SECTION 3

ASYMPTOTIC STABILITY OF EQUILIBRIUM OF A SIMPLIFIED AERODYNAMIC VEHICLE WITH FLEXIBLE TAIL

by

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In a previous paper,[1] sufficient conditions for the asymptotic stability of equilibrium of certain classes of aeroelastic systems with distributed aerodynamic load were derived via Lyapunov's direct method. In this note, similar conditions for a simplified aerodynamic vehicle with flexible tail will be derived using the same approach. The mathematical form of this aeroelastic system differs from those considered previously in the sense that the aerodynamic load can be approximated by a concentrated force applied at the domain boundary. In the previously considered systems,[1] the aerodynamic load was spatially distributed and could be approximated by introducing appropriate terms in the system equations.

Figure 3-1 shows the tail portion of a flexible vehicle. For the present analysis, it is assumed that the tail motion corresponds approximately to that of a nonuniform cantilever beam in plane bending. For this system, the dimensionless equation of the perturbed motion about its equilibrium state is given by:

$$m(x)v_o^2 \ell^2 \frac{\partial^2 w(t,x)}{\partial t^2} + v_o \ell^3 k_d(t,x) \frac{\partial w(t,x)}{\partial t} - \frac{\partial^2}{\partial x^2} EI(x) \frac{\partial^2 w(t,x)}{\partial x^2} \tag{1}$$

where both the beam deflection w and the spatial coordinate x have been normalized with respect to the beam length ℓ ; t is the dimensionless time normalized with respect to the quantity ℓ / v_o , where v_o is the free-stream velocity of air; m, k_d, and EI are linear mass density, distributed damping coefficient, and bending rigidity, respectively.

Assuming that the aerodynamic load on the tail lifting surface can be approximated by that of a thin flat plate in an incompressible flow[2], and the incremental aerodynamic moment due to tail motion is negligible, the boundary conditions have the form:

$$w(t,0) = 0 \quad , \quad \left. \frac{\partial w(t,x)}{\partial x} \right|_{x=0} = 0 \; , \tag{2}$$

$$\left. EI(x) \frac{\partial^2 w(t,x)}{\partial x^2} \right|_{x=1} = 0 \; , \quad \left. \frac{\partial}{\partial x} EI(x) \frac{\partial^2 w(t,x)}{\partial x^2} \right|_{x=1}$$

$$= 2\pi \rho_a v_o^2 \ell^2 ab \left[\frac{\partial w(t,x)}{\partial t} + \frac{\partial w(t,x)}{\partial x} \right] \Bigg|_{x=1} \tag{3}$$

where ρ_a is the mass density of the undisturbed air, a and b are the length and width of the tail lifting surfaces, respectively.

It is of interest here to derive sufficient conditions for the asymptotic stability of equilibrium of the flexible tail in the sense of Lyapunov with respect to a metric ρ defined by:

$$\rho = \left\{ \int_0^1 \left[\left(\frac{\partial w}{\partial t} \right)^2 + \sum_{n=0}^{2} \left(\frac{\partial^n w}{\partial x^n} \right)^2 \right] dx \right\}^{1/2} \tag{4}$$

Note that although the system is linear, the associated boundary-value problem is non-self-adjoint. The determination of conditions for asymptotic stability in terms of the system parameters is not a trivial task.

To apply Lyapunov's direct method to this problem, consider the following functional:

$$V = 1/2 \int_0^1 \left[m(x) v_o^2 \ell^2 \left(\frac{\partial w}{\partial t} \right)^2 + 2c_o v_o^2 \ell^2 m(x) \frac{\partial w}{\partial t} \frac{\partial w}{\partial x} + 2\pi \rho_a v_o^2 \ell^2 ab \left(\frac{\partial w}{\partial x} \right)^2 + EI(x) \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx \tag{5}$$

where c_o is a positive constant.

A sufficient condition for asymptotic stability is that V is positive definite with respect to metric ρ and $dV/dt < 0$ along any perturbed motion.[1]

The positive definiteness of V can be readily established by using the following inequality:

$$\int_0^1 G(x) \left(\frac{\partial^2 w(t,x)}{\partial x^2} \right)^2 dx \geq \left[\underset{x \,\epsilon\, (0,1)}{\text{Min}} G(x) \right] \int_0^1 \left(\frac{\partial w(t,x)}{\partial x} \right)^2 dx$$

$$\geq \left[\underset{x \,\epsilon\, (0,1)}{\text{Min}} G(x) \right] \int_0^1 w^2(t,x) \, dx \tag{6}$$

where G is a positive function of x, and choosing a constant c_η which satisfies:

$$(2\pi \rho_a ab/m(x))^{1/2} > c_o > 0 \quad \text{for all } x \,\epsilon\, (0,1) \tag{7}$$

The derivative of V with respect to t, after performing a series of partial integrations, can be expressed in the form:

$$\frac{dV}{dt} = \int_0^1 \left(m(x) v_o^2 \ell^2 \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} EI(x) \frac{\partial^2 w}{\partial x^2} \right) \frac{\partial w}{\partial t} \, dx$$

$$- \int_0^1 \left(1/2 \ c_o v_o^2 \ell^2 \frac{dm(x)}{dx} \left(\frac{\partial w}{\partial t}\right)^2 - c_o v_o^2 \ell^2 m(x) \frac{\partial^2 w}{\partial t^2} \frac{\partial w}{\partial x} \right.$$

$$\left. + \ 2 \pi \rho_a v^2 \ell^2 ab \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial t} \right) dx$$

$$+ \left[2 \pi \rho_a v^2 \ell_o^2 ab \frac{\partial w}{\partial t} \frac{\partial w}{\partial x} + 1/2 \ c_o v_o^2 \ell^2 m(x) \left(\frac{\partial w}{\partial t}\right)^2 \right.$$

$$\left. + \left(EI(x) \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x \partial t} - \frac{\partial}{\partial x} EI(x) \frac{\partial^2 w}{\partial x^2} \right) \frac{\partial w}{\partial t} \right] \Bigg|_0^1 \qquad (8)$$

where the differentiability of m has been assumed.

The above equation, in view of the system equation (1) and boundary conditions (2) and (3), reduces to:

$$\frac{dV}{dt} = - \int_0^1 \left(\left[v_o \ell^3 k_d(t,x) + 1/2 \ c_o v_o^2 \ell^2 \frac{dm(x)}{dx} \right] \left(\frac{\partial w}{\partial t}\right)^2 \right.$$

$$\left. + \ 2 \pi \rho_a v_o^2 \ell^2 ab \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial t} \right) dx$$

$$+ \int_0^1 c_o v_o^2 \ell^2 m(x) \frac{\partial^2 w}{\partial t^2} \frac{\partial w}{\partial x} dx$$

$$+ \left(1/2 \ c_o v_o^2 \ell^2 m(1) - 2 \pi \rho_a v_o^2 \ell^2 ab \right) \left(\frac{\partial w}{\partial t} \Big|_{x=1} \right)^2 \qquad (9)$$

The second integral in (9) can be rewritten in the following form by using (1)-(3) and integrating by parts:

$$\int_0^1 c_o v_o^2 \ell^2 m(x) \frac{\partial^2 w}{\partial t^2} \frac{\partial w}{\partial x} dx = - \int_0^1 \left[c_o v_o \ell^3 k_d(t,x) \frac{\partial w}{\partial t} \frac{\partial w}{\partial x} \right.$$

$$\left. - \ 1/2 \ c_o \frac{dEI(x)}{dx} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx$$

$$- 1/2 \ c_o EI(0) \left(\frac{\partial^2 w}{\partial x^2} \Big|_{x=0} \right)^2 - 2\pi c_o \ \rho_a v_o^2 \ \ell^2 \ ab \ \cdot$$

$$\cdot \left[\left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \ \frac{\partial w}{\partial t} \right] \Big|_{x=1} \qquad (10)$$

Assuming that

$$\frac{dEI(x)}{dx} < 0 \qquad \text{for all} \quad x \in (0,1) \quad , \qquad (11)$$

the following upper bound for dV/dt can be obtained by substituting (10) into (9) and applying inequality (6) :

$$\frac{dV}{dt} \leq - \int_0^1 U'(t,x) \ P \ U(t,x) \ dx - Z'(t) \ Q \ Z(t)$$

$$\cdot - c_o EI(0) \left(\frac{\partial^2 w}{\partial x^2} \Big|_{x=0} \right)^2 \qquad (12)$$

where $(\ )'$ denotes transpose, and

$$U = \text{Col} \left[\frac{\partial w}{\partial t} , \frac{\partial w}{\partial x} , \frac{\partial^2 w}{\partial x^2} \right] ,$$

$$Z = \text{Col} \left[\frac{\partial w}{\partial x} \Big|_{x=1} , \frac{\partial w}{\partial t} \Big|_{x=1} \right] ,$$

$$P = \begin{bmatrix} v_o \ell^3 k_d(t,x) + 1/2 \ c_o v_o^2 \ell^2 \ \frac{dm(x)}{dx} & c_o v_o \ \ell^3 k_d(t,x)/2 & \pi \rho_a v_o^2 \ell^2 \ ab \\ c_o v_o \ \ell^3 k_d(t,x)/2 & 1/2 \ c_o (1-\alpha) \ \underset{x}{\text{Min}} \left| \frac{dEI(x)}{dx} \right| & 0 \\ \pi \rho_a v_o^2 \ell^2 \ ab & 0 & 1/2 \ c_o \ \alpha \underset{x}{\text{Min}} \left| \frac{dEI(x)}{dx} \right| \end{bmatrix}$$

$$Q = 2\pi c_o \rho_a v_o^2 \ell^2 \ ab \begin{bmatrix} 1 & 1/2 \\ 1/2 & c_o^{-1} - m(1) \ (4 \pi \rho_a \ ab)^{-1} \end{bmatrix} ,$$

where α is a positive constant < 1 .

Clearly, dV/dt will be < 0 if Q is positive definite and P is positive definite for all $t > t_o$ and all $x \in (0,1)$, or the following Sylvester inequalities are satisfied:

$$c_o < 4 \left[1 + \frac{m(1)}{\rho_a ab} \right]^{-1} \tag{13}$$

$$\left.\begin{aligned} & 2(1-\alpha)\left(\underset{x}{\text{Min}} \left| \frac{dEI(x)}{dx} \right|\right) \left[\ell k_d(t,x) + 1/2 \; c_o v_o \frac{dm(x)}{dx} \right] > c_o v_o \ell^4 k_d^2(t,x) \\ & (1-\alpha)\left[2c_o \alpha \left(\ell k_d(t,x) + 1/2 \; v_o \frac{dm(x)}{dx} \right) \left(\underset{x}{\text{Min}} \left| \frac{dEI(x)}{dx} \right| \right) - v_o^3 (\pi \rho_a \ell ab)^2 \right] \\ & \qquad\qquad - \alpha v_o c_o^2 \ell^4 k_d^2 (t,x) > 0 \end{aligned}\right\} \begin{aligned} & \text{for all } t > t_0 \\ & \text{all } x \in (0,1) \end{aligned} \tag{14}$$

By choosing a constant c_o satisfying both (7) and (13), the inequalities (11) and (14) become a sufficient condition for asymptotic stability. Note that since $(1-\alpha)\, v_o^3 (\pi \rho_a \ell ab)^2 > 0$, only the second inequality in (14) needs to be considered.

In the case where the incremental aerodynamic moment due to tail motion is not negligible, the selection of an appropriate form for V seems to be a difficult task.

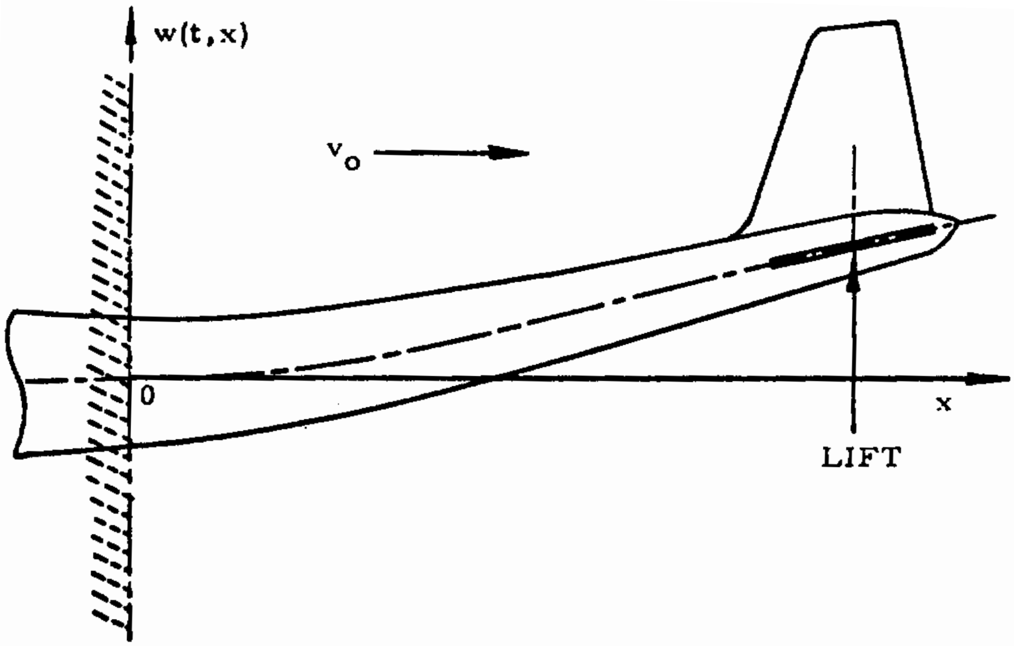

Figure 3.1

REFERENCES:

1. P. K. C. Wang: "Application of Lyapunov's Direct Method to Stability Problems in Elastic and Aeroelastic Systems," IBM Research Report RJ-305, June, 1964.

2. R. L. Bisplinghoff, H. Ashley, and R. L. Halfman: Aeroelasticity, Addison-Wesley Publishing Company, 1955.