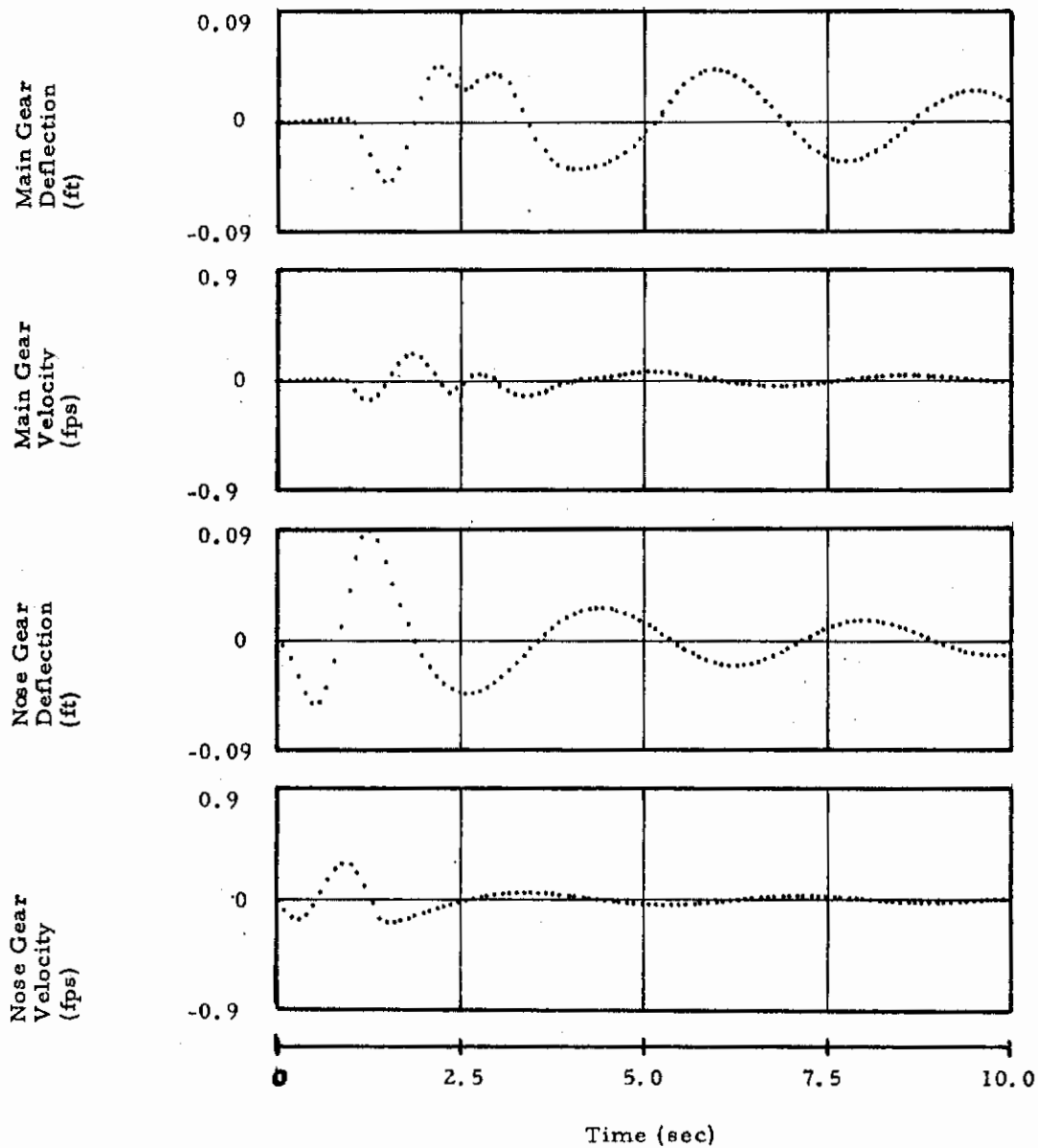


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 50 fps



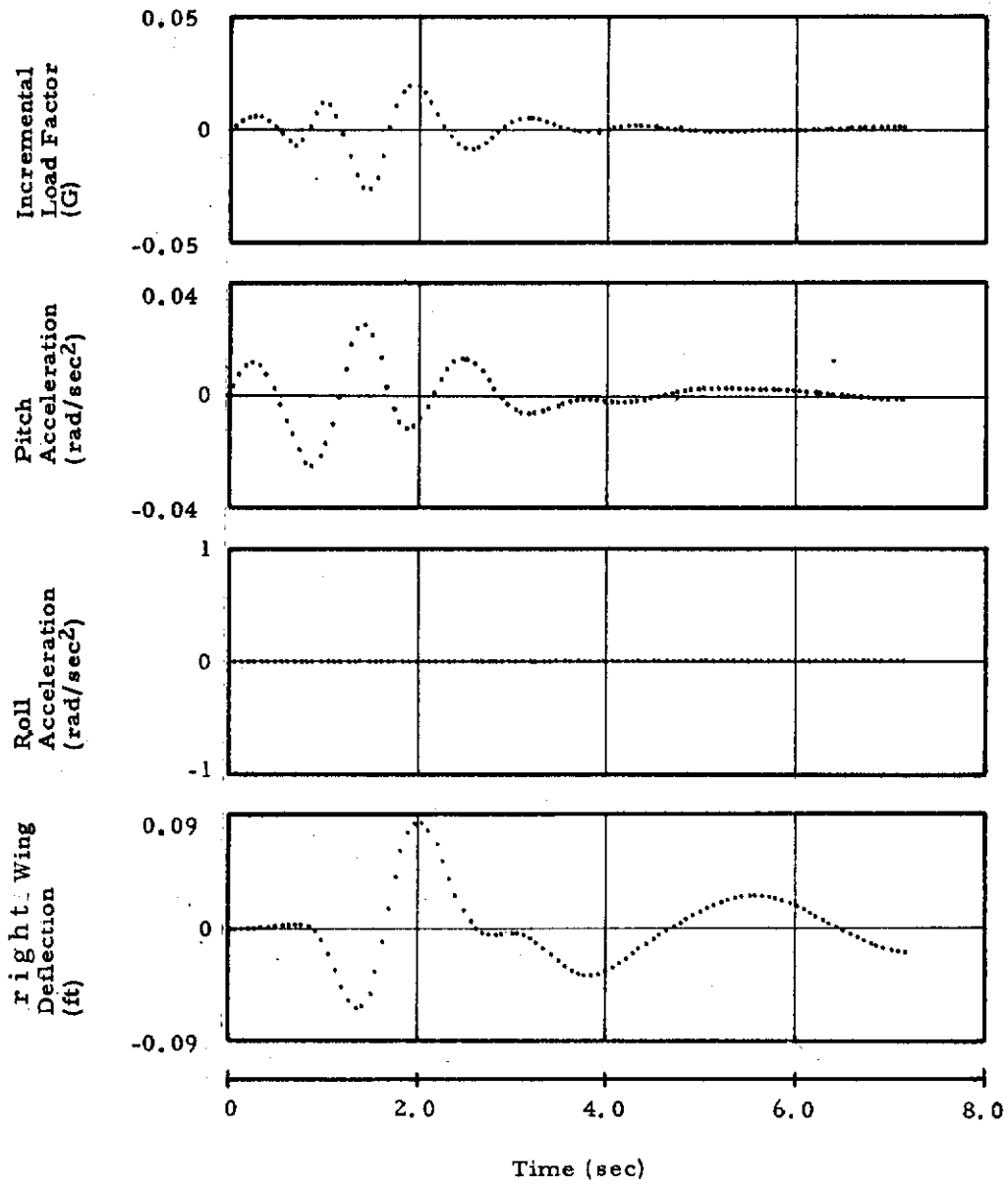


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 70 fps

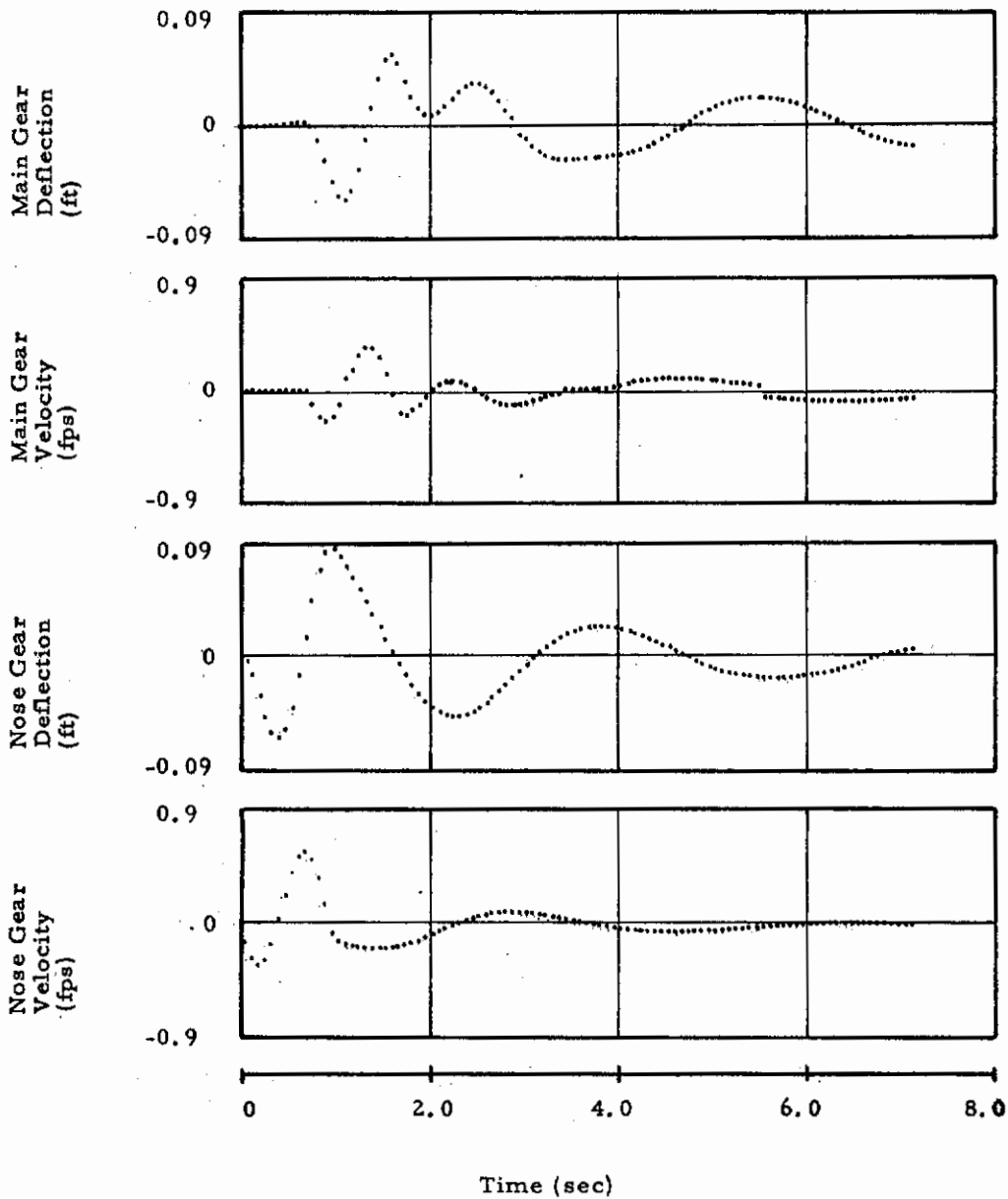


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 70 fps

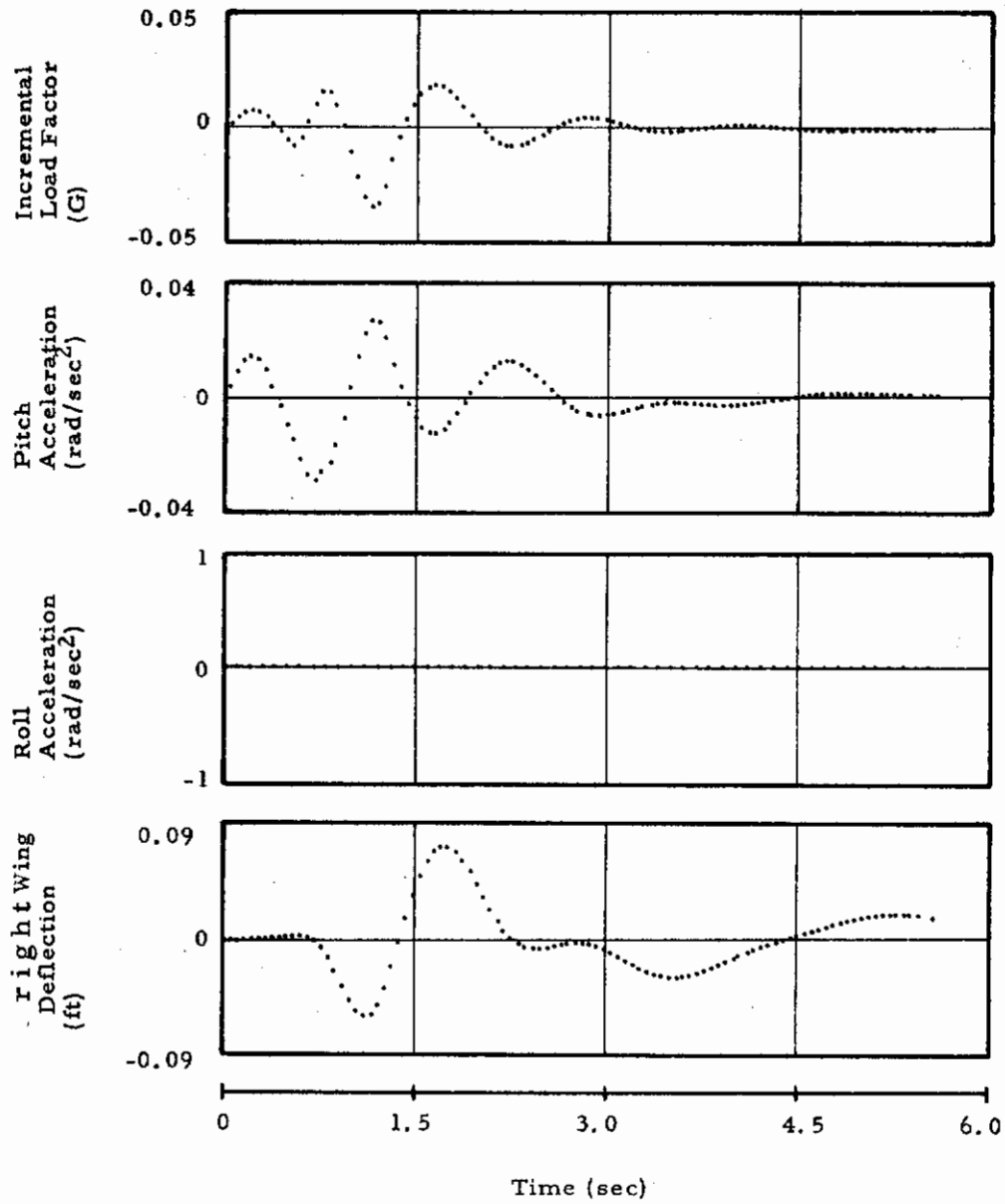


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 90 fps

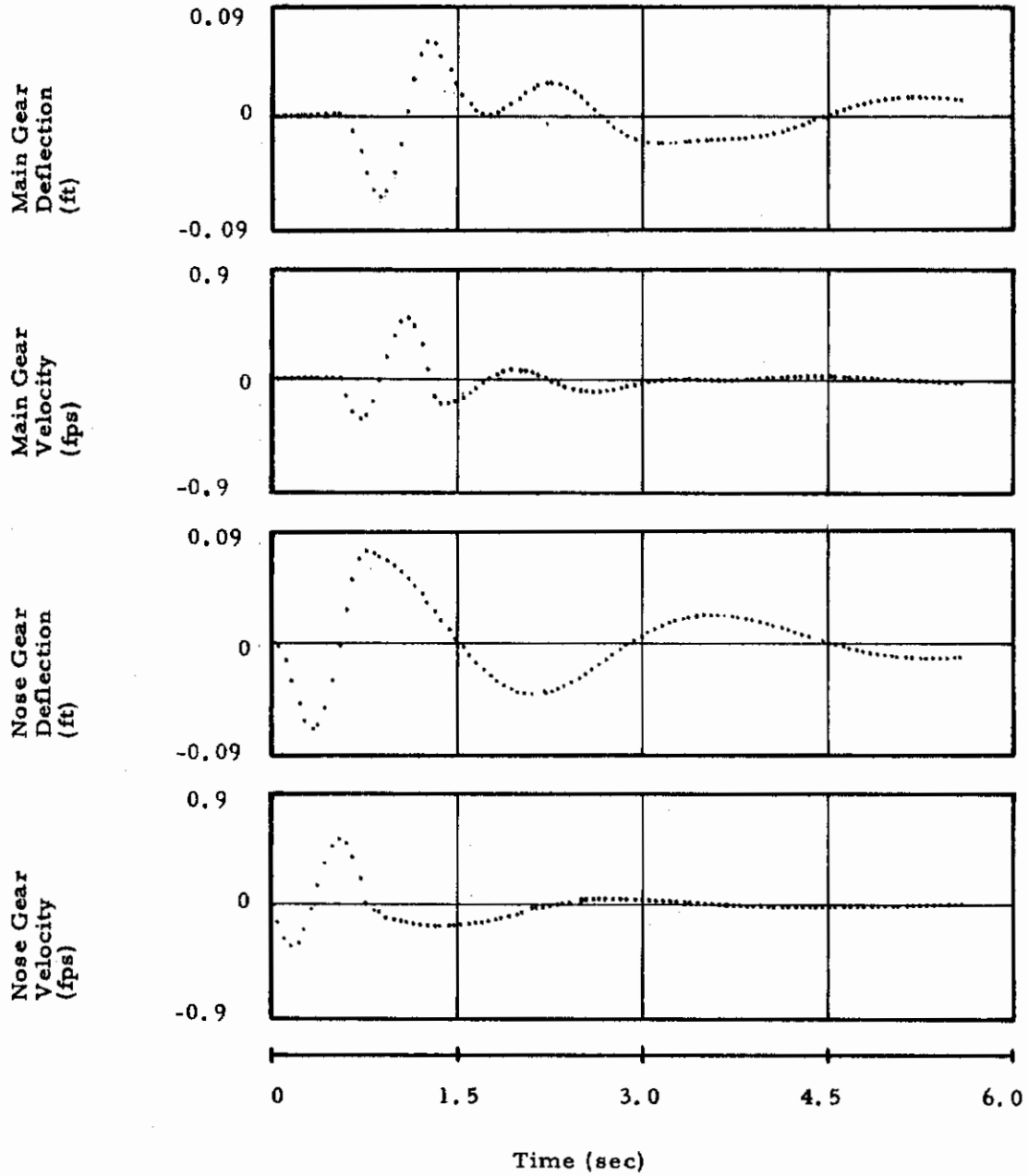


# Contrails

## Vehicle Responses

Bump Length = 70 ft

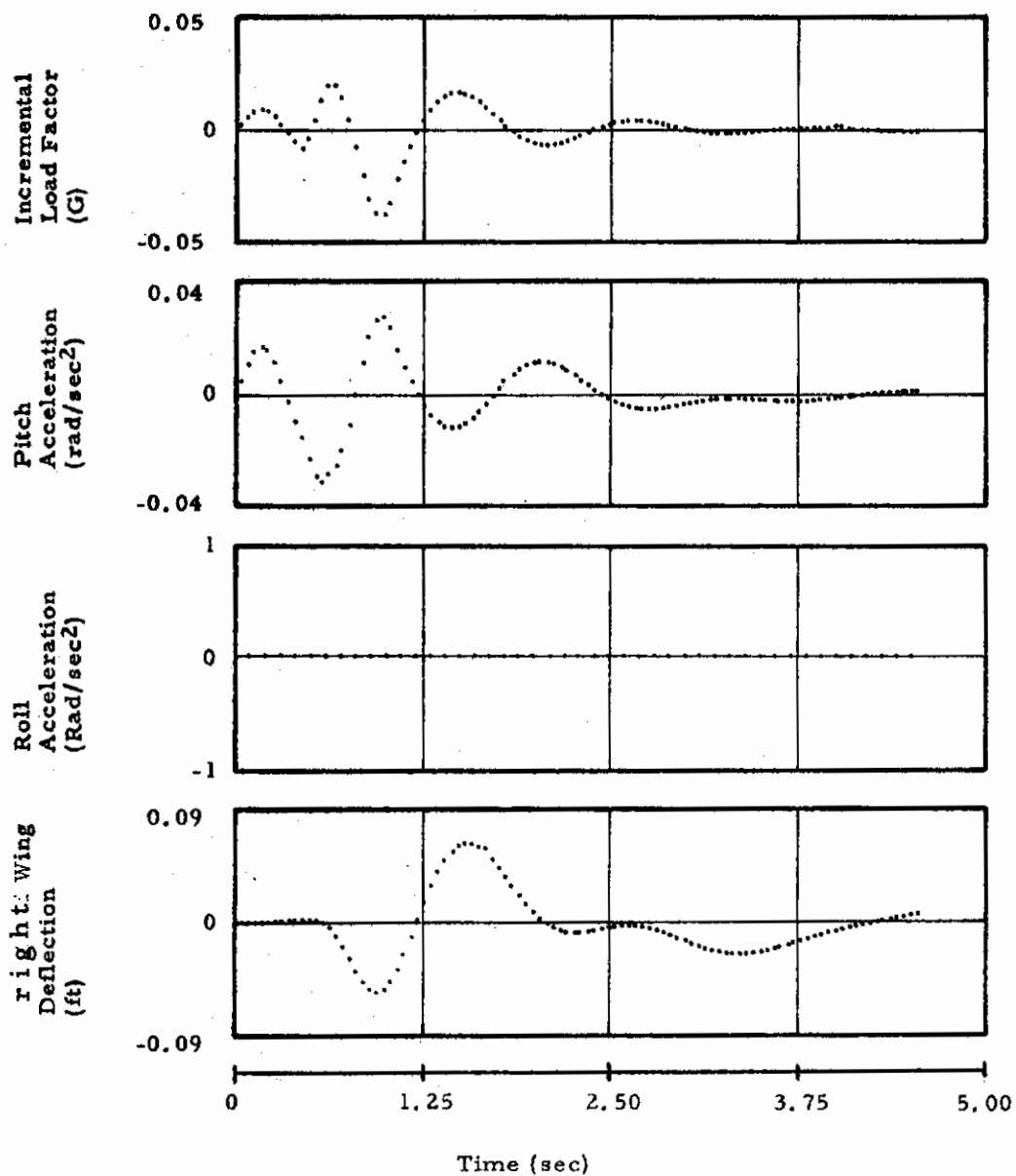
Velocity = 90 fps



## Vehicle Responses

Bump Length = 70 ft

Velocity = 110fps

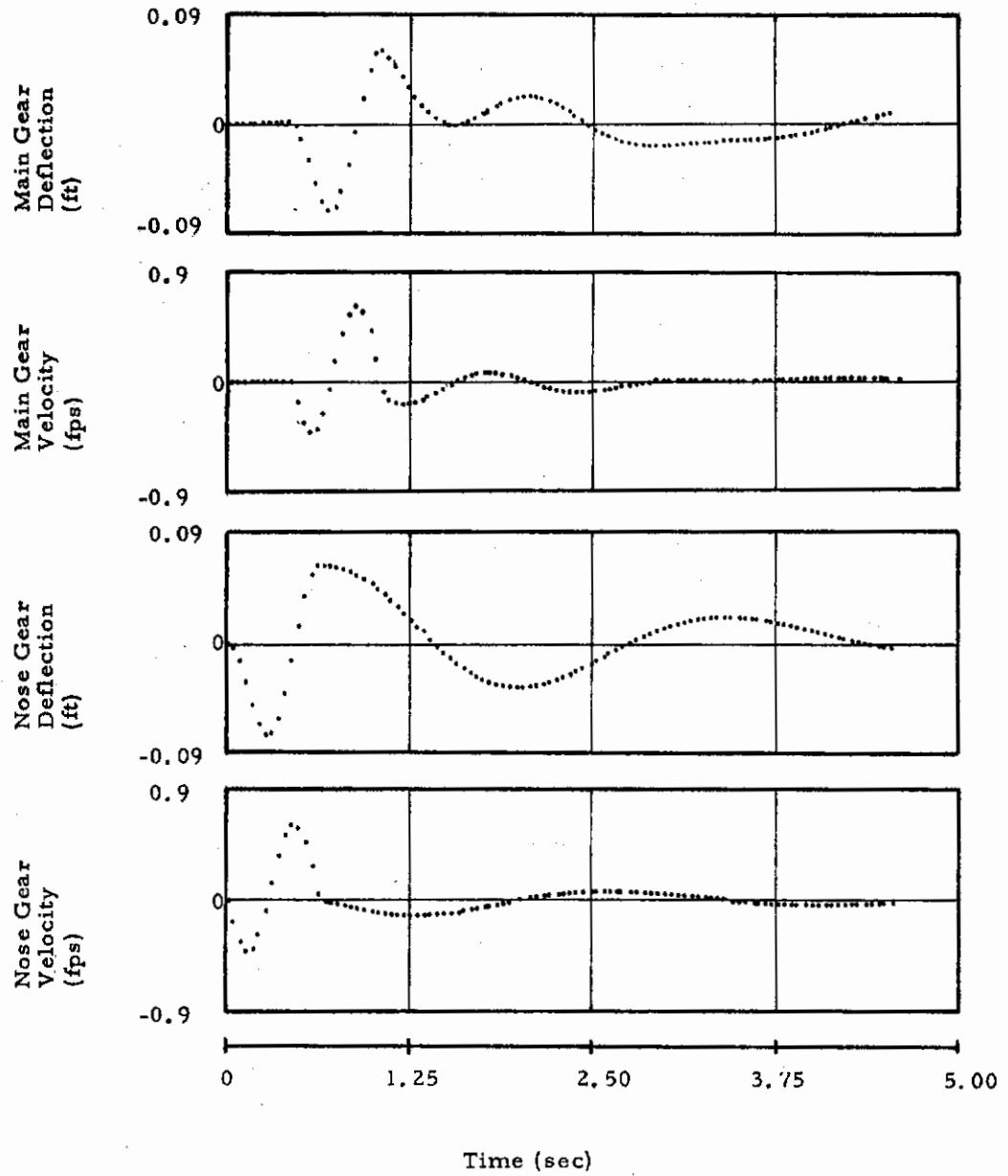


# Contrails

## Vehicle Responses

Bump Length = 70 ft

Velocity = 110 fps



APPENDIX II

STATISTICAL ANALYSIS OF A  
THREE-DEGREE-OF-FREEDOM VEHICLE

APPENDIX II  
STATISTICAL ANALYSIS  
OF A THREE-DEGREE-OF-FREEDOM VEHICLE \*

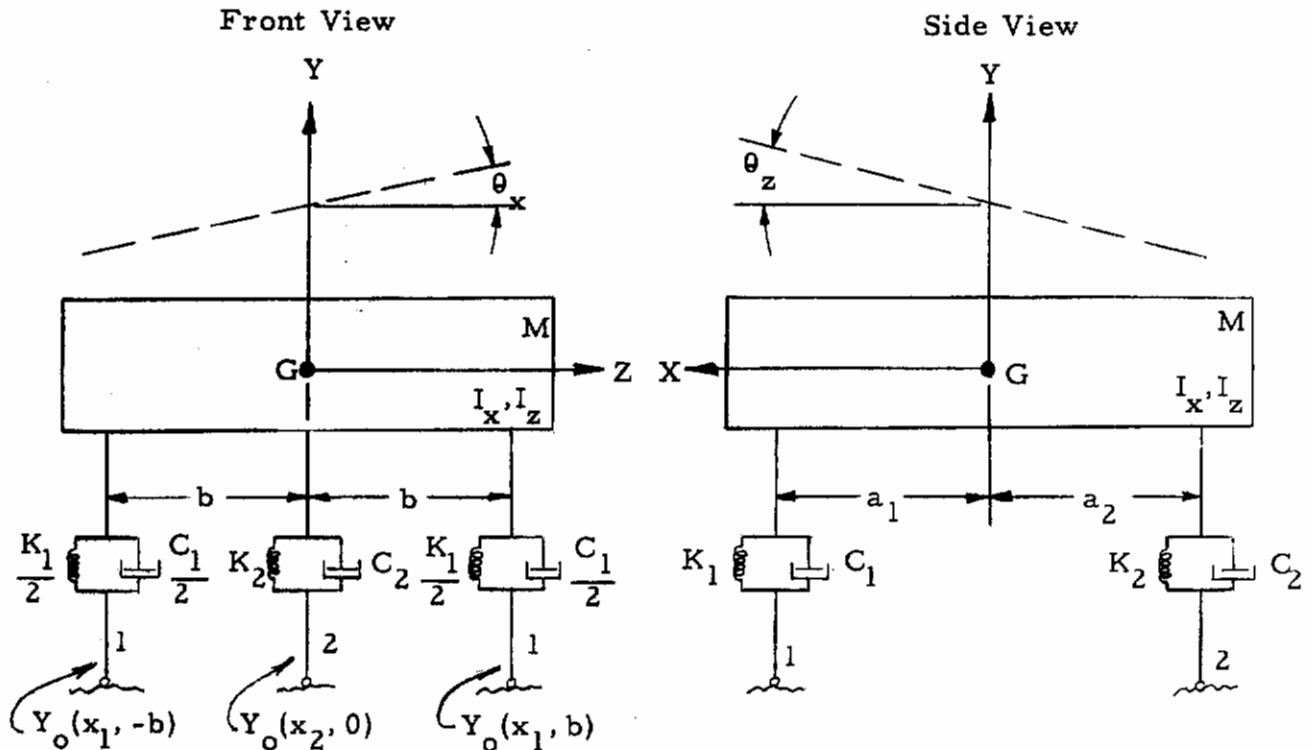


Fig. 2-14. Three-Degree-of-Freedom Vehicle

The system to be considered is shown in Figure 2-14. The rigid body of mass  $M$  is constrained to move at a constant velocity  $V_0$  along the  $X$  axis. No rotational motion about the  $Y$  axis is allowed. The rigid body has mass moment of inertia  $I_x$  about the  $X$  axis and  $I_z$  about the  $Z$  axis. The vehicle supports consist of three springs, with spring constants  $K_1/2$ ,  $K_1/2$  and

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\* References: S. H. Crandall, et al, Random Vibrations, The Technology Press and Wiley (1958).  
 J. L. Bogdanoff and F. Kozin, On the Statistical Analysis of Some Simple Vehicles Moving on a Random Track, Report No. 65, Land Locomotion Laboratory, Detroit, Michigan, Nov. 1959.



$K_2$ , and three viscous dampers, with damping constants  $C_1/2$ ,  $C_1/2$  and  $C_2$ . The geometry of the support system is given by dimensions  $a_1$ ,  $a_2$ , and  $b$ . The point followers, 1 and 2, maintain constant contact with the tracks,  $Y_0(x_1, b)$ ,  $Y_0(x_1, -b)$  and  $Y_0(x_2, 0)$ . Here  $Y_0(x, z)$  is a random function.  $X_1$  and  $X_2$  are the coordinates of followers 1 and 2 respectively. The coordinates of the mass center at time  $t$  are  $[X(t), Y(t), 0]$ . The angular orientation of the rigid body at time  $t$  is given by the angles  $\theta_x(t)$  and  $\theta_z(t)$ .

The equations of motion for this vehicle are:

$$\begin{aligned}
 M(\ddot{Y} - G) + \frac{K_1}{2} [Y + a_1 \theta_z + b\theta_x - Y_0(x_1, b)] + \frac{C_1}{2} [\dot{Y} + a_1 \dot{\theta}_z + b\dot{\theta}_x \\
 - \dot{Y}_0(x_1, b)] + \frac{K_1}{2} [Y + a_1 \theta_z - b\theta_x - Y_0(x_1, -b)] + \frac{C_1}{2} [\dot{Y} + a_1 \dot{\theta}_z \\
 - b\dot{\theta}_x - \dot{Y}_0(x_1, -b)] + K_2 [Y - a_2 \theta_z - Y_0(x_2, 0)] + C_2 [\dot{Y} - a_2 \dot{\theta}_z \\
 - \dot{Y}_0(x_2, 0)] = 0,
 \end{aligned}$$

$$\begin{aligned}
 I_{x_x} \ddot{\theta}_x + \frac{K_1}{2} b [Y + a_1 \theta_z + b\theta_x - Y_0(x_1, b)] + \frac{C_1}{2} b [\dot{Y} + a_1 \dot{\theta}_z + b\dot{\theta}_x \\
 - \dot{Y}_0(x_1, b)] - \frac{K_1}{2} b [Y + a_1 \theta_z - b\theta_x - Y_0(x_1, -b)] - \frac{C_1}{2} b [\dot{Y} + a_1 \dot{\theta}_z \\
 - b\dot{\theta}_x - \dot{Y}_0(x_1, -b)] = 0,
 \end{aligned}$$

$$\begin{aligned}
 I_{z_z} \ddot{\theta}_z + \frac{K_1}{2} a_1 [Y + a_1 \theta_z + b\theta_x - Y_0(x_1, b)] + \frac{C_1}{2} a_1 [\dot{Y} + a_1 \dot{\theta}_z + b\dot{\theta}_x \\
 - \dot{Y}_0(x_1, b)] + \frac{K_1}{2} a_1 [Y + a_1 \theta_z - b\theta_x - Y_0(x_1, -b)] + \frac{C_1}{2} a_1 [\dot{Y} + a_1 \dot{\theta}_z \\
 - b\dot{\theta}_x - \dot{Y}_0(x_1, -b)] - K_2 a_2 [Y - a_2 \theta_z - Y_0(x_2, 0)] - C_2 a_2 [\dot{Y} - a_2 \dot{\theta}_z \\
 - \dot{Y}_0(x_2, 0)] = 0.
 \end{aligned}$$

# Contrails

Let

$$l = a_1 + a_2$$

$$a_1 = a_1/l$$

$$a_2 = a_2/l$$

Then

$$a_1 + a_2 = l$$

Let

$$\rho_z^2 = I_z/M$$

Let

$$\beta_1 = C_1/(C_1 + C_2); \beta_2 = C_2/(C_1 + C_2)$$

$$\gamma_1 = K_1/(K_1 + K_2); \gamma_2 = K_2/(K_1 + K_2)$$

$$\bar{\omega}_1^2 = (K_1 + K_2)/M$$

$$\Omega^2 = K_1 b^2/I_x$$

$$\gamma_1 = (C_1 + C_2)/2\bar{\omega}_1 M$$

$$\gamma_2 = C_1 b^2/2\Omega I_x$$

Then

$$\beta_1 + \beta_2 = 1$$

$$\gamma_1 + \gamma_2 = 1.$$

If these equations are substituted into the equations of motion, then we obtain

$$\begin{aligned} Y + 2\gamma_1 \bar{\omega}_1 [\dot{Y} + (a_1 \beta_1 - a_2 \beta_2) l \dot{\theta}_z] + \bar{\omega}_1^2 [Y + (a_1 \gamma_1 - a_2 \gamma_2) l \theta_z] \\ = G + 2\gamma_1 \bar{\omega}_1 [\beta_1 Y_1(x) + \beta_2 Y_2(x)] + \bar{\omega}_1^2 [\gamma_1 Y_1(x) + \gamma_2 Y_2(x)] \end{aligned}$$

# Contrails

$$\ddot{\theta}_x + 2\gamma\Omega\dot{\theta}_x + \Omega^2\theta_x = \frac{2\gamma\Omega}{b} \dot{\Delta}(x) + \frac{\Omega^2}{b} \Delta(x)$$

$$\begin{aligned} \ddot{\theta}_z + 2\gamma_1\bar{\omega}_1 \frac{\ell}{\rho_z} [(a_1^2\beta_1 + a_2^2\beta_2)\ell\dot{\theta}_z + (a_1\beta_2 - a_2\beta_1)\dot{Y}] + \bar{\omega}_1^2 \frac{\ell}{\rho_z} \\ [(a_1^2\gamma_1 + a_2^2\gamma_2)\ell\theta_z + (a_1\gamma_1 - a_1\gamma_2)Y] = 2\gamma_1\bar{\omega}_1 \frac{\ell}{\rho_z} \\ [a_1\beta_1\dot{Y}_1(x) - a_2\beta_2\dot{Y}_2(x)] + \bar{\omega}_1^2 \frac{\ell}{\rho_z} [a_1\gamma_1Y_1(x) - a_2\gamma_2Y_2(x)] \end{aligned}$$

where

$$Y_1(x) = \frac{1}{2} [Y_0(x_1, b) + Y_0(x_1, -b)]$$

$$Y_2(x) = Y_0(x_2, 0)$$

$$\Delta(x) = \frac{1}{2} [Y_0(x_1, b) - Y_0(x_1, -b)]$$

Suppose

$$Y_1(x) = \sum_{k=-\infty}^{\infty} a_{1k} \exp(i\pi \frac{x_1}{\ell_k})$$

$$Y_2(x) = \sum_{k=-\infty}^{\infty} a_{2k} \exp(i\pi \frac{x_2}{\ell_k})$$

$$\Delta(x) = \sum_{k=-\infty}^{\infty} b_k \exp(i\pi \frac{x_1}{\Lambda_k})$$

where  $a_{1k}$ ,  $a_{2k}$  and  $b_k$  are complex random coefficients and  $\ell_k$  and  $\Lambda_k$  are real random coefficients such that

$$a_{10} = a_{20} = b_0 = 0$$

$$a_{1-k} = \overline{a_{1k}}$$

$$a_{2-k} = \overline{a_{2k}}$$

$$b_{-k} = \overline{b_k}$$

$$\ell_{-k} = -\ell_k$$

$$\Lambda_{-k} = -\Lambda_k$$

# Contrails

Let

$$x_1 = V_0 t$$

$$x_2 = V_0 t - l$$

$$\omega_k = \pi V_0 / l_k$$

$$\Omega_k = \pi V_0 / \Lambda_k$$

Then

$$Y_1(x) = \sum_{k=-\infty}^{\infty} a_{1k} \exp(i\omega_k t)$$

$$Y_2(x) = \sum_{k=-\infty}^{\infty} a_{2k} \exp(i(\omega_k t - \pi \frac{l}{l_k}))$$

$$\Delta(x) = \sum_{k=-\infty}^{\infty} b_k \exp(i\Omega_k t).$$

Let

$$\gamma_2 = \frac{a_1^2 \beta_1 + a_2^2 \beta_2}{\sqrt{a_1^2 \gamma_1 + a_2^2 \gamma_2}} \gamma_1$$

$$\bar{\omega}_2 = \sqrt{a_1^2 \gamma_1 + a_2^2 \gamma_2} \bar{\omega}_1$$

$$R_{1k} = \gamma_1 + 2i\beta_1 \gamma_1 \frac{\omega_k}{\bar{\omega}_1}$$

$$R_{2k} = \gamma_2 + 2i\beta_2 \gamma_1 \frac{\omega_k}{\bar{\omega}_1}$$

$$S_k = 1 + 2i\gamma \frac{\Omega_k}{\bar{\omega}_1}.$$

The equations of motion then become

$$\ddot{Y} + 2\gamma_1 \bar{\omega}_1 \dot{Y} + \bar{\omega}_1^2 Y + 2(a_1 \beta_1 - a_2 \beta_2) \gamma_1 \bar{\omega}_1 l \dot{\theta}_z + (a_1 \gamma_1 - a_2 \gamma_2) \bar{\omega}_1^2 l \theta_z = G + \bar{\omega}_1^2 \sum_{k=-\infty}^{\infty} [a_{1k} R_{1k} + a_{2k} R_{2k} \exp(i\pi l/l_k)] \exp(i\omega_k t)$$

$$\begin{aligned}
 & 2(a_1\beta_1 - a_2\beta_2) \gamma_1 \bar{\omega}_1 \dot{Y} + (a_1\gamma_1 - a_2\gamma_2) \bar{\omega}_1^2 Y + \frac{\rho_z^2}{l^2} \ddot{\theta}_z \\
 & + 2\gamma_2 \bar{\omega}_2 l \dot{\theta}_z + \bar{\omega}_2^2 l \theta_z = \bar{\omega}_1^2 \sum_{k=-\infty}^{\infty} [a_1 a_{1k} R_{1k} - a_2 a_{2k} R_{2k} \\
 & \exp(i\pi l/l_k)] \exp(i\omega_k t) b_{\theta_x} + 2\gamma \Omega b_{\theta_x} + \Omega^2 b_{\theta_x} \\
 & = \Omega^2 \sum_{k=-\infty}^{\infty} b_k S_k \exp(i\Omega_k t).
 \end{aligned}$$

The solutions of these equations are

$$\begin{aligned}
 Y &= \frac{G}{\bar{\omega}_1^2} + \sum_{j=1}^4 A_j \exp(\lambda_j \bar{\omega}_1 t) + \sum_{k=-\infty}^{\infty} B_k \exp(i\omega_k t) \\
 l\theta_z &= \sum_{j=1}^4 \mu_j A_j \exp(\lambda_j \bar{\omega}_1 t) + \sum_{k=-\infty}^{\infty} C_k \exp(i\omega_k t) \\
 b_{\theta_x} &= E_1 \exp(\eta_1 t) + E_2 \exp(\eta_2 t) + \sum_{k=-\infty}^{\infty} D_k \exp(i\Omega_k t)
 \end{aligned}$$

Where  $\lambda_j$  ( $j = 1, 2, 3, 4$ ) is a root of

$$\begin{aligned}
 & \frac{\rho_z^2}{l^2} \lambda_j^4 + 2[a_1^2 \beta_1 + a_2^2 \beta_2 + \frac{\rho_z^2}{l^2}] \gamma_1 \lambda_j^3 + [a_1^2 \gamma_1 + a_2^2 \gamma_2 \\
 & + 4\beta_1 \beta_2 \gamma_1^2 + \frac{\rho_z^2}{l^2}] \lambda_j^2 + 2[\beta_1 \gamma_2 + \beta_2 \gamma_1] \gamma_1 \lambda_j + \gamma_1 \gamma_2 = 0,
 \end{aligned}$$

$\mu_j$  is given by

$$\begin{aligned}
 \mu_j &= - \frac{a_1 - \gamma_2 + 2(a_1 - \beta_2) \gamma_1 \lambda_j}{1 + 2\gamma_1 \lambda_j + \lambda_j^2} \\
 &= - \frac{a_1^2 \gamma_1 + a_2^2 \gamma_2 + 2(a_1^2 \beta_1 + a_2^2 \beta_2) \gamma_1 \lambda_j + \frac{\rho_z^2}{l^2} \lambda_j}{a_1 - \gamma_2 + 2(a_1 - \beta_2) \gamma_1 \lambda_j}
 \end{aligned}$$

and  $\eta_1$  and  $\eta_2$  are given by

$$\begin{aligned}
 \eta_1 &= (-\gamma + i\sqrt{1-\gamma^2})\Omega \\
 \eta_2 &= (-\gamma - i\sqrt{1-\gamma^2})\Omega
 \end{aligned}$$

# Contrails

The constants  $A_1, A_2, A_3, A_4, E_1,$  and  $E_2$  can be evaluated from the initial conditions: the  $B_k, C_k$  and  $D_k$  are evaluated below. In passing we note that the transient solutions to our equations are

$$Y = \frac{G}{\omega_1^2} + \sum_{j=1}^4 A_j \exp(\lambda_j \bar{\omega}_1 t)$$

$$l\theta_z = \sum_{j=1}^4 \mu_j A_j \exp(\lambda_j \bar{\omega}_1 t)$$

$$b\theta_x = E_1 \exp(\eta_1 t) + E_2 \exp(\eta_2 t)$$

The steady state portions of the solutions are given by

$$Y = \sum_{k=-\infty}^{\infty} B_k \exp(i\omega_k t)$$

$$l\theta_z = \sum_{k=-\infty}^{\infty} C_k \exp(i\omega_k t)$$

$$b\theta_x = \sum_{k=-\infty}^{\infty} D_k \exp(i\Omega_k t)$$

The solutions for the  $B_k, C_k$  and  $D_k$  are

$$B_k = \frac{[a_{1k} R_{1k} + a_{2k} R_{2k} \exp(-i\pi \frac{l}{l_k})] [\omega_2/\omega_1]^2 [1 + 2\gamma_2 \omega_k/\omega_2 - \frac{\rho_z^2}{2} (\omega_k/\omega_2)^2]}{F}$$

$$C_k = \frac{[a_1 a_{1k} R_{1k} - a_2 a_{2k} R_{2k} \exp(-i\pi \frac{l}{l_k})] [a_1 R_{1k} - a_2 R_{2k}]}{F}$$

$$D_k = \frac{[a_1 a_{1k} R_{1k} - a_2 a_{2k} R_{2k} \exp(-i\pi \frac{l}{l_k})] [a_1 R_{1k} - a_2 R_{2k}]}{F}$$

$$D_k = \frac{[a_{1k} R_{1k} + a_{2k} R_{2k} \exp(-i\pi \frac{l}{l_k})] [1 + 2i\gamma_1 \omega_k/\omega_1 - (\omega_k/\omega_1)^2]}{F}$$

where

$$F = (\bar{\omega}_2/\bar{\omega}_1) \left[ 1 + 2i\gamma_2 \omega_k/\bar{\omega}_2 - \frac{\rho_z}{\ell^2} (\omega_k/\bar{\omega}_2)^2 \right] \left[ 1 + 2i\gamma_1 \omega_k/\bar{\omega}_1 - (\omega_k/\bar{\omega}_1)^2 \right] - [a_1 R_{1k} - a_2 R_{2k}]^2$$

and

$$D_k = \frac{b_k S_k}{S_k - (\Omega_k/\Omega)^2}$$

The complex frequency responses (ratios of input to output motion

as a function of frequency),  $H_{1y}$  [response  $y$  due to input  $\sum_{k=-\infty}^{\infty} a_{1k} R_{1k} \exp(i\omega_k t)$ ];  $H_{2y}$  [response  $Y$  due to input  $\sum_{k=-\infty}^{\infty} a_{2k} R_{2k} \exp(i\pi \frac{\ell}{\ell_k}) \exp(i\omega_k t)$ ],  $H_{1z}$  [response  $\theta_z$  due to input  $\sum_{k=-\infty}^{\infty} a_{1k} R_{1k} \exp(i\omega_k t)$ ],  $H_{2z}$  [response  $\theta_z$  due to input  $\sum_{k=-\infty}^{\infty} a_{2k} R_{2k} \exp(i\pi \frac{\ell}{\ell_k}) \exp(i\omega_k t)$ ], and  $H_x$  [response  $\theta_x$  due to input  $\sum_{k=-\infty}^{\infty} b_k S_k \exp(i\Omega_k t)$ ]; are given by

$$H_{1y}(\omega_k) = [a_2 R_{2k} - \frac{\rho_z}{\ell^2} u_k^2] / W$$

$$H_{2y}(\omega_k) = [a_1 R_{1k} - \frac{\rho_z}{\ell^2} u_k^2] / W$$

$$H_{1z}(\omega_k) = -[(1 - a_1^2) R_{1k} + (1 + a_1 a_2) R_{2k} - u_k^2] / W a_1 \ell$$

$$H_{2z}(\omega_k) = [(1 + a_1 a_2) R_{1k} + (1 - a_2^2) R_{2k} - u_k^2] / W a_2 \ell$$

$$H_x(\Omega_k) = \frac{1}{b(S_k - V_k^2)}$$

where

$$W = R_{1k} R_{2k} - \left[ \left( a_1^2 + \frac{\rho_z}{\ell^2} \right) R_{1k} + \left( a_2^2 + \frac{\rho_z}{\ell^2} \right) R_{2k} \right] u_k^2 + \frac{\rho_z}{\ell^2} u_k^4$$

$$u_k = \omega_k / \bar{\omega}_1$$

$$V_k = \Omega_k / \Omega$$

# Contrails

Let  $S_{1f}$  be the power spectrum of the excitation  $\sum_{k=-\infty}^{\infty} a_{1k} R_{1k} \exp(i\omega_k t)$ ,  $S_{2f}$  be the power spectrum of the excitation  $\sum_{k=-\infty}^{\infty} a_{2k} R_{2k} \exp(i\pi \frac{\ell}{\ell_k}) \exp(i\omega_k t)$ ,  $a_{2k} R_{2k} \exp(i\omega_k t)$ , and  $S_{3f}$  be the power spectrum of the excitation  $\sum_{k=-\infty}^{\infty} b_k S_k \exp(i\Omega_k t)$ , then

if there is no cross-correlation between the different excitations the power spectra of the outputs are

$$\begin{aligned} S_y(\omega_k) &= |H_{1y}(\omega_k)|^2 S_{1f}(\omega_k) + |H_{2y}(\omega_k)|^2 S_{2f}(\omega_k) \\ S_z(\omega_k) &= |H_{1z}(\omega_k)|^2 a_1^2 S_{1f}(\omega_k) + |H_{2z}(\omega_k)|^2 a_2^2 S_{2f}(\omega_k) \\ S_x(\Omega_k) &= |H_x(\Omega_k)|^2 S_{3f}(\Omega_k), \end{aligned}$$

where

$$|H(\Omega_k)|^2 = [H(\Omega_k)] [\overline{H(\Omega_k)}]$$

Let

$$\begin{aligned} a_{1k} &= |a_{1k}| \exp(i\phi_{1k}) \\ a_{2k} &= |a_{2k}| \exp(i\phi_{2k}) \\ \phi_k &= \phi_{2k} - \phi_{1k} \end{aligned}$$

Then

$$\begin{aligned} |H_{1y}(\omega_k)|^2 &= [M_{1k}^2 + N_{1k}^2] / [F_k^2 + G_k^2] [Y_1^2 + 4\beta_1^2 \gamma_1^2 u_k^2] \\ |H_{2y}(\omega_k)|^2 &= [M_{2k}^2 + N_{2k}^2] / [F_k^2 + G_k^2] [Y_2^2 + 4\beta_2^2 \gamma_1^2 u_k^2] \\ |H_{1z}(\omega_k)|^2 &= [P_{1k}^2 + Q_{1k}^2] / [F_k^2 + G_k^2] [Y_1^2 + 4\beta_1^2 \gamma_1^2 u_k^2] \ell^2 \\ |H_{2z}(\omega_k)|^2 &= [P_{2k}^2 + Q_{2k}^2] / [F_k^2 + G_k^2] [Y_2^2 + 4\beta_2^2 \gamma_1^2 u_k^2] \ell^2 \\ |H_x(\Omega_k)|^2 &= \frac{1}{b^2 [(1 - V_k^2)^2 + 4\gamma V_k^2]} \end{aligned}$$



# Contrails

Where

$$M_{1k} = a_2 \gamma_1 \gamma_2 - \left[ \gamma_1 \frac{\rho_z^2}{l^2} + 4 a_2 \beta_1 \beta_2 \gamma_1^2 \right] u_k^2$$

$$M_{2k} = a_1 \gamma_1 \gamma_2 - \left[ \gamma_2 \frac{\rho_z^2}{l^2} + 4 a_1 \beta_1 \beta_2 \gamma_2^2 \right] u_k^2$$

$$N_{1k} = 2 \left[ a_2 (\beta_1 \gamma_2 + \beta_2 \gamma_1) - \beta_1 \frac{\rho_z^2}{l^2} u_k^2 \right] \gamma_1 u_k$$

$$N_{2k} = 2 \left[ a_1 (\beta_1 \gamma_2 + \beta_2 \gamma_1) - \beta_2 \frac{\rho_z^2}{l^2} u_k^2 \right] \gamma_1 u_k$$

$$P_{1k} = \gamma_1 (1 + a_1 a_2 - a_1 \gamma_1) - \left[ \gamma_1 + 4 \beta_1 (1 + a_1 a_2 - a_1 \beta_1) \gamma_1^2 \right] u_k^2$$

$$P_{2k} = \gamma_2 (1 + a_1 a_2 - a_2 \gamma_2) - \left[ \gamma_2 + 4 \beta_2 (1 + a_1 a_2 - a_2 \beta_2) \gamma_2^2 \right] u_k^2$$

$$Q_{1k} = 2 \left[ (1 + a_1 a_2) (\beta_1 + \gamma_1) - 2 a_1 \beta_1 \gamma_1 - \beta_1 u_k^2 \right] \gamma_1 u_k$$

$$Q_{2k} = 2 \left[ (1 + a_1 a_2) (\beta_2 + \gamma_2) - 2 a_2 \beta_2 \gamma_2 - \beta_2 u_k^2 \right] \gamma_1 u_k$$

$$F_k = \gamma_1 \gamma_2 - \left[ \left( a_1^2 + \frac{\rho_z^2}{l^2} \right) \gamma_1 + \left( a_2^2 + \frac{\rho_z^2}{l^2} \right) \gamma_2 + 4 \beta_1 \beta_2 \gamma_1^2 \right] u_k^2 + \frac{\rho_z^2}{l^2} u_k^4$$

$$G_k = 2 \left\{ \beta_1 \gamma_2 + \beta_2 \gamma_1 - \left[ \left( a_1^2 + \frac{\rho_z^2}{l^2} \right) \beta_1 + \left( a_2^2 + \frac{\rho_z^2}{l^2} \right) \beta_2 \right] u_k^2 \right\} \gamma_1 u_k$$

## SYMBOLS

$A_1, A_2, A_3, A_4$	ft	Constants of Integration	$Y_o(z)$	ft	Surface Elevation
$a_1, a_2, b$	ft	Dimensions of Vehicle	$Y_1(x) f_2(x)$	ft	Input Motion
$z_{1k}, z_{2k}, b_k$	ft	Input Motion Coefficients	$\gamma, \gamma_1, \gamma_2$	no units	System Constants
$B_k, C_k, D_k$	ft	Amplitude of Forced Vibration	$a_1, a_2, \beta_1, \beta_2, \gamma_1, \gamma_2$	no units	System Constants
$C_1, C_2$	lb-sec/ft	Damping Constants	$\Delta(X)$	ft	Input Motion
$E_1, E_2$	ft	Constants of Integration	$\eta_1, \eta_2$	rad/sec	System Constants
$F, F_k, G_k$	no units	System Constant	$\theta_x, \theta_z$	rad	Angular Orientation of Vehicle
$H_x, H_y, H_{zy}, H_z, H_{2z}$	no units	Complex Frequency Response	$\Lambda_k$	ft	Input Motion Wave Length
$I_x$	slug-ft <sup>2</sup>	Mass Moment of Inertia about x	$\lambda_j, \mu_j$	no units	System Constants
$I_z$	slug-ft <sup>2</sup>	Mass Moment of Inertia about z	$\rho_z^2$	ft	System Constant
$K_1, K_2$	lb/ft	Spring Constants	$\phi_k, \phi_{1k}, \phi_{2k}$	rad	Angle
$l$	ft	Wheel Base of Vehicle	$\Omega$	rad/sec	System Constant
$l_k$	ft	Input Motion Wave Length	$\bar{\omega}_1$	rad/sec	System Constant
$M$	slugs	Mass of Vehicle	$\Omega_k, \omega_k$	rad/sec	Input Motion Frequency
$M_{1k}, M_{2k}, N_{1k}, N_{2k}$	no units	System Constants	.		refers to differentiation with respect to time
$P_{1k}, Q_{1k}, Q_{2k}$	no units	System Constants	-		signifies complex conjugate
$R_{1k}, R_{2k}, S_k$	dimensionless	System Constants			
$S_{1f}, S_{2f}, S_{3f}, S_x, S_y, S_z$	ft <sup>2</sup> /(rad/sec)	Power Spectrum of Input			
$t$	sec	Time			
$U_k, V_k$	no units	Frequency Ratio			
$V_o$	fps	Velocity of Vehicle			
$W$	no units	System Constant			
$X, Y, Z$	ft	Coordinate Axis System			