SECTION 1
EQUATIONS OF MOTION FOR ELASTIC BODIES ENTERING A PLANETARY ATMOSPHERE

by

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ABSTRACT: The equations governing the motion of elastic bodies during their entry into a planetary atmosphere at hypersonic velocities were derived under reasonable physical assumptions. The results are applied to the derivation of a mathematical model for a simplified flexible aerodynamic re-entry vehicle.

1.1 INTRODUCTION

In the recent development of space vehicles capable of entering a planetary atmosphere at orbital or escape velocities with aerodynamic controls, the aeroelastic effects become considerably intensified because of the required high payload-vehicle mass ratio and the presence of aerodynamic heating. In order to achieve proper design of the vehicle structure and control systems, it is necessary to investigate various types of forces and moments which a vehicle may encounter during its entry into the atmosphere.

In this report we derive from first principles the equations governing the motion of elastic bodies during the entry into a planetary atmosphere, under reasonable physical assumptions. These equations are then used to obtain a mathematical model for a simplified flexible aerodynamic glide vehicle during its entry into the earth's atmosphere. Hopefully, the results can provide some insight towards rational approximations in deriving useful mathematical models for more complex flexible aerodynamic re-entry vehicles.

1.2 PRELIMINARIES

Before we proceed to derive the equations of motion for an elastic body, we shall first delineate very briefly certain fundamental concepts, definitions and equations in continuum mechanics which are pertinent to the present work.

Consider a material body defined on a bounded, closed region R of Euclidean three-space E_3. The material body is said to be deformable, if there exists a transformation Φ which maps every point $Z \in R$ onto its image material point $Z' \in R'$ -- another bounded, closed region in E_3. From the Axiom of Continuity (i.e., no region of positive, finite volume is deformed into one of zero or infinite volume.), the mapping Φ and its inverse are continuous and one-to-one. Thus, the motion of a deformable body can be considered as a sequence of continuous, one-to-one transformations. In order to specify the material point of the deformable body at any instant of time, it is necessary to introduce appropriate coordinate system. In general, there are many ways for introducing a coordinate system. It is possible to select a different coordinate system at each time t, or equivalently, to view the motion in terms of a coordinate system in motion with respect to the common frame. First, we shall restrict our attention to a single, fixed Cartesian coordinate system X with x_1, x_2, x_3 as spatial coordinate variables.

Assuming that there are no body couples or couple stresses, the motion of a deformable body, as referred to an inertial Cartesian system X, is governed by the Cauchy's first law of motion[1,2]:

$$\rho \frac{Dv_i}{Dt} = \rho f_i + \frac{\partial S_{ij}}{\partial x_j} \tag{1.2-1}$$

where ρ is the local mass density; f_i is the i-th component of the extrinsic body force per unit mass; S_{ij} is the stress tensor, and v_i is the velocity. The repeated suffix j signifies summation over $j = 1, 2, 3$ and the operator D/Dt denotes material differentiation, i.e.,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \tag{1.2-2}$$

By the assumption that there are no body couples and couple stresses, the stress tensor is symmetric, i.e. $S_{ij} = S_{ji}$.

For the present work, it is convenient to use a rigid moving frame of reference. In order to describe the motion of a deformable body in this framework, it is necessary to transform (1.2-1) to the moving coordinate system.

1.2-1 Transformation to Moving Coordinates

Let us suppose that the motion is referred to another Cartesian coordinate system $\widetilde{X}$, with $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ as spatial coordinate variables, which may be accelerated with respect to the inertial system X, but which has the same length scale as the X-system. Thus, we set

$$\widetilde{x}_i = a_{ij}(x_j - b_j) \tag{1.2-3}$$

where a_{ij} and b_j are functions of time, subject to the orthonormality conditions

$$a_{ij}a_{jk} = a_{ki}a_{kj} = \delta_{ij} \tag{1.2-4}$$

where δ_{ij} denotes the Kronecker delta. We then have the inverse transformation:

$$x_i = b_i + a_{ji}\widetilde{x}_j \tag{1.2-5}$$

The body force and the stress transform to the moving coordinate system as tensors of rank 1 and rank 2 respectively. Thus, if $\widetilde{f}_i$ and $\widetilde{S}_{ij}$ denote their components as measured in the $\widetilde{X}$-system, we have

$$f_i = a_{ji}\widetilde{f}_j , \tag{1.2-6}$$

$$S_{ij} = a_{mi}\, a_{nj}\, \tilde{S}_{mn} , \tag{1.2-7}$$

$$\frac{\partial S_{ij}}{\partial x_j} = a_{mi}\, a_{nj}\, a_{pj} \frac{\partial \tilde{S}_{mn}}{\partial \tilde{x}_n} = a_{mi} \frac{\partial \tilde{S}_{mn}}{\partial \tilde{x}_n} \tag{1.2-8}$$

The velocity vector with components v_i in the X-system has components in the $\tilde{X}$-system, but these are not necessarily the velocity components measured in that system. Rather, the velocity transformation is obtained by differentiating (1.2-5):

$$v_i = \dot{b}_i + \dot{a}_{ji}\tilde{x}_j + a_{ji}\tilde{v}_j \tag{1.2-9}$$

where $\tilde{v}_j$ is the velocity component measured relative to the moving $\tilde{X}$-system, and the overdot denotes the usual differentiation with respect to time t.

To find the transformation of the acceleration, we differentiate again:

$$\frac{Dv_i}{Dt} = \ddot{b}_i + 2\dot{a}_{ji}\,\tilde{v}_j + \ddot{a}_{ji}\,\tilde{x}_j + a_{ji}\,\frac{D\tilde{v}_j}{Dt} \tag{1.2-10}$$

Since $\tilde{v}_j$ represents the velocity component as measured in tne $\tilde{X}$-system, it is a function of $\tilde{x}_i$ and t. Consequently,

$$\frac{D\tilde{v}_j}{Dt} = \frac{\partial \tilde{v}_j}{\partial t} + \frac{\partial \tilde{v}_j}{\partial \tilde{x}_i}\,\frac{D\tilde{x}_i}{Dt} = \frac{\partial \tilde{v}_j}{\partial t} + \tilde{v}_i\,\frac{\partial \tilde{v}_j}{\partial \tilde{x}_i} \quad , \tag{1.2.11}$$

i.e., $D\tilde{v}_j/Dt$ is the acceleration measured with respect to the $\tilde{X}$-system.

With the acceleration transformation (1.2-10) and the results (1.2-6)-(1.2-8), the equations of motion (1.2-1) become

$$\rho(\ddot{b}_i + 2\dot{a}_{ji}\,\tilde{v}_j + \ddot{a}_{ji}x_j + a_{ji}\,\frac{D\tilde{v}_j}{Dt}) = \rho\, a_{ji}\tilde{f}_j + a_{mi}\,\frac{\partial \tilde{S}_{mn}}{\partial \tilde{x}_n} . \tag{1.2-12}$$

This result is more meaningful if we multiply by a_{ki} and contract. Thus,

$$\rho\,\frac{D\tilde{v}_k}{Dt} = \rho(\tilde{f}_k + \tilde{d}_k) + \frac{\partial \tilde{S}_{kn}}{\partial \tilde{x}_n} \quad , \tag{1.2-13}$$

where the D'Alembert force $\tilde{d}_k$ is given by:

$$\tilde{d}_k = -a_{ki}(\ddot{b}_i + 2\dot{a}_{ji}\tilde{v}_j + \ddot{a}_{ji}\tilde{x}_j) \ . \qquad (1.2\text{-}14)$$

1.2-2 Physical Meaning of the D'Alembert Force

Several different effects comprise the net D'Alembert force $\tilde{d}_k$ measured in the moving $\tilde{X}$-system. To sort these out, it is helpful to consider the components of $\tilde{d}_k$ in the X-system.

In the inertial X-system there is, of course, no D'Alembert force. Nevertheless, it is meaningful to consider the components d_i which the vector $\tilde{d}_k$ has in the X-system. These components can be regarded as the projections of $\tilde{d}_k$ along instantaneous axes parallel to the inertial axes. Thus,

$$d_i = a_{ki}\tilde{d}_k \qquad (1.2\text{-}15)$$

and

$$\tilde{d}_k = a_{ki}d_i \ . \qquad (1.2\text{-}16)$$

In a similar manner, the velocity components $\tilde{v}_j$ measured in the $\tilde{X}$-system have components u_i in the X-system. Only in degenerate special cases do the components u_i coincide with the v_i components of velocity measured in the X-system. In general, we have:

$$u_i = a_{ji}\tilde{v}_j = v_i - \dot{b}_i - \dot{a}_{ji}\tilde{x}_i \qquad (1.2\text{-}17)$$

$$\tilde{v}_j = a_{ji}u_i \ . \qquad (1.2\text{-}18)$$

With (1.2-3) and (1.2-18), equation (1.2-14) becomes

$$\tilde{d}_k = -a_{ki}(\ddot{b}_i + 2a_{jm}\dot{a}_{ji}u_m + a_{jm}\ddot{a}_{ji}(x_m - b_m)) \ . \qquad (1.2\text{-}19)$$

Consequently, we may set

$$d_i = G_i + C_i + Q_i + T_i \ , \qquad (1.2\text{-}20)$$

where

$$G_i = -\ddot{b}_i \ , \qquad (1.2\text{-}21)$$

$$C_i = -2a_{jm}\,\dot{a}_{ji}\,u_m \;, \qquad (1.2\text{-}22)$$

$$Q_i = -\frac{1}{2}\,(a_{jm}\,\ddot{a}_{ji} + a_{ji}\,\ddot{a}_{jm})\,(x_m - b_m)\,, \qquad (1.2\text{-}23)$$

$$T_i = \frac{1}{2}\,(a_{ji}\,a_{jm} - a_{jm}\,\ddot{a}_{ji})\,(x_m - b_m)\;. \qquad (1.2\text{-}24)$$

Since the ficticious force G_i arises from the instantaneous acceleration of the $\widetilde{X}$-system origin with respect to the X-system, we term this force the D'Alembert gravity. For similar reasons, which will appear presently, the ficticious forces C_i, Q_i, T_i will be termed, respectively, the Coriolis force, the centrifigal force, and the D'Alembert twist.

Consider now a particle which is attached to the moving $\widetilde{X}$-system. Such a particle has zero velocity relative to the $\widetilde{X}$-system; according to the velocity transformation (1.2-9), its velocity v_i relative to the inertial X-system is given by:

$$v_i = \dot{b}_i + \dot{a}_{ji}\,\widetilde{x}_j \;. \qquad (1.2\text{-}25)$$

Using the coordinate transformation (1.2-3),

$$v_i - \dot{b}_i = a_{jm}\,\dot{a}_{ji}\,(x_m - b_m)\,. \qquad (1.2\text{-}26)$$

The left side of (1.2-26) represents the velocity, with respect to the X-system, of a particle fixed to the $\widetilde{X}$-system --- with the translational motion of the $\widetilde{X}$-system subtracted. Therefore, it can be expressed in terms of Ω_i, the instantaneous angular velocity of the $\widetilde{X}$-system with respect to the inertial X-system. Thus,

$$v_i - \dot{b}_i = \epsilon_{ijm}\,\Omega_j\,(x_m - b_m) \qquad (1.2\text{-}27)$$

where ϵ_{ijm} is the usual permutation symbol. Comparing (1.2-26) and (1.2-27), we have:

$$a_{jm}\,\dot{a}_{ji} = \epsilon_{ijm}\,\Omega_j \qquad (1.2\text{-}28)$$

and (1.2-22) becomes

$$C_i = -2\epsilon_{ijm}\,\Omega_j\,u_m \qquad (1.2\text{-}29)$$

or in vector notation

$$\underline{C} = 2\,\underline{U} \times \underline{\Omega} \tag{1.2-30}$$

We now proceed to show that Q_i is properly interpreted as the centrifugal force. Differentiating the orthonormality condition (1.2-4) yields:

$$a_{jm}\,\dot{a}_{ji} + a_{ji}\,\dot{a}_{jm} = 0 \quad . \tag{1.2-31}$$

Differentiating again, we have

$$a_{jm}\,\ddot{a}_{ji} + a_{ji}\,\ddot{a}_{jm} = -2\dot{a}_{jm}\,\dot{a}_{ji} \quad . \tag{1.2-32}$$

Thus, (1.2-23) becomes:

$$Q_i = \dot{a}_{jm}\,\dot{a}_{ji}\,(x_m - b_m) \quad . \tag{1.2-33}$$

With condition (1.2-4), we may write

$$\dot{a}_{ji} = a_{ki}\,a_{kn}\,\dot{a}_{jn} = -a_{ki}\,a_{jn}\,\dot{a}_{kn} \quad , \tag{1.2-34}$$

so that

$$Q_i = a_{jn} \cdot a_{jm}\,a_{ki}\,a_{kn}\,(x_m - b_m) \tag{1.2-35}$$

We can use (1.2-28) to express this result in terms of the instantaneous angular velocity of the $\tilde{X}$-system. Thus,

$$Q_i = \epsilon_{mnp}\,\epsilon_{inq}\,\Omega_p\,\Omega_q\,(x_m - b_m) \tag{1.2-36}$$

or in vector notation

$$\underline{Q} = -[\,(\underline{X} - \underline{b}\,) \times \underline{\Omega}\,] \times \underline{\Omega} \tag{1.2-37}$$

The D'Alembert twist vanishes whenever the $\tilde{X}$-system rotates steadily with respect to the inertial X-system; it represents the acceleration of a particle, fixed to the $\tilde{X}$-system, resulting from the angular acceleration of the $\tilde{X}$-system. This acceleration is unobserved in the $\tilde{X}$-system, and is replaced by a ficticious force. To show that this is a correct interpretation of T_i, we differentiate the result (1.2-28). Thus,

$$a_{jm}\,\ddot{a}_{ji} + \dot{a}_{jm}\,\dot{a}_{ji} = \epsilon_{ijm}\,\dot{\Omega}_j \quad . \tag{1.2-38}$$

Differentiating (1.2-28) again, with the roles of i and m reversed, we obtain:

$$a_{ji}\,\ddot{a}_{jm} + \dot{a}_{ji}\,\dot{a}_{jm} = -\epsilon_{ijm}\,\dot{\Omega}_j \quad . \qquad (1.2\text{-}39)$$

With T_i defined by (1.2-24), we then have by subtraction

$$T_i = \epsilon_{imj}\,\dot{\Omega}_j\,(x_m - b_m) \qquad (1.2\text{-}40)$$

or in vector notation:

$$\underline{T} = (\underline{X} - b) \times \dot{\Omega} \ . \qquad (1.2\text{-}41)$$

The results of this section indicate that the equation of motion (1.2-13) can be written in the vector form:

$$\rho \frac{DU}{Dt} = \rho\,[\,\underline{\tilde{f}} - \underline{\ddot{b}} + 2\underline{U} \times \underline{\Omega} - (\,(\underline{X} - \underline{b}) \times \underline{\Omega}) \times \underline{\Omega} + (\underline{X} - \underline{b}) \times \underline{\dot{\Omega}}\,] + \nabla \cdot \underset{\sim}{S} \, , \qquad (1.2\text{-}42)$$

where $\underset{\sim}{S}$ is the stress dyad, and $\underline{\tilde{f}}$ is the body force per unit mass, both measured in the $\tilde{X}$-system.

1.3 EQUATIONS OF MOTION

Consider a perfectly elastic body (i.e., a body whose stress depends only on the deformation and not on its deformation history) moving in a planetary atmosphere. To derive its equations of motion, it is necessary to select a suitable set of coordinate systems, and to specify all the external and internal forces and moments acting on the elastic body.

1.3-1 Coordinate Systems

For the present work, we shall consider a Cartesian "inertial" coordinate system X with its origin at the planet's center of mass, and with a set of base vectors $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$. Let $\underline{e}_3$ be aligned with the rotational axis of the planet and $\underline{e}_2$ pointing to some "fixed" star, and with $\underline{e}_1$ completing a right-handed orthogonal coordinate system as shown in Figure 1-1. For simplicity, we shall fix the origin of the moving Cartesian coordinate system $\tilde{X}$ at the center of mass of the elastic body.* Thus, the position of the origin of the $\tilde{X}$-system relative to the inertial system X can be specified by a set of spherical coordinate variables

* In many physical situations, it is advantageous to take $\tilde{X}$ to be a curvilinear coordinate system. For simplicity, the Cartesian coordinate system is used here.

r_o, Θ and ϕ, or by a vector $\underline{b}$ with components b_i given by:

$$b_1 = (r_p + h)\sin\Theta\cos\phi\,,$$

$$b_2 = (r_p + h)\sin\Theta\cos\phi\,,$$

$$b_3 = (r_p + h)\cos\Theta\,, \qquad (1.3\text{-}1)$$

where $r_o = r_p + h$, r_p being the radius of the planet, and h the altitude of the center of mass of the elastic body.

Let R and ∂R denote the instantaneous spatial region occupied by the body and its material boundary respectively. The position of a material point in R can be specified by a vector $\underline{x}$ with components x_i in the inertial system, or a vector $\underline{\tilde{x}}$ with components $\tilde{x}_j$ in the moving coordinate system. The coordinate variables x_i and $\tilde{x}_j$ are related by a transformation of the form (1.2-5) with coefficients b_i given by (1.3-1), and a_{ji} given by the following matrix:

$$||a_{ji}|| = \begin{bmatrix} \cos\psi_1\cos\psi_2-\sin\psi_1\sin\psi_2\cos\psi_3, & -\cos\psi_1\sin\psi_2-\sin\psi_1\cos\psi_2\cos\psi_3, & \sin\psi_1\sin\psi_3 \\ \sin\psi_1\cos\psi_2+\cos\psi_1\sin\psi_1\cos\psi_3, & -\sin\psi_1\sin\psi_2+\cos\psi_1\cos\psi_2\cos\psi_3, & -\cos\psi_1\sin\psi_3 \\ \sin\psi_2\sin\psi_3 & \cos\psi_2\sin\psi_3 & \cos\psi_3 \end{bmatrix} \qquad (1.3\text{-}2)$$

where ψ_i are the Eulerian angles.

In many situations, the elastic body may be composed of many parts. It may be desirable to choose a set of coordinate systems so that the motion of each part is described with respect to its own suitable coordinate system. In most cases, these coordinate systems can be related to system $\tilde{X}$ considered here by certain time-invariant transformations.

1.3-2 Extrinsic Loads

The extrinsic loads acting on the elastic body, as measured with respect to the moving coordinate system, can be separated into body loads due to gravity and D'Alembert forces, and surface loads due to aerodynamic forces. In the case of a flexible vehicle, the forces and moments due to engine thrust can be often considered as concentrated loads.

Body Loads:

For atmosphere-entry problems, it can be assumed that the planet is spherical and that the local gravitational field within the spatial domain R of the body is uniform. Thus, the gravitational force per unit mass measured with respect to the planet-centered inertial system is given by

$$\underline{f}_g = \frac{-M_p G}{(r_p + h)^3} \underline{b} \quad , \tag{1.3-3}$$

where G is the gravitational constant (6.67×10^{-8} dyne-cm/gm^2) and M_p is the planet's mass. Transforming $\underline{f}_g$ to the moving coordinate system $\widetilde{X}$, we have

$$\widetilde{f}_{(g)j} = a_{ij} f_{(g)i} = -\left(\frac{M_p G}{(r_p + h)^3}\right) a_{ij} b_i \quad , \tag{1.3-4}$$

where the coefficients a_{ij} are given by (44) .

The total gravitational force acting on the center of mass of the elastic body is given by

$$\underline{F}_g = -\left(\frac{M_p M_T G}{(r_p + h)^3}\right) \underline{b} \tag{1.3-5}$$

where M_T is the total mass of the body. Also, the total gravitational moment $\mathfrak{M}_g$ about the body's center of mass can be computed from

$$\mathfrak{M}_g = \iiint_R \left(\frac{M_p G}{(r_p + h)^3}\right) \underline{b} \times \underline{\widetilde{x}} \rho \, dR \tag{1.3-6}$$

where $\underline{\widetilde{x}}$ is the radial vector from the center of mass to an elemental mass ρdR.

The D'Alembert force components $\widetilde{d}_k$ can be computed in a straight-forward manner using equations (1.2-14), (1.3-1) and (1.3-2). Similar to (1.3-6), the total D'Alembert moment about the center of mass is given by

$$\mathfrak{M}_d = \iiint_R \underline{\widetilde{x}} \times \underline{\widetilde{d}} \rho \, dR \tag{1.3-7}$$

where $\underline{\widetilde{d}}$ is the D'Alembert force vector with components $\widetilde{d}_k$.

Surface Loads

The dominant surface loads acting on the elastic body during its entry into a planetary atmosphere arise from aerodynamic effects. Since these loads depend on the motion, the attitude and geometric configuration of the body, which in turn affect the overall motion of the body, it is generally very difficult to describe the intrinsic physical phenomena in mathematical terms. However, when the body enters the atmosphere at hypersonic velocities, the surface aerodynamic loading can be often predicted with useful accuracy via Newtonian impact theory[3]. Here, attention will be focused on this special case.

In a Newtonian flow, it is assumed that the shock waves coincide with the body surface, and the velocity of the free stream is unaltered prior to its impact on the body surface. Consequently, the aerodynamic pressure can be determined with the simple assumption that the tangential momentum of the gas stream traversing the shock layer is conserved, but its normal momentum vanishes. Note that the impact theory only specifies the aerodynamic pressure on those portions of the body surface which is exposed to the flow. Near the shielded portions of the body surface, gas-dynamics would predict expansion (Prandtl-Meyer) flows. The absolute pressure on these portions are relatively small and can often be neglected.

Now, consider an elastic body with a simple but arbitrary shape as shown in Figure 1-2. Let $\dot{\underline{b}}$ be the velocity of its center of mass relative to the inertial system X. Neglecting the atmospheric motion of the gas, and assuming that the velocity of the deforming motion of the elastic body relative to the moving coordinate system $\tilde{X}$ is small as compared to $\dot{\underline{b}}$, we can regard $-\dot{\underline{b}}$ to be the free stream velocity relative to the body's center of mass. Let us denote the portion of the body surface exposed to the flow by ∂R_f, and the unit vector inward normal to a surface element $d(\partial R_f)$ is $|\dot{b}| \cos \eta$, where η is the angle between $-\dot{\underline{b}}$ and $\underline{n}$. Also, the rate of change of momentum of the gas impinging on $d(\partial R_f)$ is directed along $-\underline{n}$ and has magnitude

$$(\rho_g |\dot{\underline{b}}| \cos \eta \, d(\partial R_f)) |\dot{\underline{b}}| \cos \eta = \rho_g |\dot{b}|^2 \cos^2 \eta \, d(\partial R_f) , \quad (1.3\text{-}8)$$

where ρ_g is the mass density of the gas which generally varies with altitude h.

According to Newtonian impact theory, the normal force $d\underline{F}_a$ acting on $d(\partial R_f)$ is given by

$$d\underline{F}_a = (p_g + 2\rho_g |\dot{\underline{b}}|^2 \cos^2 \eta) \, \underline{n} \, d(\partial R_f) , \quad (1.3\text{-}9)$$

where p_g is the free-stream static pressure depending on ρ_g and h. Note that for the sake of simplicity, we have assumed that the centrifugal force effects in the gas are negligible. For a body with a curved surface, an approximate centrifugal force correction term may be added to (1.3-9). Also, for blunt bodies with detached shocks, a modified Newtonian impact theory[3] may be used to derive expressions for the aerodynamic loading.

Now, the total aerodynamic force $\underline{F}_a$ and moment $\mathcal{M}_a$ (about the center of mass) acting on the body are given by

$$\underline{F}_a = \iint_{\partial R_f} (p_g + 2\rho_g |\dot{b}|^2 \cos^2 \eta) \, \underline{n} \, d(\partial R_f) \qquad (1.3.10)$$

$$\mathcal{M}_a = \iint_{\partial R_f} (p_g + 2\rho_g |\dot{b}|^2 \cos^2 \eta) \, \underline{x}_s \times \underline{n} \, d(\partial R_f) \qquad (1.3\text{-}11)$$

where $\underline{\tilde{x}}_s$ is a radial vector from the center of mass to the surface element $d(\partial R_f)$. Note that the surface ∂R_f depends upon the attitude of the body with respect to $\underline{\dot{b}}$, which generally varies with time. The boundary curve which separates the shielded and exposed body surfaces is determined by the condition:

$$\underline{\dot{b}} \cdot \underline{n} = 0 \, . \qquad (1.3\text{-}12)$$

1.3-3 Rigid-body Motion

From Newton's Law, we have the equation of motion for the origin of $\tilde{X}$:

$$M_T \underline{\ddot{b}} = \underline{F}_a + \underline{F}_g + \underline{F}_c \, , \qquad (1.3\text{-}13)$$

where M_T is the instantaneous total mass of the body $\underline{F}_a$ and $\underline{F}_g$ are the given by (1.3-5) and (1.3-10) respectively, and $\underline{F}_c$ is a composite of all other external forces, including forces induced by mass outflow from the body.

For the rigid-body rotation, we have

$$I \underline{\dot{\Omega}} + \underline{\Omega} \times (I \underline{\Omega}) = \mathcal{M}_g + \mathcal{M}_a + \mathcal{M}_c \, . \qquad (1.3\text{-}14)$$

where $\underline{\Omega}$ is the angular velocity of the $\tilde{X}$ system with respect to the inertial X system $\bar{I}$ is the inertia dyad, $\mathcal{M}_g$ and $\mathcal{M}_a$ are given by (1.3-6) and (1.3-11) respectively, and $\mathcal{M}_c$ is the moment induced by $\underline{F}_c$. In general, it is convenient to choose the bases vectors for $\tilde{X}$ along the principal axis of the undeformed elastic body so that I is diagonal.

1.3-4 Deforming Motion

Assuming infinitesimal strain, the deforming motion of the elastic body with respect to system $\tilde{X}$ has the form of (1.2-42) which, in view of the results given in sections 1.2-2 and 1.2-3, becomes

$$\rho \frac{DU}{Dt} = \rho[\, A\underline{f}_g - M_T^{-1}(\underline{F}_a+\underline{F}_g+\underline{F}_c) + 2\underline{U} \times \underline{\Omega} - (A^{-1}\underline{\tilde{X}} \times \underline{\Omega}) \times \underline{\Omega} + A^{-1}\underline{\tilde{X}} \times \dot{\underline{\Omega}}] + \nabla \cdot S \tag{1.3-15}$$

where A^{-1} is the matrix $||a_{ji}||$ defined by (1.3-2), $\underline{F}_a$ and $\underline{F}_g$ are defined by (1.3-10) and (1.3-5) respectively, and $\underline{F}_g$ is given by (1.3-3).

The expression $\nabla \cdot \underset{\sim}{S}$, according to linear theory of elasticity,[1,2] can be expressed in terms of displacement $\underline{W}$:

$$\nabla \cdot \underset{\sim}{S} = (2\mu + \lambda)\, \nabla^2 W + (\lambda + \mu)\, \text{curl curl } \underline{W} \tag{1.3-16}$$

where μ and λ are the Lame constants, related to Young's modulus E and Poisson's ratio ν according to

$$\mu = \frac{E}{2(1+\nu)} \; ; \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \; . \tag{1.3-17}$$

Substituting (1.3-16) into (1.3-15) yields the complete equation describing the deforming motion of the elastic body relative to the $\tilde{X}$ system:

$$\rho \frac{DU}{Dt} = \rho[\, A\underline{f}_g - M_T^{-1}(\underline{F}_a + \underline{F}_g + \underline{F}_c) + 2\underline{U} \times \underline{\Omega} - (A^{-1}\underline{\tilde{X}} \times \Omega) \times \Omega + A^{-1}\underline{\tilde{X}} \times \dot{\underline{\Omega}}] + (2\mu + \lambda)\nabla^2 \underline{W} + (\lambda + \mu)\, \text{curl curl } \underline{W} \, . \tag{1.3-18}$$

Using expression (1.3-9) for the aerodynamic loading, the boundary conditions for (1.3-18) can be given by

$$\begin{aligned} \underset{\sim}{S}\,\underline{n} &= -(p_g + 2\rho_g\, |\dot{\underline{b}}|^2 \cos^2 \eta)\, \underline{n} \qquad \text{on} \quad \partial R_f \\ \underset{\sim}{S}\,\underline{n} &= 0 \qquad \text{on} \quad \partial R - \partial R_f \end{aligned} \tag{1.3-19}$$

Thus, the equations describing the overall motion of the elastic body consist of (1.3-13), (1.3-14), (1.3-18) with boundary condition (1.3-19) and appropriate initial conditions, and equation (1.3-12) for determining the boundary curve for ∂R_f .

1.3-5 Thermoelastic Effects

In the derivation of (1.3-18), we have neglected thermoelastic effects induced by internal, aerodynamic and solar radiant heating. In situations where these effects become important, they may be included in the mathematical description by introducing an equivalent body force $\underline{F}_h$ in (1.3-18):

$$\underline{F}_h = -(2\lambda + 3\mu)\, a_t \,\text{grad}\ T(t, \underline{\tilde{X}}) \tag{1.3-20}$$

where $T(t, \tilde{X})$ is the temperature of the body (reference with respect to the temperature T_o of the stress-free body) at point $\underline{\tilde{X}}$ and time t, and a_t is the coefficient of linear expansion of the body. The variation of T is governed by the following heat conduction equation:[4]

$$\rho c \frac{DT}{Dt} = \kappa_o \nabla^2 T - T_o(2\lambda + 3\mu)\, a_t \,\text{div}\, \underline{U} + Q_{in} \tag{1.3-21}$$

where κ_o is the coefficient of internal heat conduction; c is the specific heat of the body, and Q_{in} is the amount of heat per unit volume per unit time produced by internal heat sources.

Considering only aerodynamic heating, the boundary conditions for (1.3-21) are given by:

$$\begin{aligned} \kappa_o \frac{\partial T(t, \underline{\tilde{X}})}{\partial n} &= q_a \qquad \text{on } \partial R_f \\ \kappa_o \frac{\partial T(t, \underline{\tilde{X}})}{\partial n} &= 0 \qquad \text{on } \partial R - \partial R_f \end{aligned} \tag{1.3-22}$$

where $\partial/\partial n$ denotes differentiation in the direction of the inward normal to the surface, and q_a is the aerodynamic heat flux depending on $|\dot{\underline{b}}|$, η and other parameters.

1.4 A SIMPLIFIED FLEXIBLE VEHICLE

The equations of motion for a simplified flexible hypersonic glide vehicle entering earth's atmosphere along a path in a great-circle plane will be derived. The following assumptions, in addition to those made in Section 1.3, are introduced:

(1) The vehicle has a canard configuration (see Figure 1-3) with a flexible body which can be represented by a non-uniform cantilever beam in plane bending. The free and fixed ends of the beam are taken to be coinciding with the pivot point of the canard control surface and the center of mass respectively.

(2) For simplicity, both the tail lifting and canard surfaces are taken to be flat rigid plates. The tail lifting surface is rigidly attached to the body.

(3) The velocity roll and yaw angles* are zero at all time instants.

(4) $\underline{F}_c$ and $\mathcal{M}_c$ are set equal to zero.

For convenience, we shall take both the inertial coordinate axes (x_2, x_3) and the moving coordinate axes $(\tilde{x}_2, \tilde{x}_3)$ to be in the plane of the great circle path, and the moving coordinate system $\tilde{X}$ to have its origin at the vehicle's center of mass and one of its axes coinciding with the undeformed elastic axis of the body as shown in Figure 1-4.

With the above choice of coordinate systems, the components of $\underline{b}$ corresponding to (1.3-1) are

$$b_1 = 0, \; b_2 = (r_p + h) \sin \Theta, \quad b_3 = (r_p + h) \cos \Theta \tag{1.3-1'}$$

and the Eulerian angles are:

$$\psi_1 = \psi_2 = 0, \; \psi_3 = \Theta - \alpha - \gamma \tag{1.4-1}$$

where α is the angle between the $\tilde{X}_2$ - axis and $\dot{\underline{b}}$ (or the angle of attack of the tail lifting surface), and γ is the angle between $\dot{\underline{b}}$ and the local horizon.

Also, we have the following kinematic relationships:

$$\frac{dh}{dt} = |\dot{\underline{b}}| \cdot \sin \gamma \tag{1.4-2}$$

$$\frac{d\Theta}{dt} = |\dot{\underline{b}}| \; (r_p + h)^{-1} \cos \gamma \tag{1.4-3}$$

The aerodynamic forces acting on the canard and tail lifting surfaces can be approximated by those for flat plates computed on the basis of Newtonian impact theory. Resolving these forces into lift (L_a) and drag (D_a) components along the wind axes, we have for the tail lifting surface:

$$L_{a(T)} = s_T(p_g + 2\rho_g |\dot{\underline{b}}|^2 \sin^2 \alpha) (\cos \alpha) \operatorname{Sgn} \alpha \tag{1.4-4}$$

$$D_{a(T)} = -s_T(p_g + 2\rho_g |\dot{\underline{b}}|^2 \sin^2 \alpha) \; |\sin \alpha| \tag{1.4-5}$$

*Defined on p. 45 of reference 5.

and for the canard control surface:

$$L_{a(C)} = s_C(p_g + 2\rho_g |\dot{\underline{b}}|^2 \sin^2 \Delta)(\cos\Delta)\,\mathrm{Sgn}\,\Delta \quad , \tag{1.4-6}$$

$$D_{a(C)} = -s_C(p_g + 2\rho_g |\dot{\underline{b}}|^2 \sin^2 \Delta)\,|\sin\Delta| \quad , \tag{1.4-7}$$

where s_T and s_C are the effective surface area of the tail lifting and canard surfaces respectively, and Δ is given by

$$\Delta = \alpha + \delta_C + \tan^{-1}\left(\frac{\partial w}{\partial \tilde{x}_2}\Bigg|\ \tilde{x}_2 = \ell_1\right) \tag{1.4-8}$$

with δ_C the angle between the canard surface and the elastically deformed axis of the body and w the $\tilde{X}_3$ component of displacement.

The determination of aerodynamic pressure acting on the vehicle body is generally a difficult task. Here, we shall assume that the pressure distribution can be described by a certain analytical expression p_B depending on $\tilde{x}_2$, $\partial w/\partial \tilde{x}_2$, α, $|\dot{\underline{b}}|$ and other pertinent parameters. Also, we shall denote the total lift and drag components due to p_B by $L_{a(B)}$ and $D_{a(B)}$ respectively.

Now, the dynamical equations (1.3-13) for the motion of center of mass can be resolved into the following scalar relationships defined along the wind axes:

$$M_T \frac{d|\dot{\underline{b}}|}{dt} = -D_{a(T)} - D_{a(C)} - D_{a(B)} - M_T g \sin\gamma \quad ; \tag{1.4-9}$$

$$M_T |\dot{\underline{b}}| \left(\frac{d\gamma}{dt} - |\dot{\underline{b}}|\,(r_p + h)^{-1} \cos\gamma\right) = L_{a(T)} + L_{a(C)} + L_{a(B)} - M_T g \cos\gamma \quad ; \tag{1.4-10}$$

the local value of g is related to the sea-level value g_o according to

$$g = g_o\left(\frac{r_p}{r_p + h}\right)^2 \tag{1.4-11}$$

Also, in view of assumption (3) and (4), the equation describing the rigid-body rotational motion about vehicle's center of mass as given by (1.3-14) reduces to

$$I_o \ddot{\psi}_3 = \mathcal{M}_g + \mathcal{M}_a \tag{1.4-12}$$

where

$$\mathcal{M}_g = \int_{-\ell_2}^{\ell_1} g \tilde{x}_2 \rho(\tilde{x}_2) \cos(\alpha + \gamma)\, d\tilde{x}_2 \qquad (1.4\text{-}13)$$

$$\mathcal{M}_a = s_T \ell_o (p_g + 2\rho_g |\dot{\underline{b}}|^2 \sin^2 \alpha) \mathrm{Sgn}\, \alpha - s_C \ell_1 (\mathrm{Sgn}\, \Delta) \cdot$$

$$(p_g + 2\rho_g |\dot{\underline{b}}|^2 \sin^2 \Delta) \cos \left[\delta_C + \tan^{-1} \left(\frac{\partial w}{\partial \tilde{x}_2} \bigg|_{\tilde{x}_2 = \ell_1} \right) \right] + \mathcal{M}_{a(B)} \qquad (1.4\text{-}14)$$

where $\mathcal{M}_{a(B)}$ is the total moment about the center of mass due to aerodynamic pressure p_B acting on the vehicle body, and I_o is the moment of inertia of the undeformed vehicle about the $\tilde{x}_1$-axis.

The equation governing the deforming motion of the vehicle body can be derived (1.3-18) using the relation:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi_3 & -\sin\psi_3 \\ 0 & \sin\psi_3 & \cos\psi_3 \end{bmatrix} \qquad (1.4\text{-}15)$$

By a lengthy but straight-forward computation using (1.4-2), (1.4-3) and (1.4-10) it can be shown that the equation for the bending motion can be reduced to the form:

$$\rho \frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2}{\partial \tilde{x}_2^2} EI(\tilde{x}_2) \frac{\partial^2 w}{\partial \tilde{x}_2^2} + \rho \left[g \sin \alpha \sin \gamma + \frac{d |\dot{\underline{b}}|}{dt} \sin \alpha - M_T^{-1} (L_{a(C)} \right.$$

$$\left. + L_{a(B)} + L_{a(T)}) \cos \alpha \right] + \rho (\dot{\psi}_3^2 w + \ddot{\psi}_3 \tilde{x}_2) + p_B \qquad (1.4\text{-}16)$$

where the deformation in the $\tilde{x}_2$ direction is assumed to be negligible.

The boundary conditions for (1.4-16) are:

$$w(t,0) = \frac{\partial w}{\partial \tilde{x}_2}\bigg|_{\tilde{x}_2 = 0} = 0 ,$$

$$EI(\tilde{x}_2) \frac{\partial^2 w}{\partial \tilde{x}_2^2}\bigg|_{\tilde{x}_2 = \ell_1} = 0 ,$$

$$-\frac{\partial}{\partial \tilde{x}_2} EI(\tilde{x}_2) \frac{\partial^2 w}{\partial \tilde{x}_2^2}\bigg|_{\tilde{x}_2 = \ell_1} = -s_C(p_g + 2\rho_g |\dot{\underline{b}}|^2 \sin^2 \Delta) \cos\left[\delta_C + \tan^{-1}\left(\frac{\partial w}{\partial \tilde{x}_2}\bigg|_{\tilde{x}_2 = \ell_1}\right)\right]$$

(1.4-17)

Thus, the equations describing the overall vehicle motion consist of (1.4-2), (1.4-3), (1.4-9), (1.4-10, (1.4-12) and (1.4-16).

1.5 REFERENCES

1. C. Truesdell, "The mechanical foundations of elasticity and fluid dynamics," J. Rational Mech. and Analysis, Vol. 1, pp. 125-300, 1952; Vol. 3, pp. 593-616, 1954.

2. A. C. Eringen, Nonlinear Theory of Continuous Media, McGraw-Hill Book Co., N. Y. 1962.

3. R. W. Truitt, Hypersonic Aerodynamics, The Ronald Press Co., N. Y. 1959.

4. A. E. Green and W. Zerna, Theoretical Elasticity, Oxford University Press, N. Y. 1954.

5. A. Miele, Flight Mechanics, Vol. 1, Theory of Flight Paths, Addison-Wesley Pub. Co., Mass. 1962.

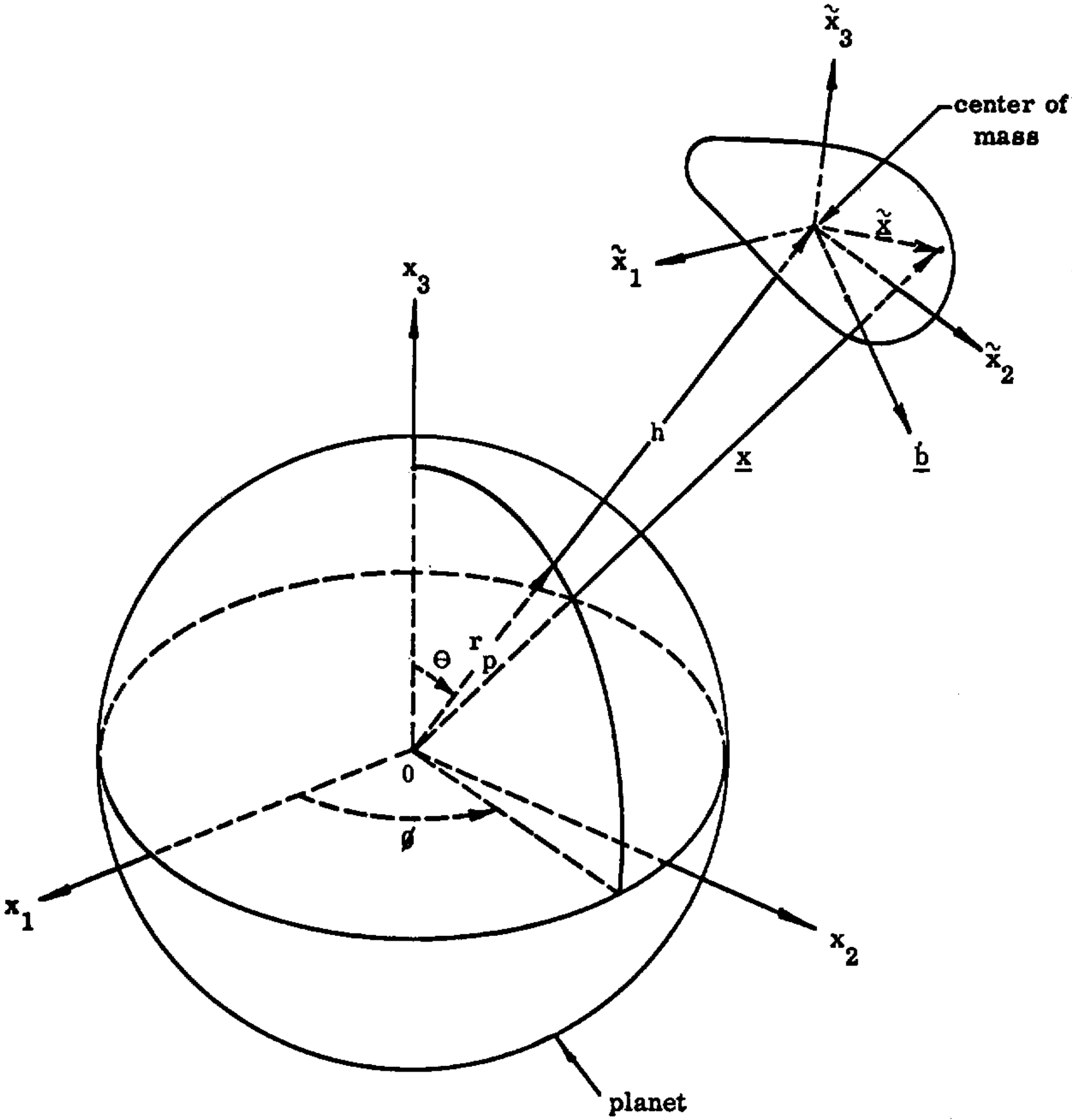

$\tilde{x}_3$
center of mass
$\tilde{\underline{x}}$
$\tilde{x}_1$
x_3
$\tilde{x}_2$
h
$\underline{x}$
$\underline{b}$
Θ
r_p
0
ϕ
x_1
x_2
planet

Figure 1.1

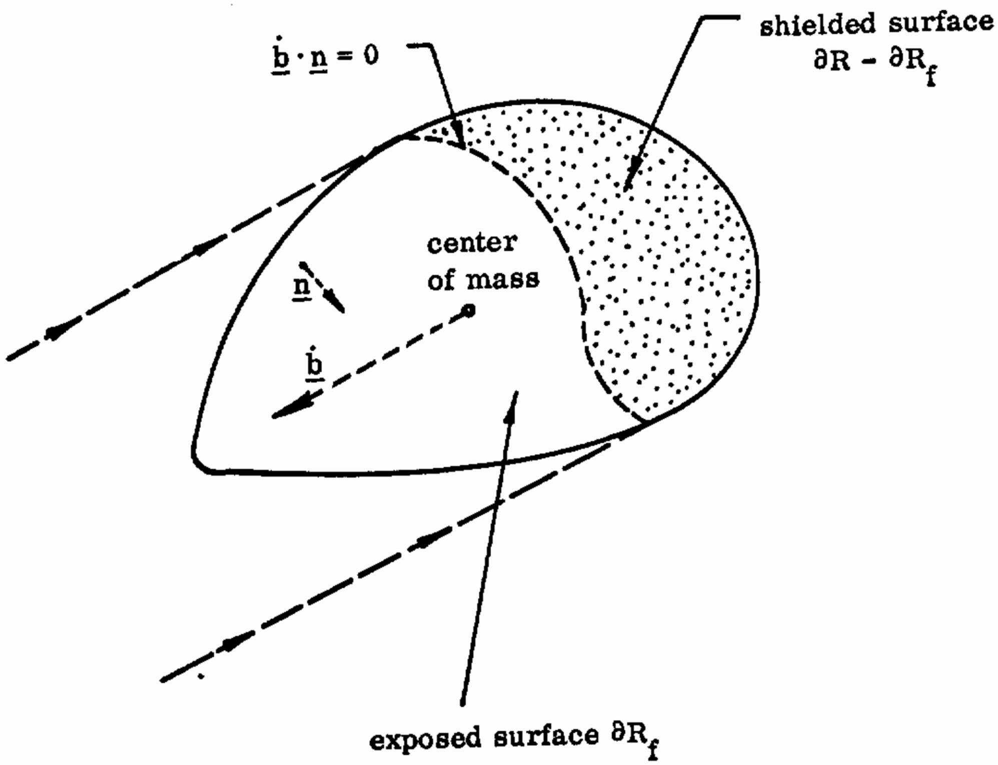

$\dot{\underline{b}} \cdot \underline{n} = 0$
shielded surface
$\partial R - \partial R_f$
center
of mass
$\underline{n}$
$\dot{\underline{b}}$
exposed surface ∂R_f

Figure 1.2

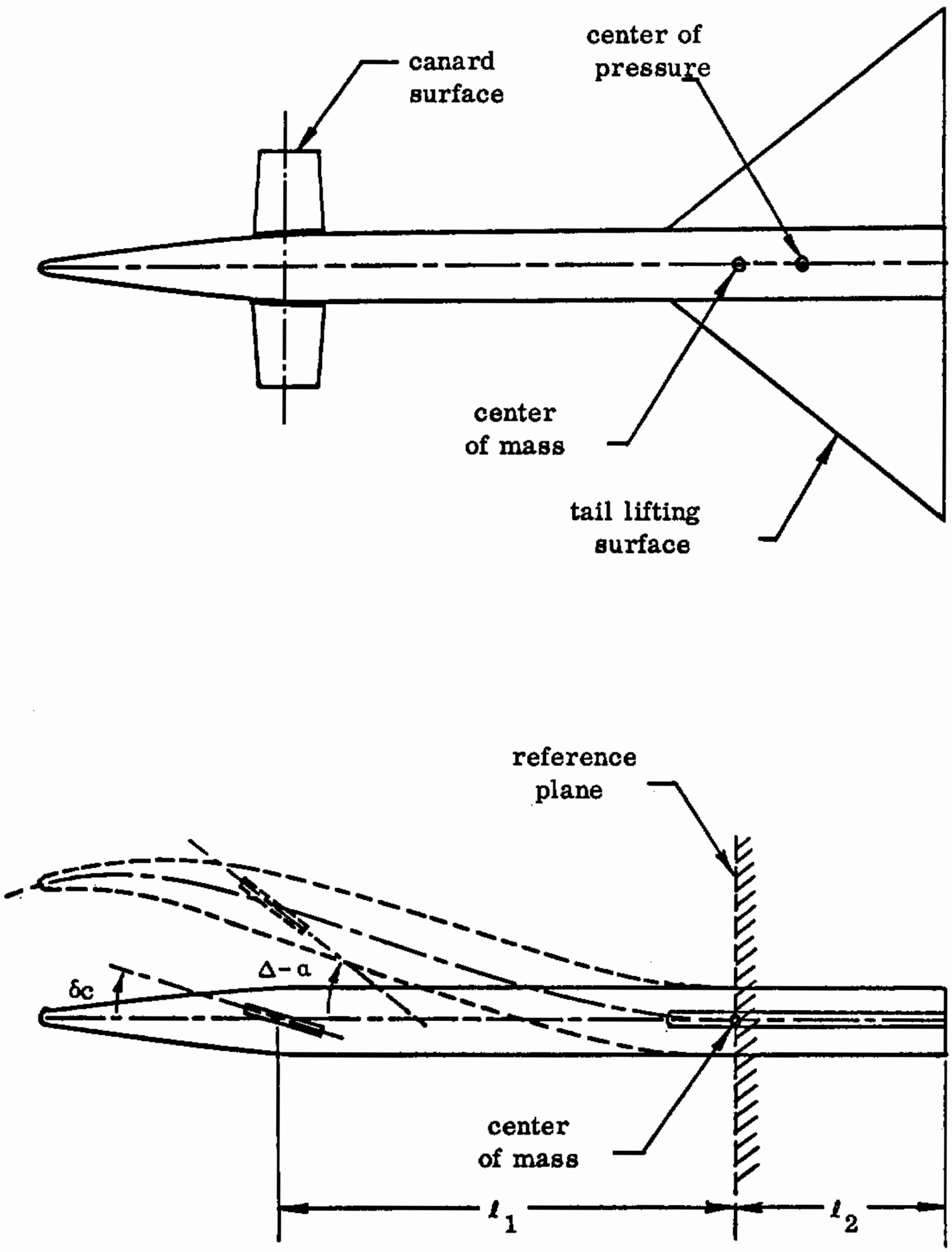
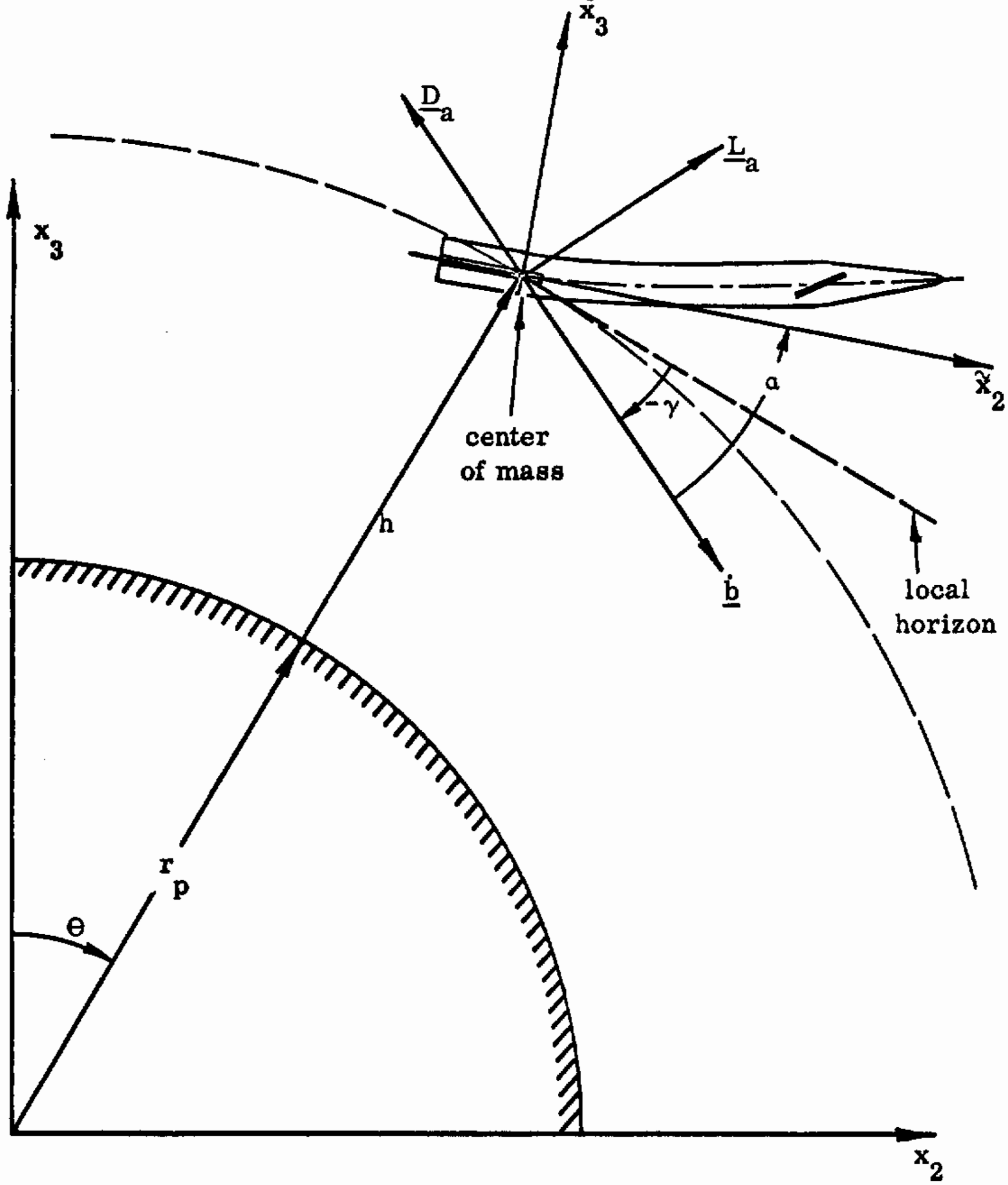

canard
surface
center of
pressure
center
of mass
tail lifting
surface
reference
plane
δc
Δ - α
center
of mass
l_1
l_2

Figure 1.3



$\tilde{x}_3$
$\underline{D}_a$
$\underline{L}_a$
x_3
$\tilde{x}_2$
γ
α
center
of mass
h
$\underline{b}$
local
horizon
r_p
θ
x_2

Figure 1.4