THE MATRIX DEFORMATION METHOD OF STRUCTURAL ANALYSIS

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Three new matrix methods of structural analysis are presented: (1) Matrix Deformation, (2) Unified Force Deformation, and (3) Static Force. These approaches are related to the matrix force method in that they require the selection of redundants.

The distinguishing feature of the matrix deformation method is that deformations of elements in the continuous structure, which are initially designated as "statically determinate," are computed before displacements and redundants. A comparison of the matrix deformation method with the matrix force approach shows that the matrix force method is most efficient when the ratio of number of redundants to number of statically determinate forces is small, while the matrix deformation method is most efficient when this ratio is greater than unity.

The order of the matrix to be solved in computing static deformations is the same as the order of the K matrix in the displacement method. Compared to the displacement method, the matrix deformation approach offers the advantage of improved matrix conditioning through proper selection of static deformations.

The unified force deformation method provides a theoretical basis from which the redundant force and matrix deformation methods can be derived.

The static force method was investigated and found to be inefficient.

SECTION I

INTRODUCTION

The matrix displacement method of structural analysis requires less computation than the force method when the number of redundants exceeds the number of statically determinate forces. However, the force method has a tendency to yield better conditioned equations and, consequently, greater accuracy. One reason for this advantage is that redundants in the force approach can be selected to minimize computational errors. The displacement method has no equivalent operation (Reference 1); consequently, the force method is a good choice when the structural configuration requires additional accuracy and the number of redundants is less than the number of statically determinate forces.

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Unfortunately, the extra computations required by the force method limits its feasibility when the number of redundants exceeds the number of statically determinate forces. Therefore, a need exists for a method that combines the speed of the displacement method and the accuracy of the force method in this regime. Such a method would be particularly useful in the analysis of structural models composed of finite elements.

The matrix deformation method provides the required capability. The equations of this method are similar to the equations of the displacement method. The major item of computation is the solution of a set of equations having a coefficient matrix K_{QQ}, which is the same size as the structural stiffness matrix K in the displacement method. The unknowns in the matrix deformation approach are deformations in the continuous structure initially designated as "statically determinate," which are analogous to the displacements of the displacement method. The difference is that statically determinate deformations can be selected to minimize computational errors in a manner analogous to the selection of redundants in the force method.

During study of the deformation approach, two other methods were discovered: (1) Unified Force Deformation, and (2) Static Force. The unified force deformation method is analogous to the unified method of structural analysis and serves as a convenient basis for deriving the matrix deformation equations. The static force method appears to have no utility.

This paper also describes the matrix deformation equations and their applications. The derivations of the methods presented in Appendix I represent an extension and refinement of work reported in Reference 2.

SECTION II

DISCUSSION

The equations of the matrix deformation method can be summarized as follows:

$$K_Q = k\, q_{FF}^T \tag{1}$$

$$K_{QQ} = q_{FF} K_Q \tag{2}$$

$$F_T = k \left(q_{RF}^T \Delta_R - e_T \right) \tag{3}$$

$$K_{QQ} e_Q = q_{F\phi}\, \phi + q_{FF}\, F_T \tag{4}$$

$$F = -K_Q e_Q + F_T \tag{5}$$

$$Q_R = q_{RF} F + q_{R\phi}\, \phi \tag{6}$$

$$\Delta = q_{F\Delta}^T\, e_Q - q_{R\Delta}^T\, \Delta_R \tag{7}$$

where

k = a square matrix of element stiffnesses (nonsingular, referred to local coordinates)

q_{FF}, q_{RF} = matrices of statically determinate element forces and reactions resulting from unit element forces applied as external loads to the statically determinate structure

$q_{F\phi}, q_{R\phi}$ = matrices of statically determinate element forces and reactions resulting from external loads on the statically determinate structure

K_{QQ} = a square matrix of stiffness coefficients

ϕ, e_T, Δ_R = column matrices of loads, unassembled element deformations, and support displacements

e_Q, F, Δ, Q_R = column matrices of statically determinate element deformations (in the continuous structure), element forces, joint displacements, and reactions

$q_{F\Delta}, q_{R\Delta}$ = matrices of statically determinate element forces and reactions resulting from unit displacement loads on the statically determinate structure.

As a simple example, consider the structure shown in Figure 1. The structure has four element forces numbered as shown. Element force No. 1 is designated as statically determinate. This force is assigned the additional symbol Q_1, while the corresponding element deformation is e_{Q1}. No unassembled deformations or support displacements are introduced; consequently $e_T = \Delta_R = 0$. The unassembled stiffness matrix is

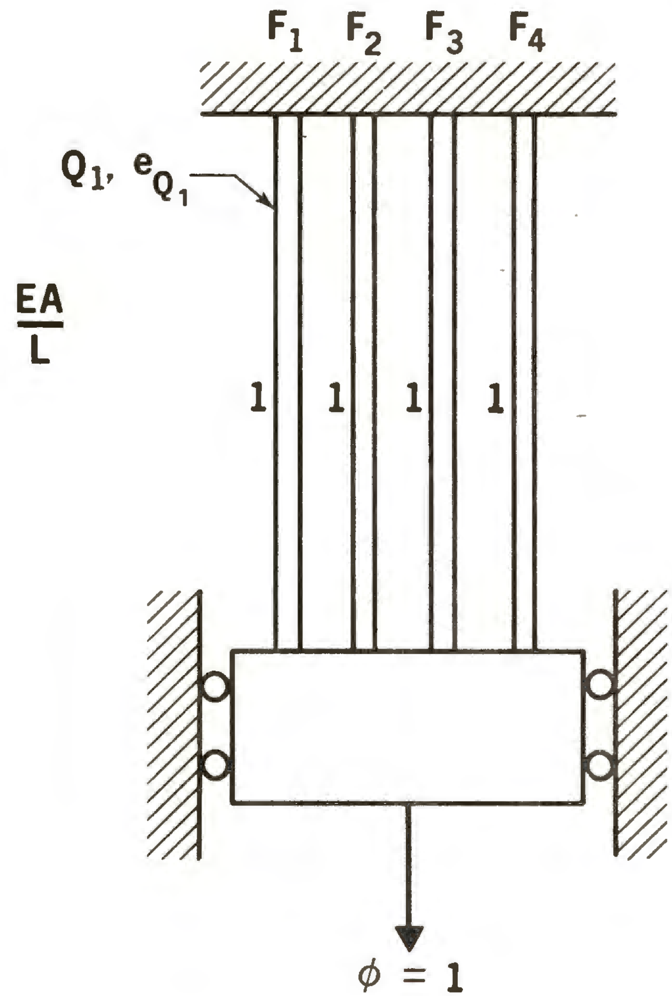
$$k = \left[\begin{array}{c|c|c|c} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \tag{8}$$

The symbol q_{FF} denotes a matrix of statically determinate forces resulting from unit element forces applied to the statically determinate structure as external loads. This matrix is

$$q_{FF} = [-1 \mid -1 \mid -1 \mid -1] \tag{9}$$

The matrix is derived as follows: When a column of q_{FF} corresponds to a statically determinate element force, a -1 is inserted in the row corresponding to the force. Thus, the -1 in column 1 can be interpreted as the force that would exist in the first element if unit tensile forces produced by the element on adjacent joints were placed upon the statically determinate structure as external loads. The remaining columns are statically determinate forces resulting from unit forces in the cut members.

FIGURE 1. A PARALLEL STRUCTURE



The symbol $q_{F\phi}$ denotes a matrix of statically determinate element forces resulting from unit loads on the cut structure. This matrix is

$$q_{F\phi} = 1 = q_{F\Delta} \tag{10}$$

From Equations 1 and 2

$$K_Q = \{-1 \mid -1 \mid -1 \mid -1\}, K_{QQ} = 4 \tag{11}$$

where the symbols $\{\ \}$ denote a column matrix.

The matrix of external loads is

$$\phi = 1 \tag{12}$$

From Equation 4

$$e_Q = 1/4(1)\,(1) = 1/4 \tag{13}$$

From Equation 5

$$F = \{1/4 \mid 1/4 \mid 1/4 \mid 1/4\} \tag{14}$$

From Equation 7

$$\Delta = 1/4 \tag{15}$$

Analysis of this structure by the matrix force method requires the solution of three simultaneous equations. This example demonstrates that the deformation method requires less calculation than the force method when the number of redundants exceeds the number of statically determinate forces.

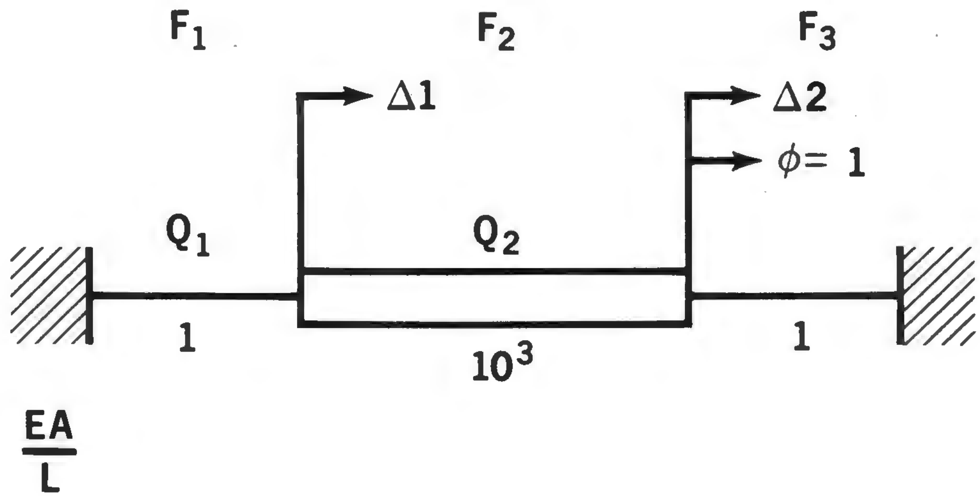
As a second example, consider the structure shown in Figure 2.

Designate the forces in the two left-hand members as statically determinate. Now

$$k = \left[\begin{array}{c|c|c} 1 & & \\ & 10^3 & \\ & & 1 \end{array}\right] \qquad \phi = 1 \tag{16}$$

$$q_{FF} = \left[\begin{array}{c|c|c} -1 & 0 & 1 \\ 0 & -1 & 1 \end{array}\right] \qquad q_{F\phi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad q_{F\Delta} = \left[\begin{array}{c|c} 1 & 1 \\ 0 & 1 \end{array}\right] \tag{17}$$

FIGURE 2. STRUCTURE WITH A STIFF MEMBER



From Equations 1 to 7

$$K_Q = \begin{bmatrix} -1 & 0 \\ 0 & -1000 \\ 1 & 1 \end{bmatrix} \qquad K_{QQ} = \begin{bmatrix} 2 & 1 \\ 1 & 1001 \end{bmatrix} \tag{18}$$

$$e_Q = \begin{bmatrix} 2 & 1 \\ 1 & 1001 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1) = \begin{Bmatrix} 0.50 \\ 0.00050 \end{Bmatrix} \tag{19}$$

$$F = \begin{Bmatrix} 0.50 \\ 0.50 \\ -0.50 \end{Bmatrix} \qquad \Delta = \begin{Bmatrix} 0.50 \\ 0.50 \end{Bmatrix} \tag{20}$$

The results shown for e_Q are obtained by solving the simultaneous equations involved by Gaussian elimination, keeping two significant figures in the calculations. The results are correct to two significant figures.

The displacement method equation for the structure shown in Figure 2 is:

$$K\Delta = P \tag{21}$$

where

$$K = \begin{bmatrix} 1001 & -1000 \\ -1000 & 1001 \end{bmatrix} \qquad P = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{22}$$

and where K is the structural stiffness matrix and P is a matrix of loads in the joint degrees of freedom.

Equation 21 cannot be solved by any procedure keeping only two significant figures because all accuracy is lost. Consequently, the deformation equations are considered better conditioned.

The improved conditioning of the deformation equations results from the proper choice of a statically determinate structure. If the stiff member in the center is selected as a redundant, the deformation equations are also poorly conditioned.

This example illustrates the ability of the deformation method to provide well conditioned equations after proper choice of statically determinate forces.

SELECTION OF STATICALLY DETERMINATE ELEMENT FORCES AND CALCULATION OF q_{FF}, $q_{F\phi}$, q_{RF} AND $q_{R\phi}$

The scope of this paper allows only a brief discussion of these subjects. The statically determinate element forces, and consequently the redundants, can be selected by a pivoting technique applied to the rectangular coefficient matrix of unknown element forces in the equilibrium equations for structural joints. This technique, known as the "Structure Cutter," is described in Reference 3. Member stiffnesses are accounted for by the introduction of weighting factors. Presently, the Structure Cutter outputs are more suitable for the force method, but necessary modifications can be easily and efficiently introduced. As the modified Structure Cutter selects statically determinate forces, it can simultaneously calculate the matrices q_{FF}, $q_{F\phi}$, q_{RF} and $q_{R\phi}$. The matrices $q_{F\Delta}$ and $q_{R\Delta}$ can be extracted from $q_{F\phi}$ and $q_{R\phi}$.

In the generation and solution of the equilibrium equations in the deformation approach, the following advantages of the force method can be preserved: (1) joint reactions can be applied in any direction; (2) the number of calculated displacements can be less than the number of unconstrained degrees of freedom; and (3) joints can have any number of degrees of freedom from 1 to 6, depending on available internal constraints.

PARALLEL ELEMENT FORCES

In the direct stiffness version of the displacement method, considerable simplification results from the use of element forces parallel to the axes of a global coordinate system. Some of this advantage can be secured for the deformation method without sacrificing the option of considering only the minimum number of joint degrees of freedom. Thus, if element forces of finite elements are selected parallel and perpendicular to element edges, then element forces of adjacent elements tend to be parallel. Columns of the matrix of coefficients in the equilibrium equations corresponding to parallel element forces are identical; consequently, duplicate columns can be deleted with significant savings in the equilibrium solution. This consideration can eliminate as much as 60 percent of the effort involved in solving equilibrium equations.

COMPARISON WITH THE DISPLACEMENT EQUATIONS

The equations of the matrix displacement method can be defined:

$$K_U = k\,P_{UF}^{T} \tag{23}$$

$$K = P_{UF}\,K_U \tag{24}$$

$$F'_T = -k\left(P_{CF}^{T}\,\Delta_R + e_T\right) \tag{25}$$

$$K\Delta = P_{U\phi}\,\phi + P_{UF}\,F'_T \tag{26}$$

$$F = -K_U\Delta + F'_T \tag{27}$$

$$Q_R = -P_{CF}\,F - P_{C\phi}\,\phi \tag{28}$$

where

P_{UF} and $P_{U\phi}$ = matrices of components in the unconstrained joint degrees of freedom of unit values of element forces and external loads

P_{CF} and $P_{C\phi}$ = matrices of components in the constrained joint degrees of freedom of unit values of the element forces and external loads.

Equations 23 through 28 are the displacement equations as they are used essentially in practice, although practices may vary. In the direct stiffness method, the calculations represented by Equations 23 and 24 are performed in scalar rather than matric form, but the same work must be accomplished. Reactions (Equation 28) can be computed from joint displacements and unassembled deformations, but the calculation is no more concise.

A comparison of Equations 1 through 7 with Equations 23 through 28 immediately shows a strong similarity. In fact, a one-to-one correspondence exists except for Equation 7. There are two major differences between the methods: (1) the deformation method requires calculation of q_{FF}, $q_{F\phi}$, q_{RF}, and $q_{R\phi}$ from P_{UF}, $P_{U\phi}$, P_{CF}, and $P_{C\phi}$, and (2) no calculation in the displacement method, analogous to Equation 7, is required since displacements are already given by Equation 26. The extra calculation required by the deformation approach can be minimized by taking advantage of the extreme sparsity of P_{UF}, $P_{U\phi}$, P_{CF}, and $P_{C\phi}$, the existence of parallel element forces, and the fact that the complete set of joint displacements is seldom required and, therefore, need not be computed.

The remaining extra calculations in the deformation method represent the minimum additional effort required to develop a well-conditioned set of matrix equations. Thus, a trade-off can exist between computing time expended to develop a well-conditioned set of equations, and computing time and reliability saved by not having to exercise extra precision in subsequent calculations.

In Appendix I, the static deformation stiffness matrix is shown to be related to the structural stiffness matrix by a simple transformation. As an example of this transformation, again consider Figure 2. If external loads are assumed to act in both joint degrees of freedom, then

$$q_{F\phi} = \left[\begin{array}{c|c} 1 & 1 \\ 0 & 1 \end{array}\right] \tag{29}$$

Now, from Equations 22, 29, and A-57

$$K_{QQ} = \left[\begin{array}{c|c} 1 & 1 \\ 0 & 1 \end{array}\right] \left[\begin{array}{c|c} 1001 & -1000 \\ -1000 & 1001 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ 1 & 1 \end{array}\right] = \left[\begin{array}{c|c} 2 & 1 \\ 1 & 1001 \end{array}\right] \tag{30}$$

Thus, the static deformation stiffness matrix is correctly computed from the structural stiffness matrix, if four significant figures are saved during the calculation.

ACCURACY OF THE DEFORMATION METHOD

The preceding section reflects a belief that the deformation method is able to provide a better conditioned set of equations in a significant number of practical cases. This belief is based on three considerations: (1) the deformation equations can be shown to be better conditioned for simple structures containing stiff members, (2) the deformation method offers a means of improving conditioning through selection of statically determinate forces (the displacement method offers no corresponding means), and (3) the example of a fuselage shell analysis (discussed in subsequent paragraphs) shows that the deformation method provides a well-conditioned set of equations with accuracy comparable to the force method.

COMPARISON OF THE FORCE AND DEFORMATION METHODS

Appendix II compares the amount of computation required by the force and deformation methods for each of two procedures: (1) Force and Deflection Analysis, in which element forces and deflections are calculated for a limited number of loading conditions, and (2) Deflection Influence Coefficients, in which only the deflection influence matrix is computed. The basis of comparison is the number of multiplications involved in the computations.

Figure 3 summarizes results of the comparison for the force and deflection procedure. The curves are based on Equations B-6 and B-13. The figure shows that the force method is most efficient when the structure has fewer redundants than statically determinate forces. Otherwise, the deformation method is most efficient.

The dotted vertical lines show the value of the ratio r for two types of idealizations. Type 1 is the lumped parameter idealization commonly used in conjunction with the force method. Type 2 represents an idealization composed of rectangular plate elements subjected to in-plane forces of the kind often used with the displacement method. These values of r were established by counting the numbers of statically determinate and redundant forces in a typical wing-box structure for the two idealizations. Figure 3 shows that the deformation method is more efficient than the force method for the type 2 idealization.

Figure 4 shows a similar comparison for the deflection influence computation. The conclusions are similar, but the advantage of the deformation method is more pronounced.

DEMONSTRATION PROBLEM

A section of stiffened cylindrical shell (Figure 5) was analyzed to demonstrate the capability of the deformation method. Both structure and applied loads were symmetric about the Y – Z plane; therefore, the idealized model was reduced to a half shell with symmetric boundary constraints on the plane of symmetry. The shell was rigidly supported at one end and was loaded by internal pressure and shear and moment applied at the free end.

The structure was modeled in terms of lumped parameter elements in a manner that produced a large ratio of redundants to statically determinate forces, simulating a model composed of finite elements. The problem size is summarized in Table 1. Element properties were based on materials and sizing of a typical airframe structure.

The analysis was performed on the IBM 360-85 computer in double precision using up to 300K of core and the FORMAT Structural Analysis System (Reference 3). FORMAT Phase 1 was used to generate the required matrices. The Phase 2 matrix abstraction capability was employed to

FIGURE 3. FORCE AND DEFLECTION ANALYSIS

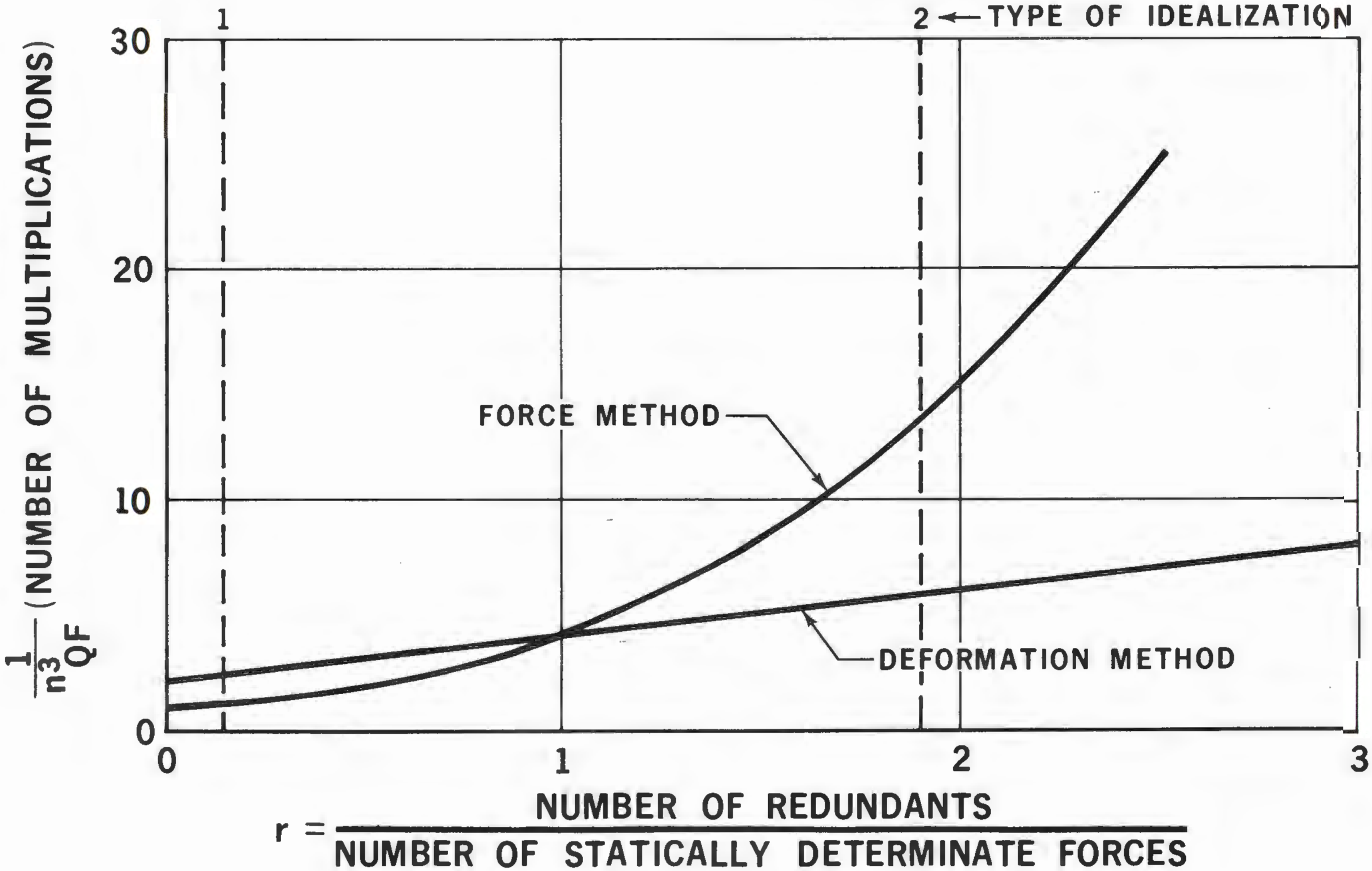


FIGURE 4. DEFLECTION INFLUENCE ANALYSIS

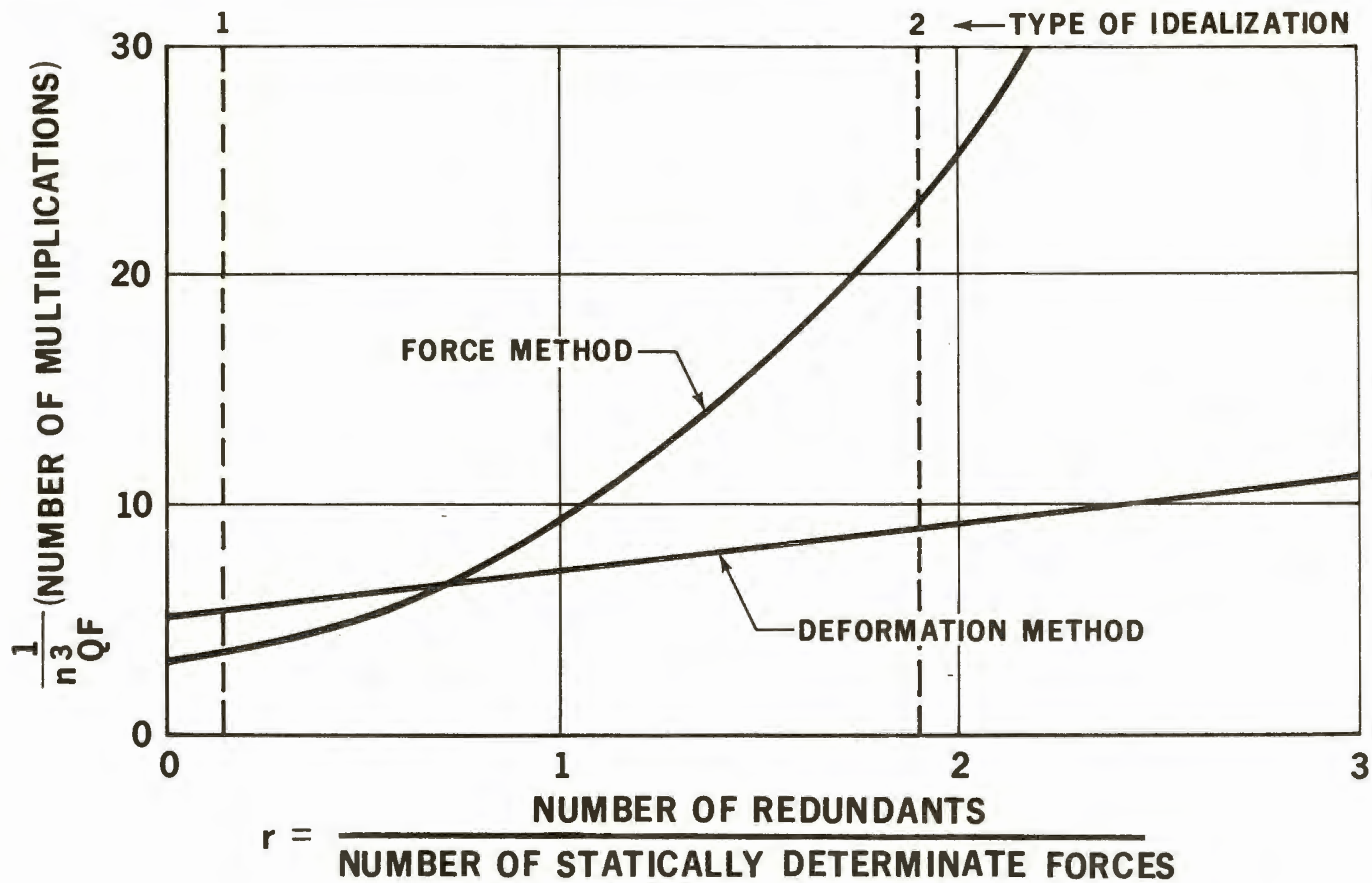
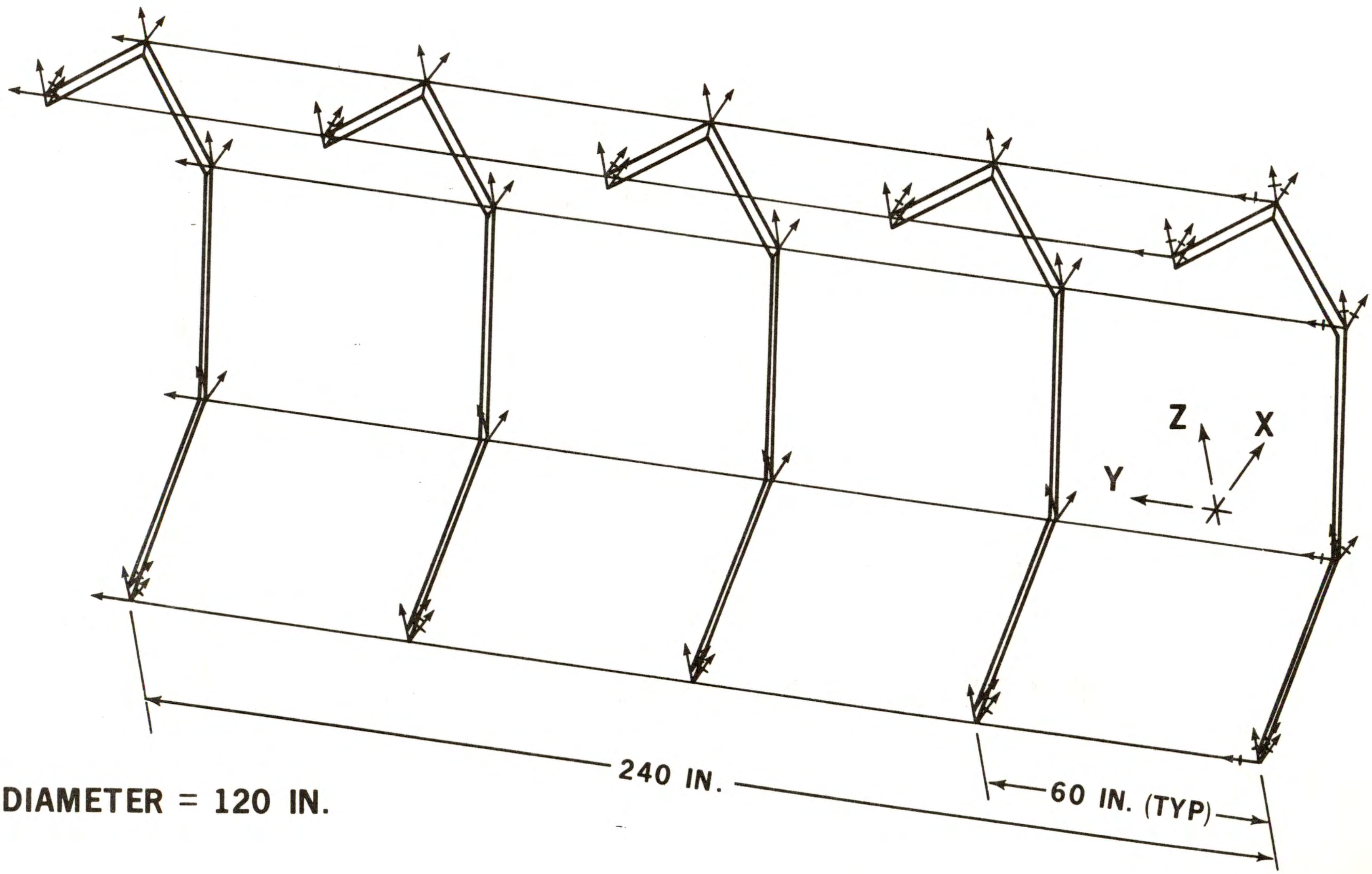


FIGURE 5. DEMONSTRATION STRUCTURE



DIAMETER = 120 IN.

TABLE 1
PROBLEM SIZE

• UNCONSTRAINED DEGREES OF FREEDOM	177
• CONSTRAINED DEGREES OF FREEDOM	33
• REACTION FORCES	33
• ELEMENT FORCES	534
• STATICALLY DETERMINATE FORCES	177
• REDUNDANT FORCES	357

TABLE 2
MACHINE TIME COMPARISON (IBM 360 - 65)

NO.	OPERATION		STATIC DEFORMATION METHOD T_{SDM} (MIN)	FORCE METHOD T_{FM} (MIN)	RATIO $\frac{T_{SDM}}{T_{FM}}$
①	MATRIX GENERATION		1.18	1.18	1.0
②	EQUILIBRIUM SOLUTION		1.427	1.427	1.0
③	CONTINUITY SOLUTION FOR DEFLECTION INFLUENCE COEFFICIENTS	DM	2.647		0.212
		FM		12.472	
④	CONTINUITY SOLUTION FOR FOUR APPLIED LOAD CONDITIONS	DM	2.685		0.257
		FM		10.436	
⑤	COMPLETE ANALYSIS	DEFL INFL COEF'S ① + ② + ③	5.254	15.079	0.348
		APPLIED LOADS ① + ② + ③	5.292	13.043	0.406

solve the matrix equations of the Force and Deformation Methods. A final operation in the Phase 2 step subtracted one solution from the other. The result showed that the solutions were identical to 10 or 12 significant figures. Computer time comparisons are shown in Table 2. The time ratios of 0.406 and 0.348 shown in Table 2 compare favorably with values of 0.393 and 0.354 computed from Equations B-6 and B-13 (see Figures 3 and 4).

UNIFIED FORCE DEFORMATION METHOD

A derivation of this method is presented in Appendix I. The approach is similar to the unified method of structural analysis, except that the unknowns in the force deformation method are redundants and static deformations, whereas the unknowns in the unified method are element forces and displacements. The force deformation method does not lend itself to practical calculation. It is included as a convenient basis for deriving the deformation method.

STATIC FORCE METHOD

Appendix I also contains a derivation of the static force method. The distinguishing feature of this method is that statically determinate element forces are computed before other unknowns. A brief examination of the equations shows that the method is not feasible because the calculations required are excessive.

SECTION III

CONCLUSIONS

The matrix deformation method is a practical approach to the analysis of complex statically indeterminate structures. This method requires less calculation than the force method when the ratio of redundants to statically determinate forces is on the order of 1 or greater. When deflection influence coefficients only are computed, the "break even" point is less than 1.

The conditioning of the deformation method equations can be improved by proper choice of statically determinate deformations. The displacement method has no corresponding capability; consequently, the deformation method can produce better conditioned equations than the displacement method for certain classes of structures.

SECTION IV

REFERENCES

1. Melosh, R.J., and Palacol, E.L., "Manipulation Errors in Finite Element Analysis of Structures," NASA CR-3875, August 1969.

2. Denke, P.H., "Some New Methods of Structural Analysis," Douglas Aircraft Company Report No. DAC-68425, August 1969.

3. Pickard, J., "FORMAT – Fortran Matrix Abstraction Technique," AFFDL-TR-66-207, Volume V, October 1968.

APPENDIX I

DERIVATION OF EQUATIONS

The equations of the unified force method, the static deformation method, and the static force method are derived in the following paragraphs.

GOVERNING EQUATIONS

Classify element forces and reactions as statically determinate or redundant. Statically determinate element forces and reactions can be expressed as follows:

$$Q_F = q_{FXF} X_F + q_{FXR} X_R + q_{F\phi} \phi \tag{A-1}$$

$$Q_R = q_{RXF} X_F + q_{RXR} X_R + q_{R\phi} \phi \tag{A-2}$$

where

Q_F and Q_R = column matrices of statically determinate element forces and reactions

X_F, X_R and ϕ = column matrices of redundant element forces, redundant reactions, and loads

q_{FXF}, q_{FXR} and $q_{F\phi}$ = statically determinate element forces resulting from unit values of redundant element forces, redundant reactions, and loads

q_{RXF}, q_{RXR} and $q_{R\phi}$ = statically determinate reactions resulting from unit values of redundant element forces, redundant reactions, and loads.

The matrices q_{FXF}, q_{FXR}, $q_{F\phi}$, q_{RXF}, q_{RXR} and $q_{R\phi}$ can be computed from statics.

Consider the joints of the structure as free bodies. Apply a unit value of the ith redundant element force F_{X_i} to the joints. Balance with the $q_{FXF_{ji}}$ and the $q_{RXF_{ki}}$ which are statically determinate element forces and reactions resulting from the unit F_{X_i}. Produce a virtual displacement. The virtual work is zero since the joints are rigid bodies in equilibrium.

$$\therefore -1 \cdot e_{X_i} - \sum_j q_{FXF_{ji}} e_{Q_j} + \sum_k q_{RXF_{ki}} \Delta_{QR_k} = 0 \qquad i = 1, 2 \cdots \tag{A-3}$$

where the summations extend over the total numbers of statically determinate element forces and reactions, and

e_{X_i} = the ith redundant element deformation

e_{Q_j} = the jth statically determinate element deformation

Δ_{QR_k} = the displacement of the kth statically determinate reaction

From Equation A-3

$$e_X = -q_{FXF}^T \; e_Q + q_{RXF}^T \; \Delta_{QR} \qquad \text{(A-4)}$$

where e_X, e_Q and Δ_{QR} are column matrices of redundant element deformations, statically determinate element deformations, and displacements of statically determinate reactions.

Similarly

$$\Delta_{XR} = q_{FXR}^T \; e_Q - q_{RXR}^T \; \Delta_{QR} \qquad \text{(A-5)}$$

where Δ_{XR} = a column matrix of displacements of redundant reactions.

Also

$$e_Q = e_{QE} + e_{QT} \qquad \text{(A-6)}$$

$$e_X = e_{XE} + e_{XT} \qquad \text{(A-7)}$$

where

e_{QE} and e_{QT} = column matrices of statically determinate elastic and unassembled deformations

e_{XE} and e_{XT} = column matrices of redundant elastic and unassembled deformations

Finally

$$\begin{Bmatrix} e_{QE} \\ e_{XE} \end{Bmatrix} = \left[\begin{array}{c|c} D_{QQ} & D_{QX} \\ D_{XQ} & D_{XX} \end{array}\right] \begin{Bmatrix} Q_F \\ X_F \end{Bmatrix} \qquad \text{(A-8)}$$

$$\begin{Bmatrix} Q_F \\ X_F \end{Bmatrix} = \left[\begin{array}{c|c} k_{QQ} & k_{QX} \\ k_{XQ} & k_{XX} \end{array}\right] \begin{Bmatrix} e_{QE} \\ e_{XE} \end{Bmatrix} \qquad \text{(A-9)}$$

where D_{QQ}, D_{QX}, D_{XQ}, and D_{XX} are partitions of the element flexibility matrix corresponding to statically determinate and redundant element forces and deformations, and k_{QQ}, k_{QX}, k_{XQ}, and k_{XX} are similar partitions of the element stiffness matrix.

UNIFIED FORCE DEFORMATION EQUATIONS

Eliminating e_Q from Equations A-5 and A-6 gives

$$q^T_{FXR}\ e_{QE} = -q^T_{FXR}\ e_{QT} + q^T_{RXR}\ \Delta_{QR} + \Delta_{XR} \tag{A-10}$$

Now,

$$Q_F = D^{-1}_{QQ}\left(D_{QQ}\ Q_F + D_{QX}\ X_F - D_{QX}\ X_F\right) \tag{A-11}$$

From Equations A-8 and A-11

$$Q_F = D^{-1}_{QQ}\left(e_{QE} - D_{QX}\ X_F\right) \tag{A-12}$$

From Equations A-1 and A-12

$$\left(q_{FXF} + D^{-1}_{QQ}\ D_{QX}\right)X_F + q_{FXR}\ X_R - D^{-1}_{QQ}\ e_{QE} = -q_{F\phi}\ \phi \tag{A-13}$$

It can be shown from Equations A-8 and A-9 that

$$k^{-1}_{XX}\ k_{XQ} + D_{XQ}\ D^{-1}_{QQ} = 0 \tag{A-14}$$

$$\therefore e_{XE} = e_{XE} + \left(k^{-1}_{XX}\ k_{XQ} + D_{XQ}\ D^{-1}_{QQ}\right) e_{QE} \tag{A-15}$$

$$= k^{-1}_{XX}\left(k_{XQ}\ e_{QE} + k_{XX}\ e_{XE}\right) + D_{XQ}\ D^{-1}_{QQ}\ e_{QE} \tag{A-16}$$

From Equations A-9 and A-16

$$e_{XE} = k^{-1}_{XX}\ X_F + D_{XQ}\ D^{-1}_{QQ}\ e_{QE} \tag{A-17}$$

Eliminating e_X, e_Q, and e_{XE} from Equations A-4, A-6, A-7 and A-17 gives

$$k^{-1}_{XX}\ X_F + \left(q^T_{FXF} + D_{XQ}\ D^{-1}_{QQ}\right)e_{QE} = -q^T_{FXF}\ e_{QT} - e_{XT} + q^T_{RXF}\ \Delta_{QR} \tag{A-18}$$

From Equations A-10, A-13, and A-18

$$\left[\begin{array}{c|c|c} k^{-1}_{XX} & 0 & q^T_{FXF} + D_{XQ}\ D^{-1}_{QQ} \\ 0 & 0 & q^T_{FXR} \\ q_{FXF} + D^{-1}_{QQ}\ D_{QX} & q_{FXR} & -D^{-1}_{QQ} \end{array}\right] \begin{Bmatrix} X_F \\ X_R \\ e_{QE} \end{Bmatrix} = \begin{Bmatrix} -q^T_{FXF}\ e_{QT} - e_{XT} + q^T_{RXF}\ \Delta_{QR} \\ -q^T_{FXR}\ e_{QT} + q^T_{RXR}\ \Delta_{QR} + \Delta_{XR} \\ -q_{F\phi}\ \phi \end{Bmatrix} \tag{A-19}$$

Equation A-19 is the unified force deformation equation. The unknowns are redundant element forces, redundant reactions, and statically determinate element deformations. Note the symmetry of the coefficient matrix of unknowns. The redundant force equations can be derived from Equation A-19 by eliminating e_{QE}. The static deformation equations are derived by eliminating X_F and X_R.

MATRIX DEFORMATION EQUATIONS

The characteristic feature of the matrix deformation method is that e_{QE} is the first matrix of unknowns to be computed. However, a study of Equation A-19 shows that e_{QE} cannot be computed before X_R. The redundant reactions are, therefore, eliminated by establishing the convention that reactions are never selected as redundants. This requirement does not compromise the method since the selection of reactions as redundants is never a good choice. The following simplified notation is introduced:

$$\begin{aligned} X &= X_F \qquad & q_{FX} &= q_{FXF} \\ \Delta_R &= \Delta_{QR} \qquad & q_{RX} &= q_{RXF} \end{aligned} \tag{A-20}$$

From Equation A-19, deleting X_R:

$$k_{XX}^{-1} X + \left(q_{FX}^T + D_{XQ} D_{QQ}^{-1}\right) e_{QE} = -q_{FX}^T e_{QT} - e_{XT} + q_{RX}^T \Delta_R \tag{A-21}$$

$$\left(q_{FX} + D_{QQ}^{-1} D_{QX}\right) X - D_{QQ}^{-1} e_{QE} = -q_{F\phi} \phi \tag{A-22}$$

Eliminate X from Equations A-21 and A-22:

$$\begin{aligned} &\left[\left(q_{FX} + D_{QQ}^{-1} D_{QX}\right) k_{XX} \left(q_{FX}^T + D_{XQ} D_{QQ}^{-1}\right) + D_{QQ}^{-1}\right] e_{QE} \\ &\qquad = q_{F\phi} \phi - \left(q_{FX} + D_{QX}^{-1} D_{QX}\right) k_{XX} \left(q_{FX}^T e_{QT} + e_{XT} - q_{RX}^T \Delta_R\right) \end{aligned} \tag{A-23}$$

From Equations A-8 and A-9 it can be shown that

$$D_{QQ}^{-1} = k_{QQ} - k_{QX} k_{XX}^{-1} k_{XQ} \tag{A-24}$$

Combining Equations A-14, A-23, and A-24 gives

$$\begin{aligned} &\left(k_{QQ} - k_{QX} q_{FX}^T - q_{FX} k_{XQ} + q_{FX} k_{XX} q_{FX}^T\right) e_{QE} \\ &\qquad = q_{F\phi} \phi + \left(k_{QX} - q_{FX} k_{XX}\right)\left(-q_{FX}^T e_{QT} + e_{XT} - q_{RX}^T \Delta_R\right) \end{aligned} \tag{A-25}$$

Equation A-25 can be written

$$\left[-I \;\middle|\; q_{FX}\right]\begin{bmatrix} k_{QQ} & k_{QX} \\ k_{XQ} & k_{XX} \end{bmatrix}\begin{bmatrix} -I \\ q_{FX}^{T} \end{bmatrix} e_{QE}$$

$$= q_{F\phi}\,\phi - \left[-I \;\middle|\; q_{FX}\right]\begin{bmatrix} k_{QQ} & k_{QX} \\ k_{XQ} & k_{XX} \end{bmatrix}\left(\begin{bmatrix} -I \\ q_{FX}^{T} \end{bmatrix} e_{QT} - \begin{bmatrix} 0 \\ q_{RX}^{T} \end{bmatrix} \Delta_R + \begin{Bmatrix} e_{QT} \\ e_{XT} \end{Bmatrix}\right) \qquad \text{(A-26)}$$

$$\therefore\; q_{F\bar{F}}\,\bar{k}\,q_{F\bar{F}}^{T}\,e_{QE} = q_{F\phi}\,\phi - q_{F\bar{F}}\,\bar{k}\left(q_{F\bar{F}}^{T}\,e_{QT} - q_{R\bar{F}}^{T}\,\Delta_R + \bar{e}_T\right) \qquad \text{(A-27)}$$

where

$$q_{F\bar{F}} = \left[-I \;|\; q_{FX}\right] \qquad \text{(A-28)}$$

$$q_{R\bar{F}} = \left[0 \;|\; q_{RX}\right] \qquad \text{(A-29)}$$

$$\bar{k} = \begin{bmatrix} k_{QQ} & k_{QX} \\ k_{XQ} & k_{XX} \end{bmatrix} \qquad \text{(A-30)}$$

$$\bar{e}_T = \begin{Bmatrix} e_{QT} \\ e_{XT} \end{Bmatrix} \qquad \text{(A-31)}$$

From Equations A-6 and A-27

$$q_{F\bar{F}}\,\bar{k}\,q_{F\bar{F}}^{T}\,e_Q = q_{F\phi}\,\phi + q_{F\bar{F}}\,\bar{k}\left(q_{R\bar{F}}^{T}\,\Delta_R - \bar{e}_T\right) \qquad \text{(A-32)}$$

Equation A-32 is the basic deformation equation. Solving for e_Q gives statically determinate deformations as functions of external loads, support displacements, and unassembled deformations.

The matrix $q_{F\bar{F}}$ can be given a physical interpretation. The rows of $q_{F\bar{F}}$ and the columns of the first partition (Equation A-28) correspond to statically determinate element forces. The columns of the second partition correspond to redundants, thus, the columns of $q_{F\bar{F}}$ correspond to element forces. The matrix can be interpreted as a matrix of statically determinate element forces resulting from unit element forces applied as external loads to the statically determinate structure. Thus, a statically determinate element carrying a unit tensile force will carry a unit compressive force if the tension is applied to the appropriate joints as an external load. In this manner the partition $-I$ is interpreted. The partition q_{FX} is simply a matrix of statically determinate forces resulting from unit redundants.

Similarly, $q_{R\bar{F}}$ is interpreted as a matrix of reactions resulting from unit element forces applied as external loads to the statically determinate structure.

From Equation A-9

$$\bar{F} = \bar{k}\,\bar{e}_E \tag{A-33}$$

where

$$\bar{F} = \begin{Bmatrix} Q_F \\ X \end{Bmatrix} \qquad \bar{e}_E = \begin{Bmatrix} e_{QE} \\ e_{XE} \end{Bmatrix} \tag{A-34}$$

From Equations A-6, A-7, A-31, and A-34

$$\bar{e} = \bar{e}_E + \bar{e}_T \tag{A-35}$$

where

$$\bar{e} = \begin{Bmatrix} e_Q \\ e_X \end{Bmatrix} \tag{A-36}$$

From Equations A-4, A-35, and A-36

$$\bar{e}_E = \begin{Bmatrix} e_Q \\ -q_{FX}^T\, e_Q + q_{RX}^T\, \Delta_R \end{Bmatrix} - \bar{e}_T \tag{A-37}$$

$$\therefore\ \bar{e}_E = -\begin{bmatrix} -I \\ q_{FX}^T \end{bmatrix} e_Q - \bar{e}_T + \begin{bmatrix} 0 \\ q_{RX}^T \end{bmatrix} \Delta_R \tag{A-38}$$

From Equations A-28, A-29, and A-38

$$\bar{e}_E = -q_{F\bar{F}}^T\, e_Q - \bar{e}_T + q_{R\bar{F}}^T\, \Delta_R \tag{A-39}$$

From Equations A-33 and A-39

$$\bar{F} = -\,\bar{k}\, q_{F\bar{F}}^T\, e_Q + \bar{k}\left(q_{R\bar{F}}^T\, \Delta_R - \bar{e}_T\right) \tag{A-40}$$

From Equation A-2, deleting X_R and introducing the simplified notation

$$Q_R = \begin{bmatrix} 0 \,|\, q_{RX} \end{bmatrix} \begin{Bmatrix} Q_F \\ X \end{Bmatrix} + q_{R\phi}\, \phi \tag{A-41}$$

From Equations A-29, A-34, and A-41

$$Q_R = q_{R\bar{F}}\, \bar{F} + q_{R\phi}\, \phi \tag{A-42}$$

The following equation can be derived from virtual work:

$$\Delta = q_{F\Delta}^T e_Q - q_{R\Delta}^T \Delta_R \qquad \text{(A-43)}$$

where

Δ =	a column matrix of joint displacements
$q_{F\Delta}, q_{R\Delta}$ =	matrices of statically determinate element forces resulting from unit loads coinciding in position and direction with the desired joint displacements

Equations A-32, A-40, A-42, and A-43 give statically determinate deformations, element forces, reactions, and displacements

Transformation of the Element Stiffness Matrix

The matrix k (Equation A-30) is divided into partitions corresponding to statically determinate and redundant element forces. This partitioning can lead to inefficiency in subsequent operations performed on the stiffness matrix. A transformation that produces the usual and more efficient banded form of the unassembled stiffness matrix is presented.

Let F = a column matrix of element forces in which the forces corresponding to any particular element appear consecutively. Then

$$F = T\bar{F} \qquad \text{(A-44)}$$

where T is a Boolean transformation that reorders the rows of $\bar{F}$. This transformation has the property

$$T T^T = T^T T = I \qquad \text{(A-45)}$$

From Equations A-32 and A-45

$$q_{F\bar{F}} T^T T \bar{k} T^T T q_{F\bar{F}}^T e_Q = q_{F\phi} \phi + q_{F\bar{F}} T^T T \bar{k} T^T T\left(q_{R\bar{F}}^T \Delta_R - \bar{e}_T\right) \qquad \text{(A-46)}$$

$$\therefore q_{FF} k q_{FF}^T e_Q = q_{F\phi} \phi + q_{FF} k\left(q_{RF}^T \Delta_R - e_T\right) \qquad \text{(A-47)}$$

where

$$q_{FF} = q_{F\bar{F}} T^T \qquad q_{RF} = q_{R\bar{F}} T^T$$
$$k = T \bar{k} T^T \qquad e_T = T \bar{e}_T \qquad \text{(A-48)}$$

The transformed element stiffness matrix k has the desired banded form.

Let

$$K_Q = k\, q_{FF}^T$$

$$K_{QQ} = q_{FF}\, K_Q \qquad \text{(A-49)}$$

$$F_T = k\left(q_{RF}^T\, \Delta_R - e_T\right)$$

From Equations A-47 and A-49

$$K_{QQ}\, e_Q = q_{F\phi}\, \phi + q_{FF}\, F_T \qquad \text{(A-50)}$$

Let

$$e = T\,\bar{e} \qquad e_E = T\,\bar{e}_E \qquad \text{(A-51)}$$

Note that e, e_E, and e_T are column matrices of element deformations, elastic element deformations, and unassembled element deformations arranged in the same order as the element forces in F. From Equations A-40 and A-45

$$T\,\bar{F} = -T\,\bar{k}\,T^T\,T\,q_{\bar{F}\bar{F}}^T\,e_Q + T\,\bar{k}\,T^T\,T\left(q_{R\bar{F}}^T\,\Delta_R - \bar{e}_T\right) \qquad \text{(A-52)}$$

From Equations A-44, A-48, A-49 and A-52

$$F = -K_Q\, e_Q + F_T \qquad \text{(A-53)}$$

From Equations A-42, A-44, A-45, and A-48

$$Q_R = q_{RF}\, F + q_{R\phi}\, \phi \qquad \text{(A-54)}$$

Statically determinate deformations, element forces, reactions, and displacements are computed from Equations A-50, A-53, A-54, and A-43.

An equation for deflection influence coefficients can be obtained by substituting e_Q from Equation A-50 into Equation A-43, and setting $\phi = I$, $F_T = 0$, $\Delta_R = 0$, and $q_{F\Delta} = q_{F\phi}$.

$$\therefore\ \delta = q_{F\phi}^T\, K_{QQ}^{-1}\, q_{F\phi} \qquad \text{(A-55)}$$

where δ is a deflection influence matrix. If $q_{F\phi}$ is a matrix of statically determinate element forces resulting from unit loads in all unconstrained joint degrees of freedom, then $q_{F\phi}$ is square and nonsingular and δ is a nonsingular matrix of deflection influence coefficients for all unconstrained degrees of freedom. In this case

$$\delta = K^{-1} \qquad \text{(A-56)}$$

where K is the structural stiffness matrix. From Equations A-55 and A-56

$$K_{QQ} = q_{F\phi}\, K\, q_{F\phi}^T \qquad \text{(A-57)}$$

The static stiffness matrix is related to the structural stiffness matrix by a simple transformation.

STATIC FORCE METHOD

The possibility of computing statically determinate element forces before other unknowns is explored.

Reactions are not selected as redundants. From Equations A-6 and A-8, introducing the simplified notation of Equation A-20

$$e_Q = D_{QQ} Q_F + D_{QX} X + e_{QT} \tag{A-58}$$

From Equations A-7 and A-8

$$e_X = D_{XQ} Q_F + D_{XX} X + e_{XT} \tag{A-59}$$

Eliminating e_Q and e_X from Equations A-4, A-58, and A-59 gives

$$\left(q_{FX}^T D_{QQ} + D_{XQ}\right) Q_F + \left(q_{FX}^T D_{QX} + D_{XX}\right) X = -q_{FX}^T e_{QT} - e_{XT} + q_{RX}^T \Delta_R \tag{A-60}$$

$$\therefore X = -AQ_F + B \tag{A-61}$$

where

$$\begin{aligned} A &= \left(q_{FX}^T D_{QX} + D_{XX}\right)^{-1} \left(q_{FX}^T D_{QQ} + D_{XQ}\right) \\ B &= \left(q_{FX}^T D_{QX} + D_{XX}\right)^{-1} \left(-q_{FX}^T e_{QT} - e_{XT} + q_{RX}^T \Delta_R\right) \end{aligned} \tag{A-62}$$

Eliminating X from Equations A-1 and A-61 and solving for Q_F gives

$$Q_F = \left(I + q_{FX} A\right)^{-1} \left(q_{FX} B + q_\phi \phi\right) \tag{A-63}$$

A study of Equations A-62 and A-63 shows that the computations involved in this approach are excessive.

APPENDIX II

COMPARISON OF COMPUTATIONAL EFFORT FOR FORCE AND DEFORMATION METHODS

The comparison is accomplished by counting the approximate number of calculations required by each method to solve a typical aircraft structural analysis problem on the following basis:

- Only multiplications are counted.
- Element stiffness and flexibility matrices are assumed to be banded and very sparse.
- The number of loading conditions is negligible compared to the number of element forces.

This basis is considered adequate to establish the "break even" point between the force and deformation methods. The following computational procedures are considered separately:

- Force and Deflection Analysis – element forces and deflections are calculated for a limited number of loading conditions.
- Deflection Influence Coefficients – only the deflections influence matrix is computed.

Force and Deflection Analysis

The required force method equations are

$$\delta_{XX} = f_X^T \, Df_X \qquad \text{(B-1)}$$

$$X = -\delta_{XX}^{-1} f_X^T \, Df_\phi \, \phi \qquad \text{(B-2)}$$

$$F = f_X X + f_\phi \, \phi \qquad \text{(B-3)}$$

$$\Delta = f_\Delta^T \, DF \qquad \text{(B-4)}$$

where f_X, f_ϕ and f_Δ are matrices of element forces in the statically determinate structure resulting from unit redundants, external loads, and displacement loads.

The number of multiplications involved in Equation B-1 is approximately $n_X^2 \, n_{QF}$, where n_X = the number of redundants and n_{QF} = the number of statically determinate forces. This estimate accounts for the sparse banded nature of D, and the existence in f_X of a partition comprising a unit matrix. The number of multiplications in Equation B-2 is approximately n_X^3. The solution of the equilibrium equations for f_X and f_ϕ is an additional source of effort. The number of multiplications involved in the solution of these equations is estimated to be less than or equal to $(n_{QF} + n_X) \, n_{QF}^2$. The remaining equations involve negligible calculation.

$$\therefore n_{RFM} = n_{QF}^3 + n_{QF}^2 \, n_X + n_{QF} \, n_X^2 + n_X^3 \qquad \text{(B-5)}$$

where n_{RFM} = the number of multiplications involved in the force method.

$$\therefore \frac{n_{RFM}}{n_{QF}^3} = 1 + r + r^2 + r^3 \qquad \text{(B-6)}$$

where

$$r = \frac{n_X}{n_{QF}} \qquad \text{(B-7)}$$

The required deformation method equations are

$$K_{QQ} = q_{FF}\, k\, q_{FF}^T \qquad \text{(B-8)}$$

$$e_Q = K_{QQ}^{-1}\, q_{F\phi}\, \phi \qquad \text{(B-9)}$$

$$F = -K_Q\, e_Q \qquad \text{(B-10)}$$

$$\Delta = q_{F\Delta}^T\, e_Q \qquad \text{(B-11)}$$

The number of multiplications involved in Equation B-8 is approximately $n_{QF}^2\, n_X$. This estimate accounts for the sparse banded nature of k and the existence in q_{FF} of a partition comprising a unit matrix. The number of multiplications in Equation B-9 is approximately n_{QF}^3. The number of multiplications involved in solving the equilibrium equations for q_{FF} and $q_{F\phi}$ is estimated to be less than or equal to $(n_{QF} + n_X)\, n_{QF}^2$. The remaining equations involve negligible calculation

$$\therefore n_{SDF} = 2n_{QF}^3 + 2n_{QF}^2\, n_X \qquad \text{(B-12)}$$

where N_{SDF} = the number of multiplications involved in the static deformation method.

$$\therefore \frac{n_{SDF}}{n_{QF}^3} = 2 + 2r \qquad \text{(B-13)}$$

Deflection Influence Coefficients

The required force method equations are again Equations B-1 to B-4 inclusive, and the equilibrium equations. Take $\phi = I$, a unit matrix, and $q_{F\Delta} = q_{F\phi}$. In this case, the effort involved in Equations B-3 and B-4 is not negligible. The estimated number of multiplications is given by

$$\frac{n_{RFM}}{n_{QF}^3} = 3 + 3r + 2r^2 + r^3 \qquad \text{(B-14)}$$

The deformation equations are

$$K_{QQ} = q_{FF}\, k\, q_{FF}^T \tag{B-15}$$

$$\Delta = q_{F\phi}^T\, K_{QQ}^{-1}\, q_{F\phi} \tag{B-16}$$

plus the equilibrium equations. The estimated number of multiplications is

$$\frac{n_{SDM}}{n_{QF}^3} = 5 + 2r \tag{B-17}$$