On the Concept of Concentrated Loads and an Extension of the Uniqueness Theorem in the Linear Theory of Elasticity

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E. Sternberg and R. A. Eubanks

A Technical Report to the

Office of Naval Research
Department of the Navy
Washington 25, D. C.

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Project No. NR 035-302

Department of Mechanics
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1. Introduction

The traditional formulation of the second boundary-value problem of the linear theory of elasticity in the presence of concentrated surface loads, rests on the following properties required of the solution to such a problem: (a) it must satisfy the field equations of the theory throughout the region occupied by the medium;[1] (b) it must conform to the boundary conditions for distributed surface tractions;[2] (c) it must be regular[3] with the exception of singularities at the points of application of concentrated loads such that the resultant of the tractions on any surface surrounding a given load-point, and lying wholly in the body, tends to the corresponding prescribed concentrated load in the limit as the surface is contracted toward the load-point.

Since the classical uniqueness theorem does not hold in the presence of singularities of the type under consideration, there is no assurance

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[1]Internal concentrated loads are excluded for the time being.

[2]If, in particular, the loading consists of concentrated forces only, the solution must clear the boundary from tractions.

[3]The precise nature of these regularity requirements, which is ordinarily not specified, will be considered later in detail.

that the foregoing formulation uniquely characterizes the solution to a concentrated-load problem. That this is not merely an idle concern has been shown previously. In [1][4] were exhibited an infinite aggregate of distinct "solutions" corresponding to the half-space and the sphere under normal concentrated loads, each of which possesses the three properties cited. There is a priori no reason to give preference to any one member of this aggregate, and the question arises as to what precisely is meant by <u>the</u> solution of a problem involving concentrated surface loads. It is hardly feasible or desirable to base this decision on experimental evidence in each individual instance; nor can the question be dismissed by a reference to the fictitious nature of concentrated loads: the point is that the fiction is convenient, provided it is made meaningful.

In order to supply an answer to the question just raised, one may uniquely define the solution to a problem involving concentrated loads as the limit of a sequence of solutions, corresponding to distributed loadings, which are covered by the classical uniqueness theorem. Speaking loosely, for the time being, a unique characterization of the solution to such a problem is reached by considering the modified problem in which each of the concentrated loads is replaced with an arbitrary distribution of surface tractions over finite surface elements (load regions) surrounding the points of application of the concentrated forces. The solution to the original concentrated-load problem is then defined as the limit of the solution to the modified problem, as the surface elements are shrunk to the load points while the resultants of the distributed "replacement loadings" are made to approach the prescribed concentrated loads.

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[4]Numbers in brackets refer to the bibliography at the end of this paper.

This limit process is spelled out precisely in Section 7, where we prove that the limit-solution so defined exists and is independent of the choice of the load regions as well as of the mode of distribution of the replacement loadings, provided these tractions on each load region are sufficiently smooth, parallel, and of the same sense.[5]

The preceding limit-definition, which is analogous to Kelvin's definition through a limit process of the solution associated with a concentrated force applied at an internal point of a medium occupying the entire space,[6] is natural on both theoretical and physical grounds. Although Boussinesq [3] based his solution for the half-space under concentrated loads on the traditional formulation of the problem, his results are in accord with the definition adopted here, as is readily verified with the aid of the appropriate limit process applied, say, to Cerutti's solution[7] for the half-space subjected to distributed tractions. The corresponding limit process for the sphere under radial concentrated loads, was carried out in [1]. The usefulness of the limit-definition ultimately depends on, and is confirmed by, experimental evidence such as that supplied by Frocht and Guernsey [4] in connection with the problem of the sphere under diametrically opposed concentrated loads.

Intuitively, one would expect the limit-solution defined earlier to possess Properties (a), (b), and (c). That this is indeed the case is proved in Section 7, where an additional property of the limit-solution is established: (d) the order of the stress-singularities at each point of application of a concentrated load is r^{-2}, where r is the distance from

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[5]Actually, a considerably weaker, but physically less transparent, restriction is found to be sufficient.

[6]See, for example, [2], art. 130.

[7]See [2], art. 166.

the load point. Also Condition (d) is intuitively plausible — at least if the boundary has a continuously turning tangent plane in a neighborhood of each load-point; similarity considerations lead one to expect that the singularity at such a point has the order of the Boussinesq singularity, induced by a concentrated load applied to a plane boundary.

We shall refer to "solutions" of concentrated-force problems which meet Conditions (a), (b), (c), but fail to agree with the limit-definition, as "pseudo-solutions".[8] The solution corresponding to a heavy sphere on a point-support, published in [5], was identified as a pseudo-solution in [1]. In the same paper other pseudo-solutions were constructed and their physical significance was examined. All of the pseudo-solutions discussed in [1] violate Requirement (d). It is, therefore, natural to enquire whether there exist pseudo-solutions which also satisfy Condition (d). According to a generalized uniqueness theorem, proved in Section 8, this is not possible, and the four properties cited represent a unique characterization of the limit-solution.

The significance of this extension of the classical uniqueness theorem to concentrated loads, which is the main result of the present paper, may be described as follows:

(1) The theorem yields an alternative unique formulation of concentrated-load problems in terms of Conditions (a), (b), (c), (d) which is equivalent to, but far more convenient than, the limit-definition from which it derives its physical motivation. In specific applications, the theorem obviates

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[8]The existence of pseudo-solutions stems from the existence of solutions of the field equations, which are regular except for self-equilibrated singularities at the boundary, and which clear the entire boundary from tractions.

the necessity for performing a limit process which is apt to be cumbersome,[9] if not prohibitive.[10]

(2) In contrast to the limit-definition, the alternative definition through Requirements (a), (b), (c), (d), permits a study of the detailed structure of the singularities encountered in concentrated-force problems. These singularities require separate treatment if one is to arrive at practically useful representations of the solution to such problems. Indeed, in order to assure results which are amenable to a complete numerical evaluation, it is essential to determine the relevant singularities in closed form, at least to the extent where the residual problem is governed by finite and continuous surface tractions. In [6] we employ Conditions (a), (b), (c), (d) to investigate the nature of the singularity at the point of application of a concentrated load acting perpendicular to a curved boundary. The boundary, in a neighborhood of the load point, is assumed to be representable by a sufficiently smooth arbitrary surface of revolution whose axis coincides with the load-axis. We show there that the singularity is, in general, not identical with the known singularity appropriate to a load applied normal to a plane boundary; furthermore, we determine the supplementary singularities needed to effect a reduction of the problem to one obeying the foregoing regularity requirements.

(3) The uniqueness theorem of Section 8, and hence the alternative formulation of concentrated force problems to which it gives rise, applies to the general anisotropic medium in the presence of a positive definite

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[9] Thus, the exceedingly tedious limit computations performed in [1], turn out to be superfluous.

[10] Particular difficulties arise in the event a concentrated load is applied at a corner of the boundary (e.g., a load applied at the vertex of a cone).

elastic potential. On the other hand, the uniqueness of the limit-definition is established in Section 7 only for the isotropic medium.[11]

The portion of the paper preceding Section 7 is, in a sense, preliminary; though partly expository in character, it still contains results which are hoped to be new as well as, perhaps, a more rigorous and systematic development of certain known results.

Following a review in Section 2 of some geometric concepts needed throughout the remainder of the paper, we re-examine in Section 3 the precise regularity limitations inherent in the classical reciprocal and uniqueness theorems. In this connection we introduce the notion of "regular states", which proves to be useful and economical in the subsequent analysis. Section 4 is devoted to Kelvin's definition through a limit process of internal concentrated loads, and should supply some conceptual clarification of this subject. In particular, we construct here a counter-example to show that Kelvin's limit process does not yield a unique definition of internal concentrated loads in the absence of a restrictive requirement which appears to have gone unnoticed.

The brief unified treatment in Section 5 of higher internal singularities (e.g., force-doublets and centers of rotation) permits some remarks which are intended to be clarifying; at the same time, this section is preparatory to the proof in Section 6 of the Lauricella-Volterra theorems concerning the representation of the solution to the second boundary-value problem in terms of the given surface tractions. The present reconsideration of these theorems might be justified on two grounds. First, a statement of

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[11]An extension of this proof to anisotropic media would require the generalization for the anisotropic stress-strain law of Kelvin's solution to the problem presented by a concentrated force at a point of a medium occupying the entire space.

either theorem, sufficiently precise for our needs, is apparently not available elsewhere; second, the proof given by Lauricella [7] for the second theorem is different and probably less direct, while a satisfactory proof of the first theorem in the formulation employed here seems to be lacking.[12] The two theorems under consideration form the basis of the limit-treatment of concentrated surface loads, given in Section 7.

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[12] The sketch of a proof appearing in [2] is not safe from objections, as we shall have occasion to point out.

2. Geometric Preliminaries

For convenient future reference, we summarize at this place certain geometric notions which are needed repeatedly in the following developments. Most of these concepts are used in the sense of Kellogg, and the corresponding definitions are quoted from [8].

A regular arc is a point set which, for some orientation of a cartesian coordinate system (x_1, x_2, x_3), admits the representation,

$$x_2 = f(x_1), \quad x_3 = g(x_1), \quad a \leq x_1 \leq b, \tag{2.1}$$

where $f(x_1)$ and $g(x_1)$ are continuously differentiable in the interval (a,b). A regular curve is a point set consisting of a finite number of regular arcs arranged in order, and such that the terminal point of each arc (other than the last) is the initial point of the following arc. The arcs have no other points in common, except that the terminal point of the last arc may coincide with the initial point of the first, in which case the curve is a closed regular curve.

A region (of space or of a surface) is a connected, not necessarily closed,[13] point set (in space or on a surface). A regular region of the plane is a bounded closed region whose boundary is a closed regular curve. A regular surface element is a point set which, for some orientation of the coordinate system (x_1, x_2, x_3), admits the representation,

$$x_3 = f(x_1, x_2), \quad (x_1, x_2) \text{ in } R, \tag{2.2}$$

where R is a regular region of the (x_1, x_2)-plane and $f(x_1, x_2)$ is

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[13] Whenever this is essential, the distinction between open and closed regions will be made explicit.

continuously differentiable in R. It follows[14] that the boundary of a regular surface element is a regular curve.

A regular surface is a point set consisting of a finite number of regular surface elements, related as follows:

(a) two of the regular surface elements may have in common either a single point, which is a vertex for both, or a single regular arc, which is an edge for both, but no other points;

(b) three or more of the regular surface elements may have at most vertices in common;

(c) any two of the regular surface elements are the first and the last of a chain, such that each has an edge in common with the next; and

(d) all the regular surface elements having a vertex in common form a chain such that each has an edge, terminating in that vertex, in common with the next; the last may, or may not, have an edge in common with the first.

The term edge here refers to one of the finite number of regular arcs of which the boundary of a regular surface element is composed, while a vertex is a point at which two edges meet. If all the edges of a regular surface belong each to two of its surface elements, the surface is a closed regular surface (otherwise it is open).

By a regular region of space we shall mean[15] a closed (not necessarily bounded) region whose boundary consists at most of a finite number of non-intersecting closed regular surfaces. Throughout what follows D + B will designate a regular region of space with the boundary B (D being the

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[14]See [8], p. 106.

[15]This definition is somewhat more general than that used by Kellogg [8], p. 113; it is more convenient for our purposes.

open region). We observe that the boundary of a regular region of space cannot extend to infinity; if D + B is not bounded, D contains all sufficiently distant points.[16] Clearly, B may have a uniquely defined tangent plane along its edges and at its vertices. On the other hand, any arc or point of B for which this is not true, is necessarily an edge or a vertex of B; in order to avoid ambiguity, we shall refer to such arcs and points as <u>singular edges</u> and <u>corners</u> of B, respectively. Any point of B at which the tangent plane exists will be called a <u>regular point</u> of B. By a <u>regular subregion</u> of B we shall mean one which contains only regular points.

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[16] Thus, the half-space bounded by a plane or the region bounded by a hyperboloid, are not regular regions of space.

3. Regular States. Limitations of the Classical Reciprocal and Uniqueness Theorems

With a view toward examining the precise circumstances under which the theorems of Betti and Kirchhoff hold, it is expedient to introduce the notion of "regular states" in the sense of the following definitions.

Definition 3.1: Let[17] $u_i(P)$, $e_{ij}(P)$, and $\tau_{ij}(P)$ be a field of displacement, strain, and stress defined for $P(x_1,x_2,x_3)$ in $D + B$. Then the ordered array of functions of position $\left[u_1,u_2,u_3; e_{11},e_{12},\ldots; \tau_{11},\tau_{12},\ldots\right]$ is said to define a state $S(P)$ in $D + B$.

The state S may be regarded as a vector with 15 components in a function space of states and is a generalization of the concept of stress-state introduced by Prager and Synge [9] for different purposes. Equality, addition, multiplication by a scalar, continuity, and differentiability of states, are defined as in ordinary vector analysis. Thus, if S and S' are states with the components u_i, e_{ij}, τ_{ij} and u_i', e_{ij}', τ_{ij}', respectively, while k is a scalar constant, $S'' = kS + S'$ is the state with the components $ku_i + u_i'$, $ke_{ij} + e_{ij}'$, $k\tau_{ij} + \tau_{ij}'$.

Definition 3.2: $S(P)$ is a regular state in $D + B$, corresponding to the body-force field $F_i(P)$, if

(a) S is continuous in $D + B$, u_i is continuously differentiable in $D + B$, and e_{ij}, τ_{ij} are piecewise continuously differentiable in $D + B$;

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[17] Throughout this paper Latin suffixes, unless otherwise specified, assume the values 1, 2, 3, and the usual summation convention for repeated suffixes is employed. The coordinates x_i are rectangular cartesians, and differentiation with respect to a coordinate is indicated by a comma.

(b) S in D satisfies the equilibrium equations

$$\tau_{ij,j} = F_i, \tag{3.1}$$

the stress-strain relations

$$e_{ij} = c_{ijmn}\tau_{mn}, \tag{3.2}$$

and the strain-displacement relations

$$u_{i,j} + u_{j,i} = 2e_{ij}; \tag{3.3}$$

(c) in case D is not bounded, $u_i = O(r^{-1})$, $\tau_{ij} = O(r^{-2})$, and $\tau_{ij,j} = O(r^{-3})$ as $r \to \infty$, where r is the distance from the origin.

It is assumed that the elastic constants in (3.2) satisfy the symmetry relations

$$c_{ijmn} = c_{jimn} = c_{ijnm} = c_{mnij}, \tag{3.4}$$

and are such that the strain-energy density

$$W = \frac{1}{2}\, c_{ijmn}\tau_{ij}\tau_{mn} \tag{3.5}$$

is positive-definite. If, in particular, the medium is isotropic, the inverted form of (3.2) becomes

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}, \tag{3.6}$$

where λ and μ are Lamé's constant and the shear modulus, respectively, while δ_{ij} denotes the Kronecker delta. In this case S will be referred to as an isotropic regular state.

If Σ is an oriented regular surface lying in D + B (in particular Σ may be a subregion of the boundary B), $n_i(Q)$ is the outer unit-normal of $\sum$ at a point Q, and S is a state regular in D + B, then the resultant surface traction T_i of S on $\sum$ at Q, is given by

$$T_i(Q) = \tau_{ij}(Q)n_j(Q). \tag{3.7}$$

We note that T_i is defined only at regular points of $\sum$.

The proof of the reciprocal and uniqueness theorems in the linear theory of elasticity rests on the divergence theorem;[18] the validity of these theorems is thus restricted by the limitations underlying the divergence theorem which we cite[19] at this place.

<u>Theorem 3.1</u>: <u>Let</u> $\bar{v}$ <u>be a vector field</u>[20] <u>continuous and piecewise continuously differentiable in</u> D + B. <u>If</u> D <u>is not bounded, let</u> $v_i = o(r^{-2})$ <u>as</u> $r \to \infty$. <u>Then,</u>

$$\int_B \bar{v} \cdot \bar{n} \, d\sigma = \int_D \nabla \cdot \bar{v} \, d\tau, \tag{3.8}$$

<u>where</u> $\bar{n}$ <u>is the outer unit-normal of</u> B.

The foregoing statement of the divergence theorem represents the strongest valid form which is relevant to our purposes. The theorem still holds if $\bar{v}$ is continuously differentiable merely in the interiors of a finite number of regular regions of which D + B is the sum, provided the volume integral in (3.8) is convergent. This generalization, however, does

[18] In the proof of the uniqueness theorem for the Dirichlet problem, the use of the divergence theorem may be avoided and a stronger theorem is obtained with the aid of the maximum principle appropriate to harmonic functions (see [8], Exercise 2, p. 224). The analogous maximum principle does not hold in elasticity theory.

[19] See [8], pp. 118, 217. The extension of Kellogg's proof to regions which are regular in our sense, is trivial.

[20] Letters carrying bars denote vectors. The symbols "•" and "x" designate scalar and vector multiplication of two vectors, respectively. Unless otherwise specified, the scalar components of a vector $\bar{v}$ are v_i. ∇ is the usual del-operator.

not result in a physically significant strengthening of the theorems of Betti and of Kirchhoff. An examination of the proofs of these theorems in the light of Theorem 3.1, with the aid of Definition 3.2, yields the following statements.

Theorem 3.2: Let S and S' be (not necessarily isotropic) regular states in $D + B$, corresponding to the body force fields $\bar{F}$ and $\bar{F}'$, respectively. Then,

$$\left.\begin{aligned} \int_B \bar{T} \cdot \bar{u}' \, d\sigma + \int_D \bar{F} \cdot \bar{u}' \, d\tau &= \int_B \bar{T}' \cdot \bar{u} \, d\sigma + \int_D \bar{F}' \cdot \bar{u} \, d\tau \\ &= \int_D \tau_{ij} e'_{ij} d\tau = \int_D \tau'_{ij} e_{ij} d\tau. \end{aligned}\right\} \quad (3.9)$$

If, in Theorem 3.2, in particular, we take $S = S'$, we reach the energy formula

$$\int_B \bar{T} \cdot \bar{u} \, d\sigma + \int_D \bar{F} \cdot \bar{u} \, d\tau = 2\int_D W \, d\tau, \quad (3.10)$$

where W is the strain-energy density given in (3.5). Equation (3.10) forms the basis for the proof of the subsequent uniqueness theorem.

Theorem 3.3: Let S' and S'' be (not necessarily isotropic) regular states in $D + B$, corresponding to the same body-force field. Let B_u, B_t be subregions of B such that $B_u + B_t = B$, $\bar{u}' = \bar{u}''$ on B_u, and $\bar{T}' = \bar{T}''$ at all regular points of B_t. Then $\tau'_{ij} = \tau''_{ij}$ in $D + B$.

The proofs of Theorems 3.2 and 3.3 available in the literature[21] rest on the assumption of a bounded region and are limited to the isotropic stress-strain relations (3.6). The adaptation of these proofs to the more general hypotheses employed here is, however, immediate. Equally trivial

[21] See, for example, [2], p. 173 and p. 170.

is the extension of Theorem 3.3 to mixed-mixed boundary-value problems (e.g., normal component of the displacement vector and tangential component of the surface traction prescribed). Furthermore, the divergence theorem, and hence the two last theorems cited, remain valid if B extends to infinity (D + B is then no longer regular),[22] provided B is sufficiently smooth.

We emphasize that the regularity requirements at infinity stipulated in Part (c) of Definition 3.2, though sufficient, are by no means necessary for the truth of the uniqueness theorem. Indeed, these conditions are artificial in character since the prescription of a definite rate of vanishing of displacements and stresses at infinity cannot, in general, be justified on physical grounds. The extent to which the conditions at infinity can be weakened, and thus made physically plausible, is still in need of investigation.[23] In case B extends to infinity, the mere requirement that the stresses vanish at infinity is evidently not sufficient for uniqueness. This is apparent from a paper by Neuber [11], containing a non-vanishing solution of the field equations for vanishing body forces, which is regular in a region bounded by a hyperboloid of revolution, clears the entire boundary from tractions, and possesses vanishing stresses at infinity.

We speak of a "unique formulation" of a particular boundary-value problem if, assuming the existence of a solution to the problem, the solution is unique. On the basis of Theorem 3.3, and with reference to the notation used in the statement of this theorem, the subsequent

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[22]See Footnote No. 16.

[23]Tiffen [10] considered the analogous issue with regard to the two-dimensional treatment of the plane problem. The authors are indebted to Dr. B. Budiansky of the NACA for calling their attention to an example which contradicts the theorem stated in the Summary of Tiffen's paper.

formulation of the mixed problem in the linear theory of elasticity, is unique:

Given $\bar{F}_*(P)$ for P in D, $\bar{u}_*(Q)$ for Q on B_u, $\bar{T}_*(Q)$ for Q on B_t, as well as the elastic constants c_{ijmn}, find a state S(P) which is regular for P in D + B, corresponding to $\bar{F} = \bar{F}_*$, such that $\bar{u} = \bar{u}_*$ on B_u and $\bar{T} = \bar{T}_*$ at all regular points of B_t.

It should be noted that the preceding statement of the problem rules out non-vanishing tractions at infinity; this case, however, is reducible to the case of vanishing tractions at infinity by means of the principle of superposition.

Theorem 3.4: The following conditions are necessary for the existence of a solution to the mixed boundary-value problem in the foregoing formulation:

(a) $\bar{u}_*(Q)$ must be continuous on B_u and continuously differentiable in any closed regular subregion of B_u;

(b) $\bar{T}_*(Q)$ must be continuous and piecewise continuously differentiable in any closed regular subregion of B_t;

(c) $\bar{F}_*(P)$ must be piecewise continuous in D + B and, if D is not bounded, $\bar{F}_* = O(r^{-3})$ as $r \rightarrow \infty$;

(d) If D is bounded and $B = B_t$, then $\bar{F}_*$ and $\bar{T}_*$ must satisfy the equilibrium conditions

$$\int_D \bar{F}_* \, d\tau + \int_B \bar{T}_* \, d\sigma = 0, \quad \int_D \bar{r} \times \bar{F}_* \, d\tau + \int_B \bar{r} \times \bar{T}_* \, d\sigma = 0.$$

Theorem 3.4 is a direct consequence of Definition 3.2. The list of necessary conditions given in this theorem could easily be augmented. Indeed, any boundary conditions which cannot be assumed by a state S which is regular in D + B (e.g., violate the symmetry of the stress tensor), are inadmissible. The determination of a set of conditions sufficient for the existence of a solution to the problem under consideration is beyond the scope of the present paper, which is primarily concerned with the uniqueness of solutions whose existence is postulated. Suffice it to say that the available existence theorems assume a degree of smoothness of the boundary which is not necessarily possessed by the boundary of a regular region of space.

Our main objective in stating Theorem 3.4 is to draw attention to the fact that the class of boundary conditions covered by the classical uniqueness theorem is far more limited than appears to be generally recognized. In particular, not only concentrated loads, but even most instances of discontinuous distributed loadings are outside the domain of validity of the traditional uniqueness theorem. It is not difficult to demonstrate the incompleteness of the customary formulation of problems involving such distributed loadings, which says nothing regarding the regularity of the solution at the boundary and in no way specifies the nature of the singularities there encountered. To illustrate this observation, we refer to the well known plane-strain solution for the half-plane under a uniformly distributed shear load applied to a finite segment of the boundary.[24] Superposition upon this solution of the traditional solution for a concentrated tangential load,[25] applied at an endpoint of the load segment, yields an entirely different stress distribution which nevertheless still conforms to the usual formulation of the original problem. Moreover, the new "solution", as well as the traditionally accepted one, exhibit infinite stresses at the endpoints of the load segment.

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[24] See, for example, [12], p. 129.

[25] See [12], p. 88.

Although a detailed study of the uniqueness of solutions to problems characterized by singular distributed loadings is beyond our present intentions, we shall briefly return to this subject at the end of the paper. We now turn to the concept of concentrated loads which is our main concern.

4. Singular States. Internal Concentrated Loads

Definition 4.1: $S(P)$ is a singular state in $D + B$ if it is not regular in the sense of Definition 3.2.

Definition 4.2: Let Q_α ($\alpha = 1, 2, \ldots N$) be a set of discrete points in $D + B$. A state $S(P)$ is regular in $D + B$ except for point singularities at Q_α if it is singular in $D + B$ but regular in every closed regular subregion of $D + B$ which does not contain the points Q_α.

If a Q_α lies in D, we shall speak of an internal point-singularity of S; if a Q_α lies on B, we shall refer to a surface point-singularity. The analogous definitions of states regular except for singularities along a surface lying in D or along an arc on B would enter naturally into the study of uniqueness questions in the presence of dislocations[26] and discontinuous distributed surface tractions.

For future convenience we recall at this place the Boussinesq-Papkovich solution of the field equations for the isotropic medium in terms of four scalar stress functions.[27]

Theorem 4.1: Let $\phi(P)$ and $\bar{V}(P)$ be a scalar and a vector field which are three times continuously differentiable for P in an arbitrary open region D. Let

$$2\mu\bar{u} = \nabla(\phi + \bar{r} \cdot \bar{V}) - 4(1 - \nu)\bar{V}, \qquad (4.1)$$

$$\nabla^2\phi = -\frac{\bar{r} \cdot \bar{F}}{2(1 - \nu)}, \quad \nabla^2\bar{V} = \frac{\bar{F}}{2(1 - \nu)} \qquad (4.2)$$

in D, where $\bar{r}$ is the position vector (x_1, x_2, x_3) of P, while μ

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[26] See [2], p. 225, second paragraph.

[27] See [3], pp. 63 and 72, as well as [13]. The solution was rediscovered later by Neuber [14]. The extension of this solution to the case of non-vanishing body forces considered here, is due to Mindlin [15].

and ν are the shear modulus and Poisson's ratio, respectively. Then the stresses associated with the displacement field $\bar{u}$ in the sense of the displacement-strain relations (3.3) and the isotropic stress-strain relations (3.6), satisfy the equilibrium equations (3.1) for the body-force field $\bar{F}(P)$.

The truth of the theorem is confirmed by direct substitution. An elegant proof of the completeness of the foregoing solution of the field equations was given by Mindlin [16], first for the case $\bar{F} = 0$, and was later extended by him to include body forces in [15].[28]

Our next objective is the definition through a limit process of the concept of internal concentrated loads. By virtue of the principle of superposition, it is sufficient here to consider the Kelvin problem presented by a concentrated load applied at a point of a medium occupying the entire space. A physically natural unique definition of the solution to this problem for an isotropic medium is supplied by the following theorem.

Theorem 4.2: Let D be the entire space and O be the origin. Let $D_n + B_n$ be a sequence of bounded regular regions of space such that each D_n contains O, and $d_n \to 0$ as $n \to \infty$, where d_n is the maximum diameter of $D_n + B_n$. Let $\bar{F}_n(P)$ be a sequence of body-force fields with the properties:

(a) $\bar{F}_n(P)$ is twice continously differentiable for P in D;

(b) $\bar{F}_n(P) = 0$ for P in D and not in $D_n + B_n$;

(c) $\int_{D_n} \bar{F}_n \, d\tau \to \bar{L}$ as $n \to \infty$;

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[28] The precise limitations regarding the nature of the region and the regularity requirements at infinity inherent in Mindlin's proof, are in need of clarification.

(d) $\int_{D_n} |\bar{F}_n| \, d\tau$ remains bounded as $n \to \infty$.

Then there exists a unique sequence of isotropic states $S_n(P)$, regular in D and corresponding to $\bar{F}_n(P)$. The sequence $S_n(P)$, for all $P \neq 0$, converges toward a limit state $S(P)$ which is independent of the particular choice of the sequences $D_n + B_n$ and $\bar{F}_n(P)$. Moreover, $S(P)$ is generated by the stress functions,

$$\phi(P) = 0, \qquad \bar{V}(P) = -\frac{\bar{L}}{8\pi(1-\nu)\, r}, \tag{4.3}$$

where r is the distance of P from the origin O.

Definition 4.3: The limit-state S of Theorem 4.2 ("Kelvin-state") is said to be the state corresponding to a concentrated force $\bar{L}$ applied at the origin to an isotropic medium occupying the entire space.

Proceeding to the proof of Theorem 4.2, we observe that the Newtonian potentials

$$\left.\begin{aligned} \phi_n(P) &= \alpha \int_{D_n} \frac{\bar{r}(Q)\cdot\bar{F}_n(Q)}{R(P,Q)}\, d\tau, \\ \bar{V}_n(P) &= -\alpha \int_{D_n} \frac{\bar{F}_n(Q)}{R(P,Q)}\, d\tau, \\ \text{where} \qquad \alpha &= 1/8\pi(1-\nu), \end{aligned}\right\} \tag{4.4}$$

$\bar{r}(Q)$ is the position vector of a point Q of D_n, and $R(P,Q)$ is the distance from Q to a point P of D, satisfy the Poisson equations (4.2) for $\bar{F} = \bar{F}_n$ throughout D. Furthermore, it follows from Hypotheses (a), (b), as well as from the properties of Newtonian potentials, that the sequence of states $S_n(P)$, generated by the stress functions ϕ_n and $\bar{V}_n$

in the sense of Theorem 4.1, is regular in D corresponding to the body-force field $\bar{F}_n$, in accordance with Definition 3.2. Thus $S_n(P)$ exists and, in view of Theorem 3.3, is unique.

In order to confirm that $S_n(P) \to S(P)$, it suffices to show that $\bar{V}_n(P) \to \bar{V}(P)$, and that the first and second partial derivatives of $\phi_n(P)$, $\bar{V}_n(P)$ tend to the corresponding derivatives of $\phi(P)$, $\bar{V}(P)$ for all $P \neq 0$. Since the argument in each instance is strictly analogous, we merely prove that $\bar{V}_n(P) \to \bar{V}(P)$. To this end, note from (4.3) and (4.4) that

$$\left.\begin{aligned} \left|\bar{V}_n(P) - \bar{V}(P)\right| = \alpha \Bigg| &\int_{D_n} \bar{F}_n(Q) \left[R^{-1}(P,Q) - r^{-1}(P)\right] d\tau \\ &+ r^{-1}(P) \left[\int_{D_n} \bar{F}_n(Q)\, d\tau - \bar{L}\right] \Bigg| , \end{aligned}\right\} \quad (4.5)$$

whence, holding $P \neq 0$ fixed, and for all n sufficiently large to insure that P is not in $D_n + B_n$,

$$\left.\begin{aligned} \left|\bar{V}_n(P) - \bar{V}(P)\right| \leq \alpha &\int_{D_n} \left|\bar{F}_n(Q)\right| \left|R^{-1}(P,Q) - r^{-1}(P)\right| d\tau \\ &+ \alpha r^{-1}(P) \left|\int_{D_n} \bar{F}_n(Q)\, d\tau - \bar{L}\right| . \end{aligned}\right\} \quad (4.6)$$

By Hypothesis (c), the second term in the right member of (4.6) tends to zero as $n \to \infty$; the first term, on the other hand, is bounded by

$$\alpha \max_{Q \text{ in } D_n + B_n} \left|R^{-1}(P,Q) - r^{-1}(P)\right| \int_{D_n} \left|\bar{F}_n(Q)\right| d\tau, \quad (4.7)$$

which approaches zero by Hypothesis (d) and since $d_n \to 0$. This completes the proof.

The Kelvin-state S, generated by the stress functions given in (4.3), was first presented by Kelvin without derivation in [17], and later deduced by a limit process in [18], p. 277, on the basis of a sequence of concentric spheres for D_n, and on the assumption of body forces which are constant within D_n. The present derivation, with the aid of the Boussinesq-Papkovich stress functions, is analogous to that employed by Mindlin [15] in connection with the problem of the half-space under a concentrated internal load.

Love's exposition [2], art. 130, of the Kelvin limit process, which no longer restricts the shape of D_n and the body-force distribution within D_n, is nevertheless open to minor objections and suffers from a certain conceptual vagueness. While the use of a sequence (or family) of regions D_n contracting toward 0, is clearly implied, it does not become fully evident that the argument involves an associated sequence (or family) of solutions of the field equations. Moreover, it is essential to make suitable smoothness requirements, such as our Hypothesis (a), regarding the body-force distributions $\bar{F}_n(P)$, if the uniqueness of the approximating states $S_n(P)$ is to be assured. Finally, lack of explicit detail in carrying out the limit process leads Love to overlook the need for a restriction on $\bar{F}_n(P)$ such as our Hypothesis (d).

Hypothesis (d) could easily be relaxed somewhat, although this would seem to serve no particular purpose; it is implied by Hypothesis (c) in the special case in which the body forces $\bar{F}_n(P)$ are parallel and of the same sense within D_n. We now show by means of a counter-example that Theorem 4.2 is false, and thus that the Kelvin limit-process does not yield a unique definition of internal concentrated loads, in the absence of Hypothesis (d).

Theorem 4.3: Theorem 4.2 is false if Hypothesis (d) is omitted.

To demonstrate this, it is sufficient to exhibit sequences D_n and $\bar{F}_n(P)$, conforming to all hypotheses of Theorem 4.2, except Hypothesis (d), such that $\lim_{n \to \infty} S_n(P) \neq S(P)$ for $P \neq 0$, where $S(P)$ is the Kelvin-state.

Let D_n be the sequence of concentric spheres

$$0 \leq r < 1/n \quad (n = 1, 2, \ldots), \tag{4.8}$$

and let $\bar{F}_n(P)$ be defined by

$$\left.\begin{aligned} \bar{F}_n(P) &= \frac{\bar{r}}{r} f_n(r) \quad \text{for} \quad 0 \leq r \leq 1/n, \\ \bar{F}_n(P) &= 0 \qquad\quad\;\; \text{for} \qquad r \geq 1/n, \end{aligned}\right\} \tag{4.9}$$

where,

$$f_n(r) = -\frac{336}{5}(1 - \nu)n^9(r - \frac{1}{n})^3(6r^2 + \frac{3}{n}r + \frac{1}{n^2}). \tag{4.10}$$

Direct computation confirms that

$$f_n'(0) = f_n''(0) = f_n(1/n) = f_n'(1/n) = f_n''(1/n) = 0, \tag{4.11}$$ [29]

whence $\bar{F}_n(P)$ meets the smoothness-hypothesis (a) of Theorem 4.2. In view of the polar symmetry of $\bar{F}_n$ about 0, clearly,

$$\int_{D_n} \bar{F}_n \, d\tau = 0 \qquad (n = 1, 2, \ldots), \tag{4.12}$$

so that $\bar{L} = 0$.

The stress-functions $\phi_n(P)$ and $\bar{V}_n(P)$, generating the sequence of states $S_n(P)$, which correspond to the body-force distributions $\bar{F}_n(P)$ and are regular throughout D, now again follow from (4.4), with $\bar{F}_n(P)$ defined

[29] The primes denote differentiation with respect to r.

as in (4.9), (4.10). Consider a fixed $P \neq 0$ and choose $n > r(P)$. Then $\phi_n(P)$, as well as the components of $\bar{V}_n(P)$, for every such fixed n, are the Newtonian potentials at a point of free-space of mass distributions over the sphere D_n, whose densities have polar symmetry about the center O. According to an elementary result in potential theory, the value of such a potential at P equals the value there of the potential associated with a single particle at O, whose mass is equal to the total mass of the distribution. Hence, and by virtue of (4.4), (4.9), we have

$$\left.\begin{aligned} \phi_n(P) &= \frac{\alpha}{r(P)} \int_{D_n} r\, f_n(r)\, d\tau, \\ \bar{V}_n(P) &= -\frac{\alpha}{r(P)} \int_{D_n} \bar{F}_n\, d\tau. \end{aligned}\right\} \qquad (4.13)$$

On the other hand, (4.10) is found to imply[30]

$$\alpha \int_{D_n} r\, f_n(r)\, d\tau = 1, \qquad (4.14)$$

and (4.13), (4.14), together with (4.12), yield

$$\phi_n(P) = 1/r, \quad \bar{V}_n(P) = 0 \qquad (n = 1, 2, \ldots). \qquad (4.15)$$

The sequence S_n, therefore, tends toward a limit state **S** which is generated by the stress-functions

$$\phi(P) = 1/r, \quad \bar{V}(P) = 0. \qquad (4.16)$$

According to (4.1) and (4.16), the displacement field belonging to S is given by

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[30] The polynomial (4.10) was actually constructed to meet conditions (4.11), (4.14).

$$\bar{\bar{u}}(P) = \nabla\, 1/r, \tag{4.17}$$

and is thus identified as appropriate to a center of compression[31] at O. If the conclusions of Theorem 4.2 were valid here, S_n should tend to the null-state, since $\bar{\bar{L}} = 0$. Indeed,

$$\int_{D_n} \left|\bar{\bar{F}}_n\right| d\tau \to \infty$$

for the sequence of body-force fields (4.9), (4.10), which thus violate Hypothesis (d) of Theorem 4.2. The proof of Theorem 4.3 is now complete and we turn to a discussion of certain properties of the Kelvin-state.

Theorem 4.4: The Kelvin-state S of Definition 4.3 has the properties:

(a) S is regular, corresponding to zero body forces, in the entire space D, except for a point-singularity at the origin O;

(b) Let Ω be any closed regular surface surrounding O and let $\bar{\bar{T}}$ be the resultant surface traction of S on that side of Ω which is oriented toward O. Then,

$$\int_{\Omega} \bar{\bar{T}}\, d\sigma = \bar{\bar{L}}; \tag{4.18}$$

(c) $\tau_{ij} = O(r^{-2})$ as $r \to 0$.

Property (a) follows from Theorem 4.1, since the stress functions (4.3) are harmonic in D except at O, and from the observation that u_i and τ_{ij} belonging to S, vanish at infinity as r^{-1} and r^{-2}, respectively. Properties (c) and (b) are established by inspection of, and an elementary computation based upon, the stresses of S. A trivial limit process confirms that (a) and (c) imply

[31] See [2], p. 187, and Definition 5.2 of this paper.

$$\text{(d)} \int_{\Omega} \bar{r} \times \bar{T} \, d\sigma = 0. \tag{4.19}$$

Thus, the tractions of S on Ω are statically equivalent to the single force $\bar{L}$ at O.

Kelvin's realization that the definition of internal concentrated loads necessitates a limit process, was apparently neglected in the subsequent treatise literature, with the exception of Love [2]. Thus, for example, in [12], art. 120, Kelvin's problem in formulated in terms of Properties (a), (b), and (d). To see that this formulation is not unique, consider the state

$$S' = S + cS^{\circ}, \tag{4.20}$$

where S is the Kelvin-state, S° is the state appropriate to a center of compression at O, whose displacement field is given by (4.17), and c is an arbitrary real constant. S' is readily found to possess Properties (a), (b), (d), and might be called a "pseudo-solution" of Kelvin's problem.

In [19], art. 32, and [20], art. 350, the problem of Kelvin is approached on the basis of Properties (a), (b), (c), and Property (c) is claimed to be a consequence of (b). This claim is not justified as is again apparent from S' which possesses Property (b) without conforming to (c). The question remains, however, whether Properties (a), (b), (c) of Theorem 4.4 uniquely characterize the Kelvin-state, and thus yield a legitimate formulation of Kelvin's problem. An affirmative answer to this question will be supplied by the uniqueness theorem proved in Section 7 of this paper.

5. Higher Internal Point-Singularities

Theorem 5.1: Let the stress-functions ϕ and $\bar{V}$ of Theorem 4.1 be harmonic in an arbitrary open region D and there generate a state S. Then the state $S_i \equiv S_{,i}$ is generated by

$$\phi_i = \phi_{,i} + V_i, \qquad \bar{V}_i = \bar{V}_{,i} \tag{5.1}$$

and in D satisfies the isotropic field equations for vanishing body forces.

This theorem follows at once from Theorem 4.1 and by inspection of Equations (3.1), (3.3), (3.6).

Through successive differentiations of the Kelvin-state S(P) with respect to the coordinates x_i of P, one obtains an infinite aggregate of states which are regular in the entire space D, except for progressively stronger point-singularities at O. In this section we deal briefly with those singular states which result from a single space-differentiation of S(P) and are needed later.[32]

Let $S_i(P,Q) \equiv S_i(x_1, x_2, x_3; \xi_1, \xi_2, \xi_3)$ be the Kelvin-state corresponding to a unit concentrated load applied at $Q(\xi_1, \xi_2, \xi_3)$ in the x_i-direction and, for brevity, write $S_i(P) \equiv S_i(x_1, x_2, x_3)$ in place of $S_i(P,O)$. We now define a set of nine states $S_{ij}(P)$ by means of

$$S_{ij}(P) = S_{i,j}(P). \tag{5.2}$$

The physical interpretation of the states S_{ij}, in terms of the Kelvin-states S_i, is apparent from the observation that

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[32] What follows is a unified treatment of material discussed in [2], art. 132.

$$\left.\begin{aligned} S_{ij}(P) &= \lim_{h\to 0} \frac{1}{h}\left[S_i(x_1,x_2,x_3) - S_i(x_1-\delta_{j1}h, x_2-\delta_{j2}h, x_3-\delta_{j3}h)\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[S_i(x_1,x_2,x_3) - S_i(x_1,x_2,x_3;\delta_{j1}h,\delta_{j2}h,\delta_{j3}h)\right], \end{aligned}\right\} \quad (5.3)$$

where δ_{ij} again designates the Kronecker delta.

<u>Definition 5.1: The state</u> $S_{ij}(P)$, <u>for</u> $i = j$, <u>is said to correspond to a force-doublet,</u>[33] <u>applied at</u> 0 <u>parallel to the</u> x_i<u>-axis. The state</u> S_{ij}, <u>for</u> $i \neq j$, <u>is said to correspond to a force-doublet with moment about the</u> x_k<u>-axis</u> $(k \neq i, j)$, <u>applied at</u> 0 <u>parallel to the</u> x_i<u>-axis.</u>[34]

We now record the Boussinesq-Papkovich stress functions, as well as the displacement fields, belonging to representative members of the set of states S_i and S_{ij}.

$$\left.\begin{aligned} &\text{For } S_1: \quad \phi_1 = 0, \quad \bar{\Psi}_1 = -\alpha\left[\frac{1}{r}, 0, 0\right], \\ &2\mu\bar{u}_1 = \alpha\left[\frac{3-4\nu}{r} + \frac{x_1^2}{r^3}, \frac{x_1x_2}{r^3}, \frac{x_1x_3}{r^3}\right], \\ &\alpha = 1/8\pi(1-\nu). \end{aligned}\right\} \quad (5.4)$$

$$\left.\begin{aligned} &\text{For } S_{11}: \quad \phi_{11} = -\frac{\alpha}{r}, \quad \bar{\Psi}_{11} = \alpha\left[\frac{x_1}{r^3}, 0, 0\right], \\ &2\mu\bar{u}_{11} = -\alpha\left[\frac{3x_1^3}{r^5} + \frac{(1-4\nu)x_1}{r^3}, \frac{3x_1^2x_2}{r^5} - \frac{x_2}{r^3}, \frac{3x_2^2x_3}{r^5} - \frac{x_3}{r^5}\right]. \end{aligned}\right\} \quad (5.5)$$

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[33] A "double force without moment", in the terminology of Love [2].

[34] Note that $S_{ij} \neq \pm S_{ji}$.

$$\left.\begin{aligned} \text{For } S_{12}: \quad & \phi_{12} = 0, \quad \bar{v}_{12} = \alpha\left[\frac{x_2}{r^3}, 0, 0\right], \\ & 2\mu\bar{u}_{12} = -\alpha\left[\frac{3x_1^2x_2}{r^5} + \frac{(3-4\nu)x_2}{r^3}, \frac{3x_1x_2^2}{r^5} - \frac{x_1}{r^3}, \frac{3x_1x_2x_3}{r^5}\right]. \end{aligned}\right\} \quad (5.6)$$

Equations (5.4), (5.5), (5.6) are obtained from (4.3) with the aid of Theorems 5.1, 4.1. The associated fields of stress follow from (3.3), (3.6), and may be omitted here. On the basis of the stress fields belonging to (5.5), (5.6), and with the aid of (5.2), (5.4), and Theorem 5.1, we arrive at the following theorem which is analogous to Theorem 4.4.

Theorem 5.2: The states S_{ij}, defined by (5.2), have the properties:

(a) S_{ij} is regular, corresponding to zero body forces, in the entire space D, except for a point-singularity at the origin 0.

(b) Let Ω be any closed regular surface surrounding 0 and let $\bar{T}_{ij}$ be the resultant surface traction of S_{ij} on that side of Ω which is oriented toward 0. Then,

$$\int_{\Omega} \bar{T}_{ij}\, d\sigma = 0. \quad (5.7)$$

For $i = j$,

$$\int_{\Omega} \bar{r} \times \bar{T}_{ij}\, d\sigma = 0, \quad (5.8)$$

while for $i \neq j$ and $k \neq i,j$,

$$\int_{\Omega} \bar{r} \times \bar{T}_{ij}\, d\sigma = \epsilon\, \bar{a}_k, \quad (5.9)$$

where $\bar{a}_k$ is a unit-vector in the x_k-direction, $\epsilon = 1$ if (i,j,k) is a cyclic permutation of $(1,2,3)$, and $\epsilon = -1$ otherwise.

(c) $\tau_{ijmn} = O(r^{-3})$ as $r \to 0$. (5.10)[35]

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[35]The last two suffixes refer to the components of the stress tensor belonging to S_{ij}.

In contrast to Properties (a), (b), (c) of Theorem 4.4, which will be shown in Section 8 to characterize the Kelvin-state uniquely, Properties (a), (b), (c) of Theorem 5.2 clearly do not supply a unique characterization of the states S_{ij}.

Theorems 4.4, 5.2 suggest a remark concerning Saint-Venant's principle. The stresses of the Kelvin-state, whose tractions on any surface Ω surrounding 0 are statically equivalent to a force, decay as r^{-2} at infinity. On the other hand, the stresses of a force-doublet state, whose tractions on Ω are self-equilibrated, decay as r^{-3}. This comparison has traditionally been cited[36] in support of Saint-Venant's principle as formulated by Boussinesq [3]. It should be observed, however, that the stresses appropriate to a force-doublet <u>with</u> moment, whose tractions on Ω are statically equivalent to a couple and thus are not self-equilibrated, also vanish as r^{-3} at infinity. Hence, the condition of self-equilibrance of the tractions on Ω does not yield a reduction in the order of vanishing of the stresses at infinity as compared to the case in which merely the resultant force on Ω is zero. This observation contradicts rather than supports Boussinesq's version of Saint-Venant's principle; it is consistent, however, with the modified version of the principle announced by von Mises [21] and proved in [22].

<u>Definition 5.2: The state</u> $S^{*} = S_{ii}$ <u>is said to correspond to a center of compression at</u> 0. <u>The state</u> $S^{k} = \frac{1}{2}(S_{ij} - S_{ji})$, <u>where</u> (i,j,k) <u>is a cyclic permutation of</u> $(1,2,3)$, <u>is said to correspond to a center of rotation at</u> 0 <u>parallel to the</u> x_k<u>-axis</u>.

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[36] See [3] and [2], art. 133.

The stress functions and displacements for S° and S^{k} follow from (5.5) and (5.6).

$$\left.\begin{aligned} &\text{For } S^{\circ}: \quad \phi^{\circ} = 2(1 - 2\nu)\frac{\alpha}{r}, \quad \bar{V}^{\circ} = 0 \\ &\mu \bar{u}^{\circ} = (1 - 2\nu)\,\alpha \nabla \frac{1}{r}. \end{aligned}\right\} \quad (5.11)^{37}$$

$$\left.\begin{aligned} &\text{For } S^{3}: \quad \phi^{3} = 0, \quad \bar{V}^{3} = \frac{\alpha}{2}\left[-\frac{\partial}{\partial x_2}\left(\frac{1}{r}\right), \frac{\partial}{\partial x_1}\left(\frac{1}{r}\right), 0\right] \\ &2\mu \bar{u}^{3} = \frac{1}{4\pi}\left[\frac{\partial}{\partial x_2}\left(\frac{1}{r}\right), -\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right), 0\right]. \end{aligned}\right\} \quad (5.12)$$

Theorem 5.2 now yields the properties,

$$\int_{\Omega} \bar{T}^{\circ}\, d\sigma = 0, \quad \int_{\Omega} \bar{r} \times \bar{T}^{\circ}\, d\sigma = 0, \quad \tau^{\circ}_{ij} = O(r^{-3}) \quad \text{as} \quad r \to 0, \qquad (5.13)$$

$$\int_{\Omega} \bar{T}^{k}\, d\sigma = 0, \quad \int_{\Omega} \bar{r} \times \bar{T}^{k}\, d\sigma = \bar{a}_k, \quad \tau^{k}_{ij} = O(r^{-3}) \quad \text{as} \quad r \to 0. \qquad (5.14)$$

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[37] Note that ϕ° and $\bar{V}^{\circ}$ are equivalent to ϕ_{ii} and $\bar{V}_{ii}$, as computed from (5.5), in the sense that they generate $\bar{u}^{\circ}$.

6. The Theorems of Lauricella and Volterra

We proceed to state, and indicate the proof of, two lemmas which are prerequisites for the proof of the two theorems to be considered in this section. These theorems, in turn, form the basis of the limit-definition of concentrated surface loads.

Lemma 6.1: *Let* $S(Q)$ *be a state regular in a neighborhood of a point* P. *Let* $S'(Q)$ *be regular in the same neighborhood, except for a singularity at* P. *Moreover, let*

$$u_i'(Q) = O(r^{-1}) \quad \text{and} \quad \tau_{ij}'(Q) = O(r^{-2}) \quad \text{as} \quad r \to 0, \tag{6.1}$$

where r *is the distance from* P *to* Q, *and let*

$$\lim_{\delta \to 0} \int_{\Sigma(\delta)} \overline{T}' \, d\sigma = \overline{L}, \tag{6.2}$$

$\Sigma(\delta)$ *being a sphere of radius* δ *centered at* P, *whose outer normal is directed toward* P. *Then,*

$$\lim_{\delta \to 0} \int_{\Sigma(\delta)} \overline{T} \cdot \overline{u}' \, d\sigma = 0, \quad \lim_{\delta \to 0} \int_{\Sigma(\delta)} \overline{T}' \cdot \overline{u} \, d\sigma = \overline{L} \cdot \overline{u}(P). \tag{6.3}$$

The first of (6.3) is immediate from the first of (6.1). To establish the second of (6.3), observe that

$$\left. \begin{aligned} \left| \int_{\Sigma(\delta)} \overline{T}' \cdot \overline{u} \, d\sigma - \overline{L} \cdot \overline{u}(P) \right| \leq & \left| \int_{\Sigma(\delta)} \overline{T}'(Q) \cdot \left[\overline{u}(Q) - \overline{u}(P) \right] d\sigma \right| \\ & + \left| \overline{u}(P) \cdot \left[\int_{\Sigma(\delta)} \overline{T}'(Q) \, d\sigma - \overline{L} \right] \right| . \end{aligned} \right\} \tag{6.4}$$

The second term of the right member of (6.4) tends to zero with δ by (6.2); the first term approaches zero as $\delta \to 0$ by virtue of the second of (6.1) and in view of the continuity of $\overline{u}(Q)$ at P.

Lemma 6.2: Let $S(Q)$ be a state regular in a neighborhood of a point P. Let $S^k(Q)$ be the state corresponding to a center of rotation[38] at P, parallel to the x_k-axis. Then,

$$\lim_{\delta \to 0} \int_{\sum(\delta)} \bar{T} \cdot \bar{u}^k \, d\sigma = 0, \quad \lim_{\delta \to 0} \int_{\sum(\delta)} \bar{T}^k \cdot \bar{u} \, d\sigma = \omega_k(P), \tag{6.5}$$

where $\sum(\delta)$ is defined as in Lemma 6.1 and $\bar{\omega} = \frac{1}{2} \nabla \times \bar{u}$ is the rotation vector belonging to $\bar{u}$.

Consider $k = 3$ as a typical case, take P as the origin, and let Q have the coordinates x_i. To prove the first of (6.5), we note that an elementary computation, based on (5.12) and (3.7), yields,

$$\left.\begin{aligned} \int_{\sum(\delta)} \bar{T} \cdot \bar{u}^3 \, d\sigma = \frac{1}{8\pi\mu\delta^4} \int_{\sum(\delta)} \Big[& \tau_{11}x_1x_2 + \tau_{12}(x_2^2 - x_1^2) \\ & + \tau_{13}x_2x_3 - \tau_{22}x_1x_2 - \tau_{23}x_1x_3 \Big] d\sigma. \end{aligned}\right\} \tag{6.6}$$

The right member of (6.6) may be written as a sum of integrals of the form

$$I(\delta) = \int_{\sum(\delta)} \tau(Q) \, f(Q) \, d\sigma, \tag{6.7}$$

where,

$$\left.\begin{aligned} & \int_{\sum(\delta)} f(Q) \, d\sigma = 0, \quad f(Q) = O(\delta^{-2}) \quad \text{as} \quad \delta \to 0, \\ & \tau(Q) = \tau(P) + g(Q), \quad g(Q) \to 0 \quad \text{as} \quad Q \to P, \end{aligned}\right\} \tag{6.8}$$

whence $I(\delta) \to 0$ as $\delta \to 0$.

We turn to the proof of the second of (6.5). With the aid of the stresses associated with (5.12), we find that

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[38]See Definition 5.2.

$$\int_{\Sigma(\delta)} \bar{T}^k \cdot \bar{u}\, d\sigma = \frac{3}{8\pi\delta^4} \int_{\Sigma(\delta)} (x_1 u_2 - x_2 u_1)\, d\sigma. \tag{6.9}$$

A Taylor expansion of $u_i(Q)$ about P gives,

$$\left. \begin{aligned} u_i(Q) &= u_i(P) + x_j \left[u_{i,j}(P) + g_{ij}(Q) \right], \\ \text{with} \quad g_{ij}(Q) &\to 0 \quad \text{as} \quad Q \to P. \end{aligned} \right\} \tag{6.10}$$

Substitution of (6.10) into (6.9) confirms the desired result after a short computation.

The tractions of the singular state S' in Lemma 6.1, on any surface Ω surrounding P and lying wholly in the neighborhood under consideration, are statically equivalent to a single force $\bar{L}$ applied at P. The tractions of S^k in Lemma 6.2, on any such Ω, are statically equivalent to a couple of moment $\bar{a}_k$, where $\bar{a}_k$ is a unit vector parallel to the x_k-axis. The work done in an "infinitesimal" displacement of a rigid body by a force system which is statically equivalent to a single force $\bar{L}$ at P together with a couple of moment $\bar{M}$, is given by

$$U = \bar{L} \cdot \bar{u}(P) + \bar{M} \cdot \bar{\omega}, \tag{6.11}$$

in which $\bar{u}(P)$ and $\bar{\omega}$ are the displacement vector of P and the rotation vector, respectively. One might suppose that the second of (6.3) and (6.5) follow trivially from (6.11); in fact, this is suggested by Love [2] on pp. 236 and 245. Such an intuitive argument, however, is not sound, as can be seen from the following observation. Let S be defined as in Lemma 6.1, and let $S^{\bullet}$ be the state corresponding to a center of compression at P, given in (5.11). Here,

$$\lim_{\delta \to 0} \int_{\Sigma(\delta)} \bar{T}^{\bullet} \cdot \bar{u} \, d\sigma = - \frac{2(1 - 2\nu)}{3(1 - \nu)} u_{i,i}(P), \qquad (6.12)$$

although, according to (5.13), the singularity of $S^{\bullet}$ at P is self-equilibrated.

We turn to the statement and proof of two theorems regarding an integral representation of the solution to the second boundary-value problem in terms of the given surface tractions. These theorems[39] were given in different form by Lauricella [23], and were attributed by him to V. Volterra.

Theorem 6.1: Let $S(Q)$ be an isotropic state, corresponding to the body force $\bar{F}(Q)$, which is regular in $D + B$ and such that $\bar{u}(P_o) = \bar{\omega}(P_o) = 0$, where P_o is a point of D. Let P be a point of D and $S_i'(Q,P,P_o)$ be a state characterized by the properties:

(a) $S_i'(Q,P,P_o) = S_i(Q,P) + S_o^i(Q,P,P_o) + S_i^*(Q,P,P_o)$, (6.13)

where $S_i(Q,P)$ is the Kelvin-state corresponding to a unit force at P in the x_i-direction,

$$\left.\begin{aligned} & S_o^i(Q,P,P_o) = - S_i(Q,P_o) + \sum_{j=1}^{3} \lambda_{ij}(P,P_o) \, S^j(Q,P_o), \\ & \text{with} \quad \lambda_{ij}(P,P_o) = \bar{R} \times \bar{a}_i \cdot \bar{a}_j, \end{aligned}\right\} \qquad (6.14)$$

$S^j(Q,P_o)$ is the state corresponding to a center of rotation at P_o parallel to the x_j-direction, $\bar{R}$ is the vector from P to P_o, and $\bar{a}_i$ is a unit vector parallel to the x_i-axis.

(b) $S_i^*(Q,P,P_o)$ corresponds to $\bar{F}(Q) = 0$ and is an isotropic regular state for Q in D + B;

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[39] See [2], art. 169, and [19], p. 122.

(c) $\bar{T}_i'(Q,P,P_o) \equiv \bar{T}_i(Q,P) + \bar{T}_o^i(Q,P,P_o) + \bar{T}_i^*(Q,P,P_o) = 0,$ (6.15)

<u>in which</u> $\bar{T}_i'$, $\bar{T}_i$, $\bar{T}_o^i$, $\bar{T}_i^*$ <u>are the surface tractions on</u> B <u>of</u> S_i', S_i, S_o^i, S_i^*, <u>respectively</u>;

(d) $\bar{u}_i^*(P_o,P,P_o) = \bar{\varpi}_i^*(P_o,P,P_o) = 0.$ (6.16)

Then,

$$u_i(P) = \int_B \bar{T}(Q) \cdot \bar{u}_i'(Q,P,P_o)\, d\sigma + \int_D \bar{F}(Q) \cdot \bar{u}_i'(Q,P,P_o)\, d\tau. \tag{6.17}$$

Observe that S_i^* is defined through (b) and (c) as the solution of a second boundary-value problem in D + B. While the existence of S_i^*, and hence of S_i', is postulated, the uniqueness of these states is assured by Theorem 3.3 and the fact that (d) precludes an arbitrary additive rigid displacement field. The singularities inherent in S_i and S_o^i, because of (6.14), (4.18), and (5.14), constitute a self-equilibrated system; thus, the boundary condition (c) for S_i^* conforms to Condition (d) of Theorem 3.4, which is necessary for the existence of S_i^* if D is bounded.

In the proof of the theorem, consider first the case in which $P = P_o$. Here, in view of (a), (b), S_i' is regular throughout D + B and, by (c), (d), is the null-state. Since $\bar{u}(P_o)$ is supposed to vanish, (6.17) clearly holds if $P = P_o$.

Now, let $P \neq P_o$, and let $\Sigma(\delta)$, $\Sigma_o(\delta)$ be spheres of radius δ, lying wholly in D and having no points in common. The region $\mathcal{D}$ bounded by B, $\Sigma(\delta)$, and $\Sigma_o(\delta)$, is then a regular region of space in which S(Q) and $S_i'(Q,P,P_o)$ are regular. An application of the reciprocal theorem, Theorem 3.2, to S and S_i in $\mathcal{D}$, in view of (c), leads to,

$$\left.\begin{aligned}\int_{\Sigma}\bar{T}_i'\cdot\bar{u}\,d\sigma+\int_{\Sigma_o}\bar{T}'\cdot\bar{u}\,d\sigma=\int_{B}\bar{T}\cdot\bar{u}_i'\,d\sigma+\int_{\Sigma}\bar{T}\cdot\bar{u}_i'\,d\sigma\\+\int_{\Sigma_o}\bar{T}\cdot\bar{u}'\,d\sigma+\int_{\mathcal{D}}\bar{F}\cdot\bar{u}'\,d\tau.\end{aligned}\right\}\quad(6.18)$$

Proceeding to the limit as $\delta\rightarrow 0$ in (6.18), we find by means of Lemmas 6.1, 6.2, Hypotheses (a), (b), and (4.18) that

$$\left.\begin{aligned}&\int_{\Sigma}\bar{T}_i'\cdot\bar{u}\;d\sigma\rightarrow u_i(P),\\&\int_{\Sigma_o}\bar{T}_i'\cdot\bar{u}\;d\sigma\rightarrow -\,u_i(P)+\bar{R}\times\bar{a}_i\cdot\bar{\omega}(P_o)=0,\\&\int_{\Sigma}\bar{T}\cdot\bar{u}_i'\,d\sigma\rightarrow 0,\qquad\int_{\Sigma_o}\bar{T}\cdot\bar{u}_i'\,d\sigma\rightarrow 0.\end{aligned}\right\}\quad(6.19)$$

Equation (6.17) now follows from (6.18) and (6.19). This completes the proof. The next theorem aims at an integral representation for the strains of a regular state in terms of the associated surface tractions, which is analogous to the representation (6.17) for the displacement components.

Theorem 6.2: Let $S(Q)$ be an isotropic state, corresponding to the body forces $\bar{F}(Q)$, which is regular in $D+B$. Let P be a point of D and $S_{ij}'(Q,P)$ be a state characterized by the properties:

(a) $$S_{ij}'(Q,P)=\frac{-1}{2}\Big[S_{ij}(Q,P)+S_{ji}(Q,P)\Big]+S_{ij}^{*}(Q,P),\qquad(6.20)$$

where $S_{ij}(Q,P)\equiv -\,S_{i,j}(Q,P)$ is the state corresponding to a force doublet[40] with or without moment, applied at P;

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[40] See Definition 5.2. Recall that all differentiations are with respect to the coordinates x_i of P.

(b) $S^*_{ij}(Q,P)$ <u>corresponds to</u> $\bar{F}(Q) = 0$ <u>and is an isotropic regular state for</u> Q <u>in</u> $D + B$;

(c) $$\bar{T}'_{ij}(Q,P) \equiv -\frac{1}{2}\left[\bar{T}_{ij}(Q,P) + \bar{T}_{ji}(Q,P)\right] + \bar{T}^*_{ij}(Q,P) = 0, \qquad (6.21)$$

<u>in which</u> $\bar{T}'_{ij}$, $\bar{T}_{ij}$, $\bar{T}^*_{ij}$ <u>are the surface tractions on</u> B <u>of</u> S'_{ij}, S_{ij}, S^*_{ij}, <u>respectively</u>. <u>Then</u>,

$$e_{ij}(P) = \int_B \bar{T}(Q) \cdot \bar{u}'_{ij}(Q,P)\, d\sigma + \int_D \bar{F}(Q) \cdot \bar{u}'_{ij}(Q,P)\, d\tau. \qquad (6.22)$$

<u>Moreover</u>,

$$S'_{ij}(Q,P) = \frac{1}{2}\left[S'_{i,j}(Q,P,P_o) + S'_{j,i}(Q,P,P_o)\right], \qquad (6.23)$$

<u>where</u> $S'_i(Q,P,P_o)$ <u>is defined as in Theorem</u> 6.1.

Note that S^*_{ij} is uniquely defined[41] as the solution of a second boundary-value problem in $D + B$. By virtue of (5.7), (5.8), (5.9), the singularity of $S_{ij} + S_{ji}$ at P is self-equilibrated; hence, the boundary condition (c) for S^*_{ij} satisfies Requirement (d) of Theorem 3.4, which is necessary for the existence of S^*_{ij} if D is bounded.

We shall deduce the present theorem from Theorem 6.1. To this end, define a state $S''_{ij}(Q,P,P_o)$ through

$$S''_{ij}(Q,P,P_o) = \frac{1}{2}(S'_{i,j} + S'_{j,i}), \qquad (6.24)$$

where $S'_i(Q,P,P_o)$ is defined in Theorem 6.1. If we compute the strains $e_{ij}(P)$ belonging to $S(Q)$ of Theorem 6.1 from the displacements (6.17), with the aid of the strain-displacement relations (3.3), we obtain[42] (6.22)

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[41]Within an arbitrary additive rigid displacement field.

[42]It is not difficult to show that a single differentiation with respect to the coordinates x_i of P of the improper volume integral in (6.17), may be performed under the integral sign. The proof of this statement is strictly similar to the proof of Theorem II on p. 152 of [8].

with $u'_{ij}(Q,P)$ replaced by $u''_{ij}(Q,P,P_o)$. It remains to be shown that $S''_{ij}(Q,P,P_o)$ coincides with the state $S'_{ij}(Q,P)$ of Theorem 6.2.

Let

$$\left.\begin{aligned} S_o^{ij}(Q,P,P_o) &\equiv \frac{1}{2}\left[S^i_{o,j} + S^j_{o,i}\right], \\ S_i^{**}(Q,P,P_o) &\equiv \frac{1}{2}\left[S^*_{i,j} + S^*_{j,i}\right], \end{aligned}\right\} \qquad (6.25)$$

$S_o^i(Q,P,P_o)$ and $S_i^*(Q,P,P_o)$ being the states defined in Theorem 6.1. Since $S_{ij}(Q,P) = - S_{i,j}(Q,P)$, it follows from (6.24), (6.25), and (6.13) that

$$\left.\begin{aligned} S''_{ij}(Q,P,P_o) = &- \frac{1}{2}\left[S_{ij}(Q,P) + S_{ji}(Q,P)\right] \\ &+ S_o^{ij}(Q,P,P_o) + S_i^{**}(Q,P,P_o). \end{aligned}\right\} \qquad (6.26)$$

Comparing (6.26) with (6.20), we note that $S'_{ij}(Q,P) = S''_{ij}(Q,P,P_o)$ follows if we show that $S_o^{ij}(Q,P,P_o)$ is the null-state and that $S_i^*(Q,P) = S_i^{**}(Q,P,P_o)$. Indeed, an elementary computation, based on (6.14), yields,

$$\bar{u}^i_{o,j} = (\bar{a}_j \times \bar{a}_k \cdot \bar{a}_i)\, \bar{u}^k(Q,P). \qquad (6.27)$$

whence $\bar{u}^i_{o,j} + \bar{u}^j_{o,i} = 0$ and thus, according to (6.25), S_o^{ij} is the null-state.

With a view toward verifying that $S_i^*(Q,P) \equiv S_i^{**}(Q,P,P_o)$, observe that both states are regular in D + B and, by Theorem 3.3, must be identical[43] if their surface tractions on B coincide. A direct computation, involving (3.7), (6.15), and (6.25), confirms that $\bar{T}^*(Q,P) = \bar{T}^{**}(Q,P,P_o)$ on B, which completes the proof of the theorem.

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[43]See Footnote No. 41.

Lauricella [7] establishes (6.22) directly and then merely states the usual line-integral representation for the displacement field of $S(Q)$ in terms of the components of strain. Love [2], art. 169, and Trefftz [19], p. 124, present a rough sketch of the proof of Theorem 6.1. Both these authors specify merely the stress-resultants of the singularity of $S_i'(Q,P,P_o)$ at P_o; in view of our observations following Theorem 5.2, this is insufficient for a unique characterization of S_i'.

The formation of $e_{ii}(P)$ from (6.22), at once leads to Betti's formula for the dilatation in terms of the surface tractions.[44] The singular states S_i' and S_{ij}' of Theorems 6.1 and 6.2 play a role which is analogous to that played by Green's function of the first kind in connection with the Dirichlet problem. Formulas (6.17) and (6.22) reduce the solution of the second boundary-value problem in a given region $D + B$ to the determination of the complementary states S_i^* and S_{ij}^* appropriate to $D + B$.

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[44] See [2], p. 234.

7. Limit-Definition of Concentrated Surface Loads

We turn now to the definition through a limit process of the solution to problems involving concentrated surface loads and, without loss in generality, confine ourselves to the case in which the body forces are absent.

Theorem 7.1: Let $D + B$ be a regular region of space and let $\{Q_\alpha\}$ ($\alpha = 1, 2, \ldots N$) be N distinct points on B. Let $\Lambda_\alpha^{(n)}$ ($n = 1, 2, \ldots$) be N sequences of closed subregions of B ("load regions") such that $\Lambda_\alpha^{(n)}$, for all n and $\alpha = 1, 2, \ldots N$, is a regular surface containing Q_α in its interior, and $\delta_\alpha^{(n)} \to 0$ as $n \to \infty$, where $\delta_\alpha^{(n)}$ is the maximum diameter of $\Lambda_\alpha^{(n)}$.

Let $\{\bar{L}_\alpha\}$ ($\alpha = 1, 2, \ldots N$) be a set of vectors ("concentrated loads") and $\bar{f}^{(n)}(Q)$ be a sequence of functions ("replacement loadings") defined for Q on B and such that

(1) $\bar{f}^{(n)}(Q) = 0$ for Q not in $\Lambda_\alpha^{(n)}$ ($\alpha = 1, 2, \ldots N$);

(2) $\displaystyle\int_{\Lambda_\alpha^{(n)}} \bar{f}^{(n)}\, d\sigma \to \bar{L}_\alpha$ as $n \to \infty$;

(3) $\displaystyle\int_{\Lambda_\alpha^{(n)}} \left| \bar{f}^{(n)} \right| d\sigma$ is bounded as $n \to \infty$.

Let $S^{(n)}(P)$ be a sequence of states with the properties:

(a) $S^{(n)}(P)$ is regular and isotropic in $D + B$ for $\bar{F} = 0$;

(b) $\bar{T}^{(n)}(Q) = \bar{T}_*(Q) + \bar{f}^{(n)}(Q)$ for Q on B, with $\bar{T}_*(Q)$ continuous on B;

(c) $\bar{u}^{(n)}(P_o) = \bar{\omega}^{(n)}(P_o) = 0$, where P_o is a point of D.

Then $S^{(n)}(P)$, together with its first space-derivatives, as $n \to \infty$, is uniformly convergent in every closed subregion of $D + B$ which does not contain any point Q_α. The limit state $S(P) = \lim_{n \to \infty} S^{(n)}(P)$ is independent of the particular choice of the sequences $\Lambda_\alpha^{(n)}$ and $\bar{f}^{(n)}$. Moreover, for P in D, the limit state S admits the representation:

$$u_i(P) = \int_B \bar{T}_*(Q) \cdot \bar{u}_i'(Q,P,P_o)\, d\sigma + \sum_{\alpha=1}^{N} \bar{L}_\alpha \circ \bar{u}_i'(Q_\alpha,P,P_o), \tag{7.1}$$

$$e_{ij}(P) = \int_B \bar{T}_*(Q) \circ \bar{u}_{ij}'(Q,P)\, d\sigma + \sum_{\alpha=1}^{N} \bar{L}_\alpha \circ \bar{u}_{ij}'(Q_\alpha,P), \tag{7.2}$$

$$\tau_{ij}(P) = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}, \tag{7.3}$$

where $\bar{u}_i'(Q,P,P_o)$ and $\bar{u}_{ij}'(Q,P)$ are defined as in Theorems 6.1 and 6.2, respectively.

Here $\bar{T}_*(Q)$ are the distributed surface tractions prescribed in the concentrated-load problem under consideration. We proceed to the proof of the theorem. Applying Theorem 6.1 to the regular states $S^{(n)}(P)$, we have,

$$u_i^{(n)}(P) = \int_B \bar{T}_*(Q) \cdot \bar{u}_i'(Q,P,P_o)\, d\sigma + \sum_{\alpha=1}^{N} \int_{\Lambda_\alpha^{(n)}} \bar{f}^{(n)}(Q) \cdot \bar{u}_i'(Q,P,P_o)\, d\sigma \tag{7.4}$$

for all P in D. Since $u_i^{(n)}(P)$ is continuous in $D + B$,

$$\lim_{P \to Q_o} u_i^{(n)}(P) = u_i^{(n)}(Q_o), \tag{7.5}$$

where Q_o is a point of B. We assume henceforth that Q_o is not in $\{Q_\alpha\}$ ($\alpha = 1, 2, \ldots N$). Then there exists $M > 0$ such that $n > M$ implies that Q_o is not in $\Lambda_\alpha^{(n)}$. For such a choice of n, the integrand of the second integral in (7.4) is a continuous function of P at $P = Q_o$

and

$$\left.\begin{aligned} &\lim_{P \to Q_o} \sum_{\alpha=1}^{N} \int_{\Lambda_\alpha^{(n)}} \bar{f}^{(n)}(Q) \cdot \bar{u}_i'(Q,P,P_o)\, d\sigma \\ &\qquad = \sum_{\alpha=1}^{N} \int_{\Lambda_\alpha^{(n)}} \bar{f}^{(n)}(Q) \cdot \bar{u}_i'(Q,Q_o,P_o)\, d\sigma. \end{aligned}\right\} \quad (7.6)$$

Equations (7.4), (7.5), (7.6) assure that

$$\lim_{P \to Q_o} \int_B \bar{T}_*(Q) \cdot \bar{u}_i'(Q,P,P_o)\, d\sigma \qquad (7.7)$$

exists.

By (7.1) and (7.4),

$$\left.\begin{aligned} u_i^{(n)}(P) - u_i(P) = \sum_{\alpha=1}^{N} \Bigg\{ &\int_{\Lambda_\alpha^{(n)}} \bar{f}^{(n)}(Q) \cdot \bar{u}_i'(Q,P,P_o)\, d\sigma \\ &- \bar{L}_\alpha \cdot \bar{u}_i'(Q_\alpha,P,P_o) \Bigg\}, \end{aligned}\right\} \quad (7.8)$$

in which $u_i(P)$ is given by (7.1), and provided $n > M$. Furthermore, according to (7.1), (7.4), (7.5), (7.6), (7.7), $\lim_{P \to Q_o} u_i(P) \equiv u_i(Q_o)$ exists, and (7.8) remains valid for all P in $D + B$, but not in $\{Q_\alpha\}$ ($\alpha = 1, 2, \ldots N$).

Thus, for P in any closed subregion E of $D + B$ which does not contain any of the points Q_α, and for $n > M$, we may write,

$$\left.\begin{aligned} \left| u_i^{(n)}(P) - u_i(P) \right| \leq \sum_{\alpha=1}^{N} \Bigg\{ &\int_{\Lambda_\alpha^{(n)}} \left|\bar{f}_\alpha^{(n)}(Q)\right| \left| \bar{u}_i'(Q,P,P_o) - \bar{u}_i'(Q_\alpha,P,P_o) \right| d\sigma \\ &+ \left| \bar{u}_i'(Q_\alpha,P,P_o) \cdot \left[\int_{\Lambda_\alpha^{(n)}} \bar{f}^{(n)}(Q)\, d\sigma - \bar{L}_\alpha \right] \right| \Bigg\}. \end{aligned}\right\} \quad (7.9)$$

The second term in (7.9) tends to zero uniformly for P in E as $n \to \infty$ since $\bar{u}_i'(Q_\alpha,P,P_o)$ is bounded for P in E, and in view of Hypothesis (2).

Moreover, clearly, the integral in the first term of (7.9), for all P in E, is bounded by

$$\max_{Q \text{ on } \Lambda_\alpha^{(n)}} \left| \bar{u}_i'(Q,P,P_o) - \bar{u}_i'(Q_\alpha,P,P_o) \right| \int_{\Lambda_\alpha^{(n)}} \left| \bar{F}_\alpha^{(n)}(Q) \right| d\sigma, \tag{7.10}$$

which uniformly tends to zero by Hypothesis (3) and because $\bar{u}_i'(Q,P,P_o)$ is continuous with respect to Q for Q in $\Lambda_\alpha^{(n)}$ and P in E if $n > M$, while $\delta_\alpha^{(n)} \to 0$ as $n \to \infty$.

We have shown that $u_i^{(n)}(P)$ uniformly converges toward $u_i(P)$ of (7.1), for P in E. To complete the proof of the theorem, we have yet to establish the uniform convergence of the strains $e_{ij}^{(n)}(P)$ toward $e_{ij}(P)$ of (7.2), as well as the uniform convergence in E of the first space-derivatives of $u_i^{(n)}(P)$ and $e_{ij}^{(n)}(P)$; this, however, is accomplished by strictly analogous means.

Theorem 7.1 is the counterpart, in connection with concentrated surface-loads, of Theorem 4.2, which supplies the limit definition of internal concentrated loads. It should be noted that while the existence of the sequence of approximating states $S_n(P)$ in Theorem 4.2 was demonstrated, the existence of the approximating states $S^{(n)}$ in Theorem 7.1 is postulated. Hypothesis (3) of the present theorem is analogous to Hypothesis (d) of Theorem 4.2; it could easily be weakened and is implied by Hypothesis (2) in the event that the replacement loading on each load region constitutes a system of tractions which are parallel and of the same sense. Theorem 7.1 is readily extended to the limit definition of the solution to a mixed boundary-value problem involving concentrated surface loads.

<u>Definition 7.1</u>: <u>The limit state</u> S(P) <u>of Theorem 7.1 is said to constitute the solution of the second boundary-value problem for the region</u> D + B,

characterized by the surface tractions $\bar{T}_*(Q)$, the concentrated loads $\bar{L}_\alpha$ applied at the points Q_α, and by vanishing body forces.

Theorem 7.2: The limit state $S(P)$ of Theorem 7.1 has the properties:

(a) $S(P)$ is a regular isotropic state, corresponding to vanishing body forces, in $D + B$, except for point-singularities at every point Q_α for which $\bar{L}_\alpha \neq 0$.

(b) $\bar{T}(Q) = \bar{T}_*(Q)$ at all regular points of B which are not in $\{Q_\alpha\}$ $(\alpha = 1, 2, \ldots N)$.

(c) $\lim_{\delta \to 0} \int_{\Sigma_\alpha(\delta)} \bar{T}\, d\sigma = \bar{L}_\alpha$ $(\alpha = 1, 2, \ldots N)$, where $\Sigma_\alpha(\delta)$ is the intersection with $D + B$ of a sphere of radius δ, centered at Q_α, the outer normal of Σ_α being directed toward Q_α.

(d) $\tau_{ij}(P) = O(r_\alpha^{-2})$ as $r_\alpha \to 0$, where r_α is the distance from P to Q_α.

Property (a) follows at once from Definitions 3.2, 4.1, 4.2, as well as from the fact that the sequence of regular, isotropic states $S^{(n)}$ of Theorem 7.1, together with its first space derivatives, converges uniformly in any closed subregion of $D + B$ which contains no points in $\{Q_\alpha\}$.

Consider a regular point Q_0 of B, which is not in $\{Q_\alpha\}$ $(\alpha = 1, 2, \ldots N)$, and recall that the maximum diameter $\delta_\alpha^{(n)}$ of the load region $\Lambda_\alpha^{(n)}$, in Theorem 7.1, tends to zero as $n \to \infty$. By Hypotheses (1) and (b) of Theorem 7.1, Property (a) in the present theorem, and since $S^{(n)}(Q_0) \to S(Q_0)$ as $n \to \infty$, given $\epsilon > 0$, there exists $M > 0$ such that $n > M$ implies

$$\bar{T}^{(n)}(Q_0) = \bar{T}_*(Q_0), \qquad \left|\bar{T}^{(n)}(Q_0) - \bar{T}(Q_0)\right| < \epsilon,$$

which confirms Property (b) of the limit-state.

Let $\Sigma_\alpha(\delta)$, in the statement of Property (c), contain no points in $\{Q_\alpha\}$ ($\alpha = 1, 2, \ldots N$). It follows from the uniform convergence of $S^{(n)}(P)$ toward $S(P)$ in every closed subregion of $D + B$ which excludes the set $\{Q_\alpha\}$ that

$$\int_{\Sigma_\alpha} \bar{T}\, d\sigma = \int_{\Sigma_\alpha} \lim_{n \to \infty} \bar{T}^{(n)}\, d\sigma = \lim_{n \to \infty} \int_{\Sigma_\alpha} \bar{T}^{(n)}\, d\sigma. \tag{7.11}$$

Next, let $\Pi_\alpha(\delta)$ be the intersection with B of the solid sphere of radius δ, centered at Q_α. Choose δ small enough so that Π_α contains no member of the set $\{Q_\beta\}$ ($\beta = 1, 2, \ldots N$) for which $\beta \neq \alpha$. Since $S^{(n)}(P)$ satisfies the homogeneous equilibrium equations in $D + B$, and from Hypothesis (b) of Theorem 7.1, we have

$$\int_{\Sigma_\alpha} \bar{T}^{(n)}\, d\sigma = \int_{\Pi_\alpha} \bar{T}^{(n)}\, d\sigma = \int_{\Pi_\alpha} \bar{T}_*\, d\sigma + \int_{\Lambda_\alpha^{(n)}} \bar{f}^{(n)}\, d\sigma, \tag{7.12}$$

for all n sufficiently large to insure that $\Lambda_\alpha^{(n)}$ is wholly contained in $\Pi_\alpha(\delta)$. Hypothesis (2) and (7.12) yield,

$$\lim_{n \to \infty} \int_{\Sigma_\alpha} \bar{T}^{(n)}\, d\sigma = \int_{\Pi_\alpha} \bar{T}_*\, d\sigma + \bar{L}_\alpha, \tag{7.13}$$

which, together with (7.11), implies

$$\lim_{\delta \to 0} \int_{\Sigma_\alpha(\delta)} \bar{T}\, d\sigma = \bar{L}_\alpha. \tag{7.14}$$

Thus Property (c) is verified.

According to Theorem 6.2 and Equations (5.5), (5.6), $\bar{u}'_{ij} = O(r^{-2})$ as $r \to 0$, if r is the distance from Q to P. Hence, Property (d) follows directly from (7.2) and (3.6). This completes the proof of Theorem 7.2, which is the analogue of Theorem 4.4.

Let $A_\alpha = \int_{\Sigma_\alpha} d\sigma$ be the area of Σ_α. Then Property (c) in Theorem 7.2 implies

$$\lim_{\delta \to 0} \delta^{-2} A_\alpha(\delta) > 0, \tag{7.15}$$

unless $L_\alpha = 0$. Condition (7.15) is certainly met if the load point Q_α is a regular point of B. This condition need not be satisfied, however, in case Q_α lies on a singular edge or on a corner of B. To illustrate this eventuality, let Q_α be the origin and let B, in a neighborhood of Q_α, be a surface of revolution whose axis is the x_3-axis. Thus, let B locally admit the representation,

$$\rho = f(x_3) \quad \text{for} \quad 0 \leq x_3 \leq a, \tag{7.16}$$

where $\rho = (x_1^2 + x_2^2)^{1/2}$, $f(x_3)$ is continuously differentiable in $0 \leq x_3 \leq a$, $f(0) = f'(0) = 0$, and $f(x_3) > 0$ in $0 < x_3 \leq a$. For sufficiently small δ we have here,

$$A_\alpha(\delta) = 2\pi\delta(\delta - x_3), \tag{7.17}$$

and a trivial computation yields,

$$\lim_{\delta \to 0} \delta^{-2} A_\alpha(\delta) = 0. \tag{7.18}$$

In the sense of the foregoing observation, the body is "incapable of supporting a concentrated load" at the point Q_α under consideration.

It will become apparent in the following section that Properties (a), (b), (c), and (d) in Theorem 7.2 uniquely characterize the limit-state S(P) of Theorem 7.1. On the other hand, as pointed out in the Introduction and demonstrated in [1], there exist "pseudo-solutions" of concentrated-load problems which possess Properties (a), (b), and (c), without being identical with the limit-state S(P), defined in Theorem 7.1. Thus, the traditional formulation of concentrated-load problems in terms of Properties (a), (b), and (c), is incomplete. Moreover, any expectation, based on an appeal to Saint-Venant's principle, that pseudo-solutions represent useful approximations to S(P), is unfounded, as was shown in [1]. This observation is not in conflict with a rigorous formulation of the principle [21], [22].

8. Extension of the Uniqueness Theorem to Concentrated Loads

In this section we state and prove a uniqueness theorem appropriate to problems involving internal concentrated loads as well as concentrated surface loads.

Theorem 8.1: Let $D + B$ be a regular region of space. Let B_u and B_t be subregions of B such that $B_u + B_t = B$. Let $\{Q_\alpha\}$ $(\alpha = 1, 2, \ldots N)$ be a set of N distinct points such that each Q_α lies either in D or in the interior of B_t. Let $\sum_\alpha(\delta)$ be the intersection with $D + B$ of a sphere with radius δ, centered at Q_α, the outer normal of $\sum_\alpha$ being directed toward Q_α.

Let S and S' be two states with the following properties:

(a) S and S' are (not necessarily isotropic) regular states in $D + B$, corresponding to the same body-force field, except possibly for point singularities at $Q_\alpha (\alpha = 1, 2, \ldots N)$;

(b) $\bar{u}' = \bar{u}''$ on B_u, $\bar{T}' = \bar{T}''$ at all regular points of B_t at which S and S' are non-singular;

(c)
$$\lim_{\delta \to 0} \int_{\sum_\alpha(\delta)} \bar{T}' \, d\sigma = \lim_{\delta \to 0} \int_{\sum_\alpha(\delta)} \bar{T}'' \, d\sigma; \qquad (8.1)$$

(d) $\tau'_{ij}(P) = O(r_\alpha^{-2})$ and $\tau''_{ij}(P) = O(r_\alpha^{-2})$ as $r_\alpha \to 0$, where r_α is the distance from P to Q_α.

Then $\tau'_{ij}(P) = \tau''_{ij}(P)$ for all P in $D + B$ and not in $\{Q_\alpha\}$ $(\alpha = 1, 2, \ldots N)$.

We note that Theorem 8.1 reduces to the classical uniqueness theorem, Theorem 3.3, in case S and S' are free from singularities in $D + B$.

To prove the theorem, consider the state

$$S(P) \equiv S'(P) - S''(P). \qquad (8.2)$$

In view of Hypotheses (a), (b), (c), and (d), $S(P)$ has the properties:

(A) $S(P)$ is regular in $D + B$, corresponding to $\overline{F}(P) = 0$, with the possible exception of point singularities at $Q_\alpha (\alpha = 1, 2, \ldots N)$;

(B) $\overline{u} = 0$ on B_u and $\overline{T} = 0$ at all regular points of B_t at which S is non-singular;

$$\text{(C)} \quad \lim_{\delta \to 0} \int_{\Sigma_\alpha(\delta)} \overline{T} \, d\sigma = 0 \quad (\alpha = 1, 2, \ldots N); \tag{8.3}$$

(D) $\tau_{ij}(P) = O(r_\alpha^{-2})$ as $r_\alpha \to 0$.

Evidently, there exists $\delta_o > 0$ such that $0 < \delta < \delta_o$ implies:

(α) no two members of the set of solid spheres $0 \leq r_\alpha \leq \delta$ ($\alpha = 1, 2, \ldots N$) intersect;

(β) if Q_α is in D, the solid sphere $0 \leq r_\alpha \leq \delta$ does not intersect B;

(γ) if Q_α is on B_t, the solid sphere $0 \leq r_\alpha \leq \delta$ intersects neither B_u nor any edge or vertex of B_t which does not contain Q_α, and the intersection of $\Sigma_\alpha(\delta)$ with B_t is a closed regular curve.

Figure 1 shows a schematic diagram of $D + B$ together with the points Q_α and the associated spherical surfaces $\Sigma_\alpha(\delta)$. Let $\mathcal{D}(\delta) + \mathcal{B}(\delta)$ be the closed region, with the boundary $\mathcal{B}(\delta)$, consisting of all points in $D + B$ and not in $0 \leq r_\alpha < \delta$ ($\alpha = 1, 2, \ldots N$). By (α), (β), and (γ), $\mathcal{D}(\delta) + \mathcal{B}(\delta)$ is a regular subregion[45] of $D + B$, in which $S(P)$, according to (A), is regular. We may, therefore, apply the energy formula (3.10) to $S(P)$ in $\mathcal{D} + \mathcal{B}$, and, by virtue of (A), (B), (α), (β), (γ), obtain,

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[45] See the definition of a "regular region of space" given in Section 2. Conditions (α), (β), and (γ) assure that $\mathcal{B}(\delta)$ consists of a finite number of non-intersecting closed regular surfaces.

$$\left.\begin{aligned}&\int_{B(\delta)} \bar{T} \cdot \bar{u}\, d\sigma = \sum_{\alpha=1}^{N} \int_{\Sigma_\alpha(\delta)} \bar{T} \cdot \bar{u}\, d\sigma = 2 \int_{\mathcal{D}(\delta)} W\, d\tau . \\ &(0 < \delta < \delta_0).\end{aligned}\right\} \quad (8.4)$$

Furthermore, the intersection of $0 < a \leq r_\alpha < \delta < \delta_0$ with $D + B$, for every fixed α, is also a regular region of space in which $S(P)$, in view of (A), satisfies the homogeneous equilibrium conditions. This observation, together with (B), (C), (α), (β), and (γ), yields,

$$\int_{\Sigma_\alpha(\delta)} \bar{T}\, d\sigma = 0 \quad \text{for} \quad 0 < \delta < \delta_0 \quad (\alpha = 1, 2, \ldots N). \qquad (8.5)$$

To establish the theorem, we need to show that $\tau_{ij}(P) = 0$ for every P in D and not in $\{Q_\alpha\}$ $(\alpha = 1, 2, \ldots N)$. To this end it suffices to show that

$$\lim_{\delta \to 0} \int_{\mathcal{D}(\delta)} W\, d\tau = 0, \qquad (8.6)$$

since W is a positive definite function of the components of stress. Thus, we need to prove that the improper strain-energy integral $\int_D W\, d\tau$ is convergent and has the value zero.

We turn to the proof of (8.6). According to (8.4), we merely have to demonstrate that given $\epsilon > 0$, there exists $\delta_1 > 0$ such that $0 < \delta < \delta_1$ implies,

$$\left.\begin{aligned}&\left| \int_{\Sigma_\alpha(\delta)} T_i\, u_i\, d\sigma \right| < \epsilon \quad \text{(no summation)} \\ &\qquad (\alpha = 1, 2, \ldots N).\end{aligned}\right\} \quad (8.7)$$

For convenience, we shall henceforth write $\sum(\delta)$, T, and u, in place of $\sum_\alpha(\delta)$, T_i, and u_i, respectively. Thus, examine the integral,

$$I(\delta) = \int_{\sum(\delta)} T(Q,\delta)\, u(Q,\delta)\, d\sigma \quad (0 < \delta < \delta_o), \tag{8.8}$$

where Q is a point on $\sum(\delta)$ and, from (8.5),

$$\int_{\sum(\delta)} T(Q,\delta)\, d\sigma = 0 \qquad (0 < \delta < \delta_o). \tag{8.9}$$

Let $\sum^{(1)}(\delta)$ and $\sum^{(2)}(\delta)$ be the subsets of $\sum(\delta)$ which are characterized by the requirements,

$$\left.\begin{aligned} T(Q,\delta) &\geq 0 \quad \text{for} \quad Q \quad \text{in} \quad \textstyle\sum^{(1)}(\delta), \\ T(Q,\delta) &\leq 0 \quad \text{for} \quad Q \quad \text{in} \quad \textstyle\sum^{(2)}(\delta). \end{aligned}\right\} \tag{8.10}$$

Then,

$$\int_{\sum(\delta)} Tu\, d\sigma = \int_{\sum^{(1)}(\delta)} Tu\, d\sigma + \int_{\sum^{(2)}(\delta)} Tu\, d\sigma, \tag{8.11}$$

and, by (8.9),

$$\int_{\sum^{(1)}(\delta)} T\, d\sigma = -\int_{\sum^{(2)}(\delta)} T\, d\sigma. \tag{8.12}$$

Equations (8.10), (8.11), and (8.12), in conjunction with the generalized first mean-value theorem for surface integrals, assure the existence of two points $Q_1(\delta)$ and $Q_2(\delta)$ in $\sum(\delta)$, such that

$$\left.\begin{aligned} &\left|I(\delta)\right| \leq \left|u(Q_1,\delta) - u(Q_2,\delta)\right| \int_{\sum^{(1)}(\delta)} T(Q,\delta)\, d\sigma \\ &(0 < \delta < \delta_o). \end{aligned}\right\} \tag{8.13}$$

By (D), there exist $M > 0$, $\delta_2 > 0$ $(\delta_2 < \delta_1)$, such that $|T(Q,\delta)| < M/\delta^2$ for $0 < \delta < \delta_2$ and all Q in $\Sigma(\delta)$. Hence,

$$\int_{\Sigma^{(1)}(\delta)} T(Q,\delta)\, d\sigma < 4\pi M. \tag{8.14}$$

Moreover, by (A), $u(Q,\delta)$ is continuous on $\Sigma(\delta)$ so that, given $\epsilon > 0$, there exists $\delta_1 > 0$ $(\delta_1 < \delta_2)$ such that $0 < \delta < \delta_1$ implies,

$$|u(Q_1,\delta) - u(Q_2,\delta)| < \epsilon/4\pi M. \tag{8.15}$$

Equations (8.13), (8.14), and (8.15) yield

$$|I(\delta)| < \epsilon \quad \text{if} \quad 0 < \delta < \delta_1, \tag{8.16}$$

which completes the proof of the theorem.

Theorem 8.1 is readily extended to mixed-mixed boundary-value problems, as well as to the case in which the boundary B, assumed to be suitably smooth, extends to infinity. The theorem confirms, in particular, that Properties (a), (b), (c) in Theorem 4.4 and Properties (a), (b), (c), (d) in Theorem 7.2, uniquely characterize the Kelvin-state and the limit-state of Theorem 7.1, respectively. On the basis of the present generalized uniqueness theorem, we arrive at the following unique formulation of mixed boundary-value problems, involving internal concentrated loads as well as concentrated surface loads:

Let $D + B$, B_u, B_t, $\{Q_\alpha\}$, r_α, and $\Sigma_\alpha(\delta)$ $(\alpha = 1, 2, \ldots N)$ be defined as in Theorem 8.1. Given $\bar{F}_*(P)$ for P in D, $\bar{u}_*(Q)$ for Q on B_u, $\bar{T}_*(Q)$ for Q on B_t, $\bar{L}_\alpha$ $(\alpha = 1, 2, \ldots N)$, and the elastic constants c_{ijmn}, find a state $S(P)$ with the properties:

(a) $S(P)$ _is regular in_ $D + B$, _corresponding to_ $\bar{\bar{F}} = \bar{\bar{F}}_*$, _except for point singularities at each_ Q_α _for which_ $\bar{L}_\alpha \neq 0$.

(b) $\bar{u} = \bar{u}_*$ _on_ B_u _and_ $\bar{\bar{T}} = \bar{\bar{T}}_*$ _at all regular points of_ B_t _which are not in_ $\{Q_\alpha\}$.

(c) $\lim_{\delta \to 0} \int_{\Sigma_\alpha(\delta)} \bar{\bar{T}}\, d\sigma = \bar{L}_\alpha \; (\alpha = 1, 2, \ldots N)$.

(d) $\tau_{ij}(P) = O(r_\alpha^{-2})$ _as_ $r_\alpha \to 0$.

Necessary conditions for the existence of the solution to the foregoing problem, analogous to Conditions (a), (b), (c), (d) of Theorem 3.4, are immediate and need not be listed here explicitly. To these we add Condition (7.15).

The significance of this alternative formulation of concentrated-load problems, which derives its physical motivation from the limit-definitions contained in Theorems 4.4 and 7.1, was discussed in the Introduction to the present paper.

9. Concluding Remarks

As emphasized in the discussion following Theorem 3.4, not only concentrated loads but also most instances of discontinuous distributed loadings, are beyond the range of validity of the classical uniqueness theorem, Theorem 3.2. The traditional formulation of problems characterized by discontinuous distributed surface tractions is, in general, not unique, which fact was illustrated in Section 3.

A natural and unique definition of the solution to the second boundary-value problem for an isotropic medium, in the presence of merely piecewise continuous surface tractions, may be based on the Lauricella-Volterra formulas which were deduced in Section 6 on the assumption that $S(P)$ is regular in $D + B$. Thus, we may use (6.17) and (6.22) to define the displacement and strain field of the solution, although $\bar{T}(Q)$ are here no longer the surface tractions of a regular state.[46]

Such a formal definition of the solution to a problem characterized by piecewise continuous surface tractions does not, however, satisfactorily dispose of the uniqueness questions here involved. First, from the viewpoint of applications, it is analytically awkward to be limited to a formulation of the problem which is tied to a particular integral representation of the solution. Second, any specific application of the foregoing definition presupposes an explicit knowledge of the states S_i' and S_{ij}'', defined in Theorems 6.1 and 6.2; this, in turn, necessitates the solution of two usually highly complicated boundary-value problems.

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[46] Analogously, but for lack of motivation, we could have directly adopted (7.1), (7.2), and (7.3) as a definition of the solution to the second boundary-value problem in the presence of concentrated loads.

A possible program to be pursued in connection with the class of problems under consideration, consists in first adopting (6.17), (6.22) as a definition of the solution; one may then study the properties of the solution so defined -- in particular, the character of the singularities present -- with a view toward reaching an extension of the classical uniqueness theorem, analogous to Theorem 8.1 for concentrated loads. Such a generalization of the uniqueness theorem would give rise to a practically useful alternative formulation of the problem in terms of intrinsic properties of the solution. This task is, however, beyond the scope of the present paper. Similarly, uniqueness questions related to geometrically induced singularities, are in need of further attention.

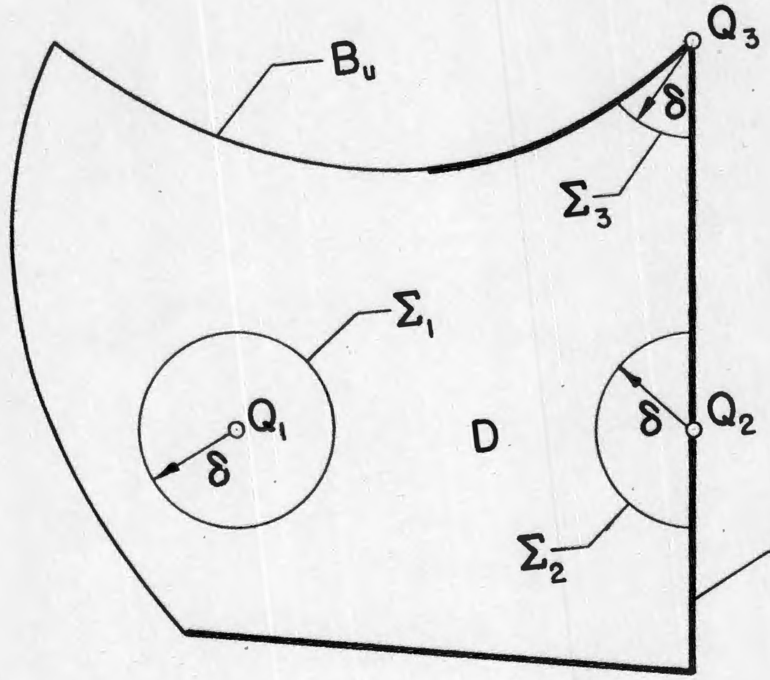

Figure 1

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